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Some trace inequalities for matrix means

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Abstract
In this short note, we present some trace inequalities for matrix means. Our results are generalizations of the ones shown by Bhatia, Lim, and Yamazaki.

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1 Introduction

Let M_n be the space of $n \times n$ complex matrices. Let $A, B \in M_n$ be positive definite, the weighted geometric mean of A and B, denoted by $A\#B$, is defined as

$$A\#_t B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^t A^{1/2}.$$

When $t = \frac{1}{2}$, this is the geometric mean, denoted by $A\#B$. For $A \in M_n$, we denote the vector of eigenvalues by $\lambda(A) = (\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$, and we assume that the components of $\lambda(A)$ are in descending order. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n.

Recently, Bhatia, Lim, and Yamazaki proved in [1] that if $A, B \in M_n$ are positive definite, then

$$\operatorname{tr}\big(A + B + 2(A\#B)\big) \le \operatorname{tr}\big(\big(A^{1/2} + B^{1/2}\big)^2\big) \tag{1.1}$$

and

$$\operatorname{tr}\big(\big(A + B + 2(A\#B)\big)^2\big) \le \operatorname{tr}\big(\big(A^{1/2} + B^{1/2}\big)^4\big). \tag{1.2}$$

These authors also have shown in [1] that if $A, B \in M_n$ are positive definite and $0 < t < 1$, then

$$\operatorname{tr}(A\#_t B + B\#_t A) \le \operatorname{tr}\big(A^{1-t}B^t + A^t B^{1-t}\big) \tag{1.3}$$

and

$$\operatorname{tr}\big((A\#_t B + B\#_t A)^2\big) \le \operatorname{tr}\big(\big|\big(A^{1-t}B^t + A^t B^{1-t}\big)\big|^2\big). \tag{1.4}$$

In this short note, we first obtain a trace inequality, which is similar to inequality (1.1). Meanwhile, we also obtain generalizations of inequalities (1.1), (1.2), (1.3), and (1.4).

2 Main results

In this section, we first give a trace inequality, which is similar to inequality (1.1). To do this, we need the following lemmas.

Lemma 2.1 ([2]) *Let $A, B \in M_n$ be positive definite. Then*

$$\prod_{j=1}^{k} \lambda_j(AB) \le \prod_{j=1}^{k} \lambda_j^{1/2}\left(A^2B^2\right), \quad 1 \le k \le n.$$

Lemma 2.2 ([3]) *Let $A, B \in M_n$. If $\lambda(A), \lambda(B) > 0$ such that*

$$\prod_{j=1}^{k} \lambda_j(A) \le \prod_{j=1}^{k} \lambda_j(B), \quad 1 \le k \le n,$$

then

$$\det(I + A) \le \det(I + B).$$

Theorem 2.1 *Let A and B be positive definite. Then*

$$\operatorname{tr}\left(\log\left(A^{1/2} + B^{1/2}\right)^2\right) \le \operatorname{tr}\left(\log\left(A + B + 2(A\#B)\right)\right).$$

Proof By Lemma 2.1, we have

$$\begin{aligned}\prod_{j=1}^{k} \lambda_j\left(A^{-1/2}B^{1/2}\right) &= \prod_{j=1}^{k} \lambda_j\left(B^{1/2}A^{-1/2}\right)\\ &\le \prod_{j=1}^{k} \lambda_j\left(\left(A^{-1/2}BA^{-1/2}\right)^{1/2}\right), \quad 1 \le k \le n.\end{aligned}$$

Using Lemma 2.2, we get

$$\det\left(I + A^{-1/2}B^{1/2}\right) \le \det\left(I + \left(A^{-1/2}BA^{-1/2}\right)^{1/2}\right) \tag{2.1}$$

and

$$\det\left(I + B^{1/2}A^{-1/2}\right) \le \det\left(I + \left(A^{-1/2}BA^{-1/2}\right)^{1/2}\right). \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\det\left(I + A^{-1/2}B^{1/2}\right)\det\left(I + B^{1/2}A^{-1/2}\right) \le \det\left(I + 2\left(A^{-1/2}BA^{-1/2}\right)^{1/2} + A^{-1/2}BA^{-1/2}\right),$$

which is equivalent to

$$\begin{aligned}&\det\left(I + A^{-1/2}B^{1/2} + B^{1/2}A^{-1/2} + A^{-1/2}BA^{-1/2}\right)\\ &\quad \le \det\left(I + 2\left(A^{-1/2}BA^{-1/2}\right)^{1/2} + A^{-1/2}BA^{-1/2}\right).\end{aligned} \tag{2.3}$$

