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Least-squares Hermitian problem of complex matrix equation $(AXB, CXD) = (E, F)$

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Abstract
In this paper, we present a direct method to solve the least-squares Hermitian problem of the complex matrix equation $(AXB, CXD) = (E, F)$ with complex arbitrary coefficient matrices A, B, C, D and the right-hand side E, F. This method determines the least-squares Hermitian solution with the minimum norm. It relies on a matrix-vector product and the Moore-Penrose generalized inverse. Numerical experiments are presented which demonstrate the efficiency of the proposed method.

Keywords: matrix equation; least-squares solution; Hermitian matrices; Moore-Penrose generalized inverse

1 Introduction

Let A, B, C, D, E, and F are given matrices of suitable sizes defined over the complex number field. We are interested in the analysis of the linear matrix equation

$$(AXB, CXD) = (E, F) \tag{1}$$

to be solved for $X \in \mathbf{C}^{n\times n}$. Here and in the following $\mathbf{C}^{m\times n}$ denotes the set of all $m \times n$ complex matrices, while the set of all $m \times n$ real matrices is denoted by $\mathbf{R}^{m\times n}$. In particular, we focus on the least-squares Hermitian solution with the least norm of (1), which can be described as follows.

Problem 1 Given $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times s}$, $C \in \mathbf{C}^{m\times n}$, $D \in \mathbf{C}^{n\times t}$, $E \in \mathbf{C}^{m\times s}$, and $F \in \mathbf{C}^{m\times t}$, let

$$H_L = \Big\{X | X \in \mathbf{HC}^{n\times n}, \\ \|AXB - E\|^2 + \|CXD - F\|^2 = \min_{X_0 \in \mathbf{HC}^{n\times n}} \big(\|AX_0B - E\|^2 + \|CX_0D - F\|^2\big)\Big\}.$$

Find $X_H \in H_L$ such that

$$\|X_H\| = \min_{X \in H_L} \|X\|, \tag{2}$$

where $\mathbf{HC}^{n\times n}$ denotes the set of all $n \times n$ Hermitian matrices.

Correspondingly, the set of all $n \times n$ real symmetric matrices and the set of all $n \times n$ real anti-symmetric matrices are denoted by $\mathbf{SR}^{n\times n}$ and $\mathbf{ASR}^{n\times n}$, respectively.

Nowadays, matrix equations are very useful in numerous applications such as control theory [1, 2], vibration theory [3], image processing [4, 5] and so on. Therefore it is an active area of research to solve different matrix equations [2, 6–15]. For the real matrix equation (1), in [16], least-squares solutions with the minimum-norm were obtained by using generalized singular value decomposition and canonical correlation decomposition of matrices (CCD). In [17], the quaternion matrix equation (1) was considered and the least-squares solution with the least norm was given through the use of the Kronecker product and the Moore-Penrose generalized inverse. Though the matrix equations of the form (1) are studied in the literature, less or even no attention was paid to the least-squares Hermitian solution of (1) over the complex field which is studied in this paper. A special vectorization, as we defined in [18], of the matrix equation (1) is carried out and Problem 1 is turned into the least-squares unconstrained problem of a system of real linear equations.

The notations used in this paper are summarized as follows: For $A \in \mathbf{C}^{m\times n}$, the symbols A^T, A^H and A^+ denote the transpose matrix, the conjugate transpose matrix and the Moore-Penrose generalized inverse of matrix A, respectively. The identity matrix is denoted by I. For $A = (a_{ij}) \in \mathbf{C}^{m\times n}$, $B = (b_{ij}) \in \mathbf{C}^{p\times q}$, the Kronecker product of A and B is defined by $A \otimes B = (a_{ij}B) \in \mathbf{C}^{mp\times nq}$. For matrix $A \in \mathbf{C}^{m\times n}$, the stretching operator $\operatorname{vec}(A)$ is defined by $\operatorname{vec}(A) = (a_1, a_2, \ldots, a_n)^T$, where a_i is the ith column of A.