Multiplying $\det A^{1/2}$ both sides in inequality (2.3), we have

$$\det\bigl(A + B + A^{1/2}B^{1/2} + B^{1/2}A^{1/2}\bigr) \le \det\bigl(A + B + 2A^{1/2}\bigl(A^{-1/2}BA^{-1/2}\bigr)^{1/2}A^{1/2}\bigr). \tag{2.4}$$

Note that $\log\det X = \operatorname{tr}\log X$, inequality (2.4) implies

$$\operatorname{tr}\bigl(\log\bigl(A^{1/2} + B^{1/2}\bigr)^2\bigr) \le \operatorname{tr}\bigl(\log\bigl(A + B + 2(A\#B)\bigr)\bigr).$$

This completes the proof. □

Next, we show generalizations of inequalities (1.1), (1.2), (1.3), and (1.4). To do this, we need the following lemma.

Lemma 2.3 ([2]) *Let $A, B \in M_n$ and $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 0$. Then*

$$\|AB\| \le \bigl\||A|^p\bigr\|^{1/p}\bigl\||B|^q\bigr\|^{1/q}.$$

This is the Hölder inequality of unitary invariant norms for matrices. For more information on this inequality and its applications the reader is referred to [4] and the references therein.

Theorem 2.2 *Let A and B be positive definite and $1 \le r \le 2$. Then*

$$\operatorname{tr}\bigl(\bigl(A + B + 2(A\#B)\bigr)^r\bigr) \le (2 - r)\operatorname{tr}\bigl(A^{1/2} + B^{1/2}\bigr)^2 + (r - 1)\operatorname{tr}\bigl(A^{1/2} + B^{1/2}\bigr)^4. \tag{2.5}$$

Proof Let

$$p = \frac{1}{2 - r}, \qquad q = \frac{1}{r - 1},$$

then

$$\frac{1}{p} + \frac{1}{q} = 1, \quad p, q > 0.$$

By Lemma 2.3, we obtain

$$\begin{aligned}
&\operatorname{tr}\bigl(\bigl(A + B + 2(A\#B)\bigr)^r\bigr) \\
&\quad = \operatorname{tr}\bigl(\bigl(A + B + 2(A\#B)\bigr)^{2-r}\bigl(A + B + 2(A\#B)\bigr)^{2r-2}\bigr) \\
&\quad \le \bigl(\operatorname{tr}\bigl(A + B + 2(A\#B)\bigr)^{p(2-r)}\bigr)^{1/p}\bigl(\operatorname{tr}\bigl(A + B + 2(A\#B)\bigr)^{q(2r-2)}\bigr)^{1/q} \\
&\quad = \bigl(\operatorname{tr}\bigl(A + B + 2(A\#B)\bigr)\bigr)^{2-r}\bigl(\operatorname{tr}\bigl(A + B + 2(A\#B)\bigr)^2\bigr)^{r-1}.
\end{aligned} \tag{2.6}$$

It follows from (1.1), (1.2), and (2.6) that

$$\operatorname{tr}\bigl(\bigl(A + B + 2(A\#B)\bigr)^r\bigr) \le \bigl(\operatorname{tr}\bigl(A^{1/2} + B^{1/2}\bigr)^2\bigr)^{2-r}\bigl(\operatorname{tr}\bigl(A^{1/2} + B^{1/2}\bigr)^4\bigr)^{r-1}.$$

By Young's inequality, we have

$$\operatorname{tr}\bigl(\bigl(A + B + 2(A\#B)\bigr)^r\bigr) \le (2-r)\operatorname{tr}\bigl(A^{1/2} + B^{1/2}\bigr)^2 + (r-1)\operatorname{tr}\bigl(A^{1/2} + B^{1/2}\bigr)^4.$$

This completes the proof. □

Remark 2.1 Putting $r = 1$ in (2.5), we get (1.1). Putting $r = 2$ in (2.5), we get (1.2). Therefore, inequality (2.5) is a generalization of inequalities (1.1) and (1.2).

Remark 2.2 Let A and B be positive definite. By the concavity of $f(x) = x^r$, $x \ge 0$, $0 < r < 1$, then we have

$$n^{r-1}\operatorname{tr} f(X) \le f(\operatorname{tr} X),$$

where X is positive definite. It follows from this last inequality and inequality (1.1) that

$$\begin{aligned} n^{r-1}\operatorname{tr}\bigl(A + B + 2(A\#B)\bigr)^r &\le \bigl(\operatorname{tr}\bigl(A + B + 2(A\#B)\bigr)\bigr)^r \\ &\le \bigl(\operatorname{tr}\bigl(\bigl(A^{1/2} + B^{1/2}\bigr)^2\bigr)\bigr)^r. \end{aligned}$$

Meanwhile, we also have

$$f(\operatorname{tr} X) \le \operatorname{tr} f(X),$$

which implies

$$n^{r-1}\operatorname{tr}\bigl(A + B + 2(A\#B)\bigr)^r \le \operatorname{tr}\bigl(\bigl(A^{1/2} + B^{1/2}\bigr)^{2r}\bigr).$$

This is a complement of (1.1) for $0 < r < 1$.

Theorem 2.3 *Let A and B be positive definite and $1 \le r \le 2$. Then*

$$\operatorname{tr}\bigl((A\#_t B + B\#_t A)^r\bigr) \le (2-r)\operatorname{tr}\bigl(A^{1-t}B^t + A^t B^{1-t}\bigr) + (r-1)\operatorname{tr}\bigl(\bigl|\bigl(A^{1-t}B^t + A^t B^{1-t}\bigr)\bigr|^2\bigr). \quad (2.7)$$

Proof Let

$$p = \frac{1}{2-r}, \qquad q = \frac{1}{r-1},$$

then

$$\frac{1}{p} + \frac{1}{q} = 1, \quad p, q > 0.$$

By Lemma 2.3, we obtain

$$\begin{aligned} \operatorname{tr}\bigl((A\#_t B + B\#_t A)^r\bigr) &= \operatorname{tr}\bigl((A\#_t B + B\#_t A)^{2-r}(A\#_t B + B\#_t A)^{2r-2}\bigr) \\ &\le \bigl(\operatorname{tr}(A\#_t B + B\#_t A)^{p(2-r)}\bigr)^{1/p}\bigl(\operatorname{tr}(A\#_t B + B\#_t A)^{q(2r-2)}\bigr)^{1/q} \\ &= \bigl(\operatorname{tr}(A\#_t B + B\#_t A)\bigr)^{2-r}\bigl(\operatorname{tr}(A\#_t B + B\#_t A)^2\bigr)^{r-1}. \end{aligned} \quad (2.8)$$

It follows from (1.3), (1.4), and (2.8) that

$$\operatorname{tr}\big((A\#_t B + B\#_t A)^r\big) \le \big(\operatorname{tr}\big(A^{1-t}B^t + A^t B^{1-t}\big)\big)^{2-r}\big(\operatorname{tr}\big(\big|\big(A^{1-t}B^t + A^t B^{1-t}\big)\big|^2\big)\big)^{r-1}.$$

By Young's inequality, we have

$$\operatorname{tr}\big((A\#_t B + B\#_t A)^r\big) \le (2-r)\operatorname{tr}\big(A^{1-t}B^t + A^t B^{1-t}\big) + (r-1)\operatorname{tr}\big(\big|\big(A^{1-t}B^t + A^t B^{1-t}\big)\big|^2\big).$$

This completes the proof. □

Remark 2.3 Putting $r = 1$ in (2.7), we get (1.3). Putting $r = 2$ in (2.7), we get (1.4). Therefore, inequality (2.7) is a generalization of inequalities (1.3) and (1.4).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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