For all $A, B \in \mathbf{C}^{m\times n}$, we define the inner product $\langle A, B\rangle = \operatorname{tr}(A^H B)$. Then $\mathbf{C}^{m\times n}$ is a Hilbert inner product space and the norm of a matrix generated by this inner product is the matrix Frobenius norm $\|\cdot\|$. Further, denote the linear space $\mathbf{C}^{m\times n} \times \mathbf{C}^{m\times t} = \{[A, B] | A \in \mathbf{C}^{m\times n}, B \in \mathbf{C}^{m\times t}\}$, and for the matrix pairs $[A_i, B_i] \in \mathbf{C}^{m\times n} \times \mathbf{C}^{m\times t}$ $(i = 1, 2)$, we can define their inner product as follows: $\langle [A_1, B_1], [A_2, B_2]\rangle = \operatorname{tr}(A_2^H A_1) + \operatorname{tr}(B_2^H B_1)$. Then $\mathbf{C}^{m\times n} \times \mathbf{C}^{m\times t}$ is also a Hilbert inner space. The Frobenius norm of the matrix pair $[A, B] \in \mathbf{C}^{m\times n} \times \mathbf{C}^{m\times t}$ can be derived:

$$\begin{aligned} \bigl\|[A, B]\bigr\| &= \sqrt{\bigl\langle [A, B], [A, B]\bigr\rangle} \\ &= \sqrt{\operatorname{tr}\bigl(A^H A\bigr) + \operatorname{tr}\bigl(B^H B\bigr)} \\ &= \sqrt{\|A\|^2 + \|B\|^2}. \end{aligned}$$

The structure of the paper is the following. In Section 2, we deduce some results in Hilbert inner product $\mathbf{C}^{m\times n} \times \mathbf{C}^{m\times t}$ which are important for our main results. In Section 3, we introduce a matrix-vector product for the matrices and vectors of $\mathbf{C}^{m\times n}$ based on which we consider the structure of (AXB, CXD) over $X \in \mathbf{HC}^{n\times n}$. In Section 4, we derive the explicit expression for the solution of Problem 1. Finally we express the algorithm for Problem 1 and perform some numerical experiments.

2 Some preliminaries

In this section, we will prove some theorems which are important for the proof of our main result. Given $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^n$, the solution of the linear equations $Ax = b$ involves two cases: inconsistent and consistent. The former leads to the solution in the least-squares sense, which can be expressed as $\operatorname{argmin}_{x\in\mathbf{R}^n} \|Ax - b\|$. This problem can be solved by solving the corresponding normal equations:

$$A^T A x = A^T b, \tag{3}$$

moreover, we have

$$\{x | x \in \mathbf{R}^n, \|Ax - b\| = \min\} = \{x | x \in \mathbf{R}^n, A^T A x = A^T b\}.$$

As for the latter, (3) still holds, further, the solution set of $Ax = b$ and that of (3) is the same, that is,

$$\{x | x \in \mathbf{R}^n, Ax = b\} = \{x | x \in \mathbf{R}^n, A^T A x = A^T b\}.$$

It is should be noticed that (3) is always consistent. Therefore, solving $Ax = b$ is usually translated into solving the corresponding consistent equations (3). In the following, we will extend the conclusion to a more general case. To do this, we first give the following problem.

Problem 2 Given complex matrices $A_1, A_2, \ldots, A_l \in \mathbf{C}^{m\times n}$, $B_1, B_2, \ldots, B_l \in \mathbf{C}^{m\times n}$, $C \in \mathbf{C}^{m\times n}$, and $D \in \mathbf{C}^{m\times n}$, find $k = (k_1, k_2, \ldots, k_l)^T \in \mathbf{R}^l$ such that

$$\sum_{i=1}^{l} k_i [A_i, B_i] = \left[\sum_{i=1}^{l} k_i A_i, \sum_{i=1}^{l} k_i B_i\right] = [C, D]. \tag{4}$$

Theorem 1 *Assume the matrix equation* (4) *in Problem* 2 *is consistent*. *Let*

$$E_i = \begin{bmatrix} \mathrm{Re}(A_i) \\ \mathrm{Im}(A_i) \\ \mathrm{Re}(B_i) \\ \mathrm{Im}(B_i) \end{bmatrix}, \qquad F = \begin{bmatrix} \mathrm{Re}(C) \\ \mathrm{Im}(C) \\ \mathrm{Re}(D) \\ \mathrm{Im}(D) \end{bmatrix}.$$

Then the set of vectors k that satisfies (4) *is exactly the set that solves the following consistent system*:

$$\begin{bmatrix} \langle E_1, E_1\rangle & \langle E_1, E_2\rangle & \cdots & \langle E_1, E_l\rangle \\ \langle E_2, E_1\rangle & \langle E_2, E_2\rangle & \cdots & \langle E_2, E_l\rangle \\ \vdots & \vdots & & \vdots \\ \langle E_l, E_1\rangle & \langle E_l, E_2\rangle & \cdots & \langle E_l, E_l\rangle \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_l \end{bmatrix} = \begin{bmatrix} \langle E_1, F\rangle \\ \langle E_2, F\rangle \\ \vdots \\ \langle E_l, F\rangle \end{bmatrix}. \tag{5}$$

Proof By (3), we have

$$\begin{aligned} &\left[\sum_{i=1}^{l} k_i A_i, \sum_{i=1}^{l} k_i B_i\right] = [C, D] \\ &\quad\Longleftrightarrow \quad \sum_{i=1}^{l} k_i [A_i, B_i] = [C, D] \end{aligned}$$

$$\iff \quad \sum_{i=1}^{l} k_i \begin{bmatrix} \mathrm{Re}(A_i) \\ \mathrm{Im}(A_i) \\ \mathrm{Re}(B_i) \\ \mathrm{Im}(B_i) \end{bmatrix} = \begin{bmatrix} \mathrm{Re}(C) \\ \mathrm{Im}(C) \\ \mathrm{Re}(D) \\ \mathrm{Im}(D) \end{bmatrix}$$

$$\iff \begin{bmatrix} \mathrm{vec}(\mathrm{Re}\,A_1)^T & \mathrm{vec}(\mathrm{Im}\,A_1)^T & \mathrm{vec}(\mathrm{Re}\,B_1)^T & \mathrm{vec}(\mathrm{Im}\,B_1)^T \\ \mathrm{vec}(\mathrm{Re}\,A_2)^T & \mathrm{vec}(\mathrm{Im}\,A_2)^T & \mathrm{vec}(\mathrm{Re}\,B_2)^T & \mathrm{vec}(\mathrm{Im}\,B_2)^T \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{vec}(\mathrm{Re}\,A_l)^T & \mathrm{vec}(\mathrm{Im}\,A_l)^T & \mathrm{vec}(\mathrm{Re}\,B_l)^T & \mathrm{vec}(\mathrm{Im}\,B_l)^T \end{bmatrix}$$

$$\times \begin{bmatrix} \mathrm{vec}(\mathrm{Re}\,A_1) & \mathrm{vec}(\mathrm{Re}\,A_2) & \cdots & \mathrm{vec}(\mathrm{Re}\,A_l) \\ \mathrm{vec}(\mathrm{Im}\,A_1) & \mathrm{vec}(\mathrm{Im}\,A_2) & \cdots & \mathrm{vec}(\mathrm{Im}\,A_l) \\ \mathrm{vec}(\mathrm{Re}\,B_1) & \mathrm{vec}(\mathrm{Re}\,B_2) & \cdots & \mathrm{vec}(\mathrm{Re}\,B_l) \\ \mathrm{vec}(\mathrm{Im}\,B_1) & \mathrm{vec}(\mathrm{Im}\,B_2) & \cdots & \mathrm{vec}(\mathrm{Im}\,B_l) \end{bmatrix} k$$

$$= \begin{bmatrix} \mathrm{vec}(\mathrm{Re}\,A_1)^T & \mathrm{vec}(\mathrm{Im}\,A_1)^T & \mathrm{vec}(\mathrm{Re}\,B_1)^T & \mathrm{vec}(\mathrm{Im}\,B_1)^T \\ \mathrm{vec}(\mathrm{Re}\,A_2)^T & \mathrm{vec}(\mathrm{Im}\,A_2)^T & \mathrm{vec}(\mathrm{Re}\,B_2)^T & \mathrm{vec}(\mathrm{Im}\,B_2)^T \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{vec}(\mathrm{Re}\,A_l)^T & \mathrm{vec}(\mathrm{Im}\,A_l)^T & \mathrm{vec}(\mathrm{Re}\,B_l)^T & \mathrm{vec}(\mathrm{Im}\,B_l)^T \end{bmatrix}$$

$$\times \begin{bmatrix} \mathrm{vec}(\mathrm{Re}\,C) \\ \mathrm{vec}(\mathrm{Im}\,C) \\ \mathrm{vec}(\mathrm{Re}\,D) \\ \mathrm{vec}(\mathrm{Im}\,D) \end{bmatrix}$$

$$\iff \begin{bmatrix} \langle E_1, E_1\rangle & \langle E_1, E_2\rangle & \cdots & \langle E_1, E_l\rangle \\ \langle E_2, E_1\rangle & \langle E_2, E_2\rangle & \cdots & \langle E_2, E_l\rangle \\ \vdots & \vdots & & \vdots \\ \langle E_l, E_1\rangle & \langle E_l, E_2\rangle & \cdots & \langle E_l, E_l\rangle \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_l \end{bmatrix} = \begin{bmatrix} \langle E_1, F\rangle \\ \langle E_2, F\rangle \\ \vdots \\ \langle E_l, F\rangle \end{bmatrix}.$$

Thus we have (5). □

We now turn to the case that matrix equation (4) in Problem 2 is inconsistent, just as (3), the related least-squares problem should be considered.

Problem 3 Given matrices $A_1, A_2, \ldots, A_l \in \mathbf{C}^{m\times n}$, $B_1, B_2, \ldots, B_l \in \mathbf{C}^{s\times t}$, $C \in \mathbf{C}^{m\times n}$, and $D \in \mathbf{C}^{s\times t}$, find

$$k = (k_1, k_2, \ldots, k_l)^T \in \mathbf{R}^l$$

such that

$$\|k_1A_1 + k_2A_2 + \cdots + k_lA_l - C\|^2 + \|k_1B_1 + k_2B_2 + \cdots + k_lB_l - D\|^2 = \min. \tag{6}$$

Based on the results above, we list the following theorem which concludes that solving Problem 3 is equivalent to solving the consistent matrix equation system (5).

Theorem 2 *Suppose that the notations and conditions are the same as in Theorem* 1. *Then the solution set of Problem* 3 *is the solution set of system* (5).

Proof By (4), we have

$$\begin{aligned}
&\left\| \left[\sum_{i=1}^{l} k_i A_i - C, \sum_{i=1}^{l} k_i B_i - D \right] \right\|^2 \\
&= \left\| \left(\sum_{i=1}^{l} k_i (\operatorname{Re} A_i) - (\operatorname{Re} C) \right) + \sqrt{-1} \left(\sum_{i=1}^{l} k_i (\operatorname{Im} A_i) - (\operatorname{Im} C) \right) \right\|^2 \\
&\quad + \left\| \left(\sum_{i=1}^{l} k_i (\operatorname{Re} B_i) - (\operatorname{Re} D) \right) + \sqrt{-1} \left(\sum_{i=1}^{l} k_i (\operatorname{Im} B_i) - (\operatorname{Im} D) \right) \right\|^2 \\
&= \left\| \sum_{i=1}^{l} k_i \begin{bmatrix} \operatorname{Re}(A_i) \\ \operatorname{Im}(A_i) \\ \operatorname{Re}(B_i) \\ \operatorname{Im}(B_i) \end{bmatrix} - \begin{bmatrix} \operatorname{Re}(C) \\ \operatorname{Im}(C) \\ \operatorname{Re}(D) \\ \operatorname{Im}(D) \end{bmatrix} \right\|^2 \\
&= \left\| \sum_{i=1}^{l} k_i E_i - F \right\|^2.
\end{aligned}$$

It follows that (5) holds. This implies the conclusion. □

3 The structure of (AXB, CXD) over $X \in \mathrm{HC}^{n\times n}$

Based on the discussion in [18], we first recall a matrix-vector product for vectors and matrices in $\mathbf{C}^{m\times n}$.

Definition 1 Let $x = (x_1, x_2, \dots, x_k)^T \in \mathbf{C}^k$, $y = (y_1, y_2, \dots, y_k)^T \in \mathbf{C}^k$ and $A = (A_1, A_2, \dots, A_k)$, $A_i \in \mathbf{C}^{m\times n}$ $(i = 1, 2, \dots, k)$. Define

(i) $A \circ x = x_1 A_1 + x_2 A_2 + \cdots + x_k A_k \in \mathbf{C}^{m\times n}$;

(ii) $A \circ (x, y) = (A \circ x, A \circ y)$.

From Definition 3.1, we list the following facts which are useful for solving Problem 1. Let $y = (y_1, y_2, \dots, y_k)^T \in \mathbf{C}^k$, $P = (P_1, P_2, \dots, P_k)$, $P_i \in \mathbf{C}^{m\times n}$ $(i = 1, 2, \dots, k)$, and $a, b \in \mathbf{C}$.

(i) $x^H \circ y = x^H y = (x, y)$;

(ii) $A \circ x + P \circ x = (A + P) \circ x$;

(iii) $A \circ (ax + by) = a(A \circ x) + b(A \circ y)$;

(iv) $(aA + bP) \circ x = a(A \circ x) + b(P \circ x)$;

(v) $(A, P) \circ \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = A \circ x + P \circ y$;

(vi) $\left[\begin{smallmatrix} A \\ P \end{smallmatrix}\right] \circ x = \left[\begin{smallmatrix} A\circ x \\ P\circ x \end{smallmatrix}\right]$;

(vii) $\operatorname{vec}(A \circ x) = (\operatorname{vec}(A_1), \operatorname{vec}(A_2), \dots, \operatorname{vec}(A_k))x$;

(viii) $\operatorname{vec}((aA + bP) \circ x) = a \operatorname{vec}(A \circ x) + b \operatorname{vec}(P \circ x)$.

Suppose $B = (B_1, B_2, \dots, B_s) \in \mathbf{C}^{k\times s}$, $B_i \in \mathbf{C}^k$ $(i = 1, 2, \dots, s)$, $C = (C_1, C_2, \dots, C_t) \in \mathbf{C}^{k\times t}$, $C_i \in \mathbf{C}^k$ $(i = 1, 2, \dots, t)$, $D \in \mathbf{C}^{l\times m}$, $H \in \mathbf{C}^{n\times q}$. Then

(ix) $D(A \circ x) = (DA) \circ x$;

(x) $(A \circ x)H = (A_1H, A_2H, \dots, A_kH) \circ x$.

For $X = \operatorname{Re} X + \sqrt{-1}\operatorname{Im} X \in \mathbf{HC}^{n\times n}$, by $X^H = X$, we have $(\operatorname{Re} X + \sqrt{-1}\operatorname{Im} X)^H = \operatorname{Re} X + \sqrt{-1}\operatorname{Im} X$. Thus we get $\operatorname{Re} X^T = \operatorname{Re} X$, $\operatorname{Im} X^T = -\operatorname{Im} X$.

Definition 2 For matrix $A \in \mathbf{R}^{n\times n}$, let $a_1 = (a_{11}, a_{21}, \dots, a_{n1})$, $a_2 = (a_{22}, a_{32}, \dots, a_{n2})$, $\dots$, $a_{n-1} = (a_{(n-1)(n-1)}, a_{n(n-1)})$, $a_n = a_{nn}$, the operator $\operatorname{vec}_S(A)$ is denoted

$$\operatorname{vec}_S(A) = (a_1, a_2, \dots, a_{n-1}, a_n)^T \in \mathbf{R}^{\frac{n(n+1)}{2}}. \tag{7}$$

Definition 3 For matrix $B \in \mathbf{R}^{n\times n}$, let $b_1 = (b_{21}, b_{31}, \dots, b_{n1})$, $b_2 = (b_{32}, b_{42}, \dots, b_{n2})$, $\dots$, $b_{n-2} = (b_{(n-1)(n-2)}, b_{n(n-2)})$, $b_{n-1} = b_{n(n-1)}$, the operator $\operatorname{vec}_A(B)$ is denoted

$$\operatorname{vec}_A(B) = (b_1, b_2, \dots, b_{n-2}, b_{n-1})^T \in \mathbf{R}^{\frac{n(n-1)}{2}}. \tag{8}$$

Let

$$E_{ij} = (e_{st}) \in \mathbf{R}^{n\times n}, \tag{9}$$

where

$$e_{st} = \begin{cases} 1 & (s,t) = (i,j), \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\begin{aligned} K_S = {}& (E_{11}, E_{21} + E_{12}, \dots, E_{n1} + E_{1n}, \\ & E_{22}, E_{32} + E_{23}, \dots, E_{n2} + E_{2n}, \dots, E_{(n-1)(n-1)}, E_{n(n-1)} + E_{(n-1)n}, E_{nn}). \end{aligned} \tag{10}$$

Note that $K_S \in \mathbf{R}^{n\times \frac{n^2(n+1)}{2}}$.

Let

$$K_A = (E_{21} - E_{12}, \dots, E_{n1} - E_{1n}, E_{32} - E_{23}, \dots, E_{n2} - E_{2n}, \dots, E_{n(n-1)} - E_{(n-1)n}). \tag{11}$$

Note that $K_A \in \mathbf{R}^{n\times \frac{n^2(n-1)}{2}}$.

Based on Definition 2, Definition 3, (10) and (11) we get the following lemmas which are necessary for our main results.

Lemma 1 *Suppose $X \in \mathbf{R}^{n\times n}$, then*

(i)

$$X \in \mathbf{SR}^{n\times n} \iff X = K_S \circ \operatorname{vec}_S(X), \tag{12}$$

(ii)

$$X \in \mathbf{ASR}^{n\times n} \iff X = K_A \circ \operatorname{vec}_A(X). \tag{13}$$

Lemma 2 *Suppose* $X = \operatorname{Re} X + \sqrt{-1}\operatorname{Im} X \in \mathbf{C}^{n\times n}$, *then*

$$X \in \mathbf{HC}^{n\times n} \quad \Longleftrightarrow \quad X = K_S \circ \operatorname{vec}_S(\operatorname{Re} X) + \sqrt{-1}K_A \circ \operatorname{vec}_A(\operatorname{Im} X). \tag{14}$$

Lemma 3 *Given* $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times s}$, $C \in \mathbf{C}^{p\times n}$, *and* $D \in \mathbf{C}^{n\times q}$, *and* $X = \operatorname{Re} X + \sqrt{-1}\operatorname{Im} X \in \mathbf{HC}^{n\times n}$. *The complex matrices* $F_{ij} \in \mathbf{C}^{m\times s}$, $G_{ij} \in \mathbf{C}^{m\times s}$, $H_{ij} \in \mathbf{C}^{p\times q}$ *and* $K_{ij} \in \mathbf{C}^{p\times q}$ $(i,j = 1,2,\dots,n; i \geq j)$ *are defined by*

$$F_{ij} = \begin{cases} A_iB_j, & i = j, \\ A_iB_j + A_jB_i, & i > j, \end{cases} \qquad G_{ij} = \begin{cases} 0, & i = j, \\ \sqrt{-1}(A_iB_j - A_jB_i), & i > j, \end{cases}$$
$$H_{ij} = \begin{cases} C_iD_j, & i = j, \\ C_iD_j + C_jD_i, & i > j, \end{cases} \qquad K_{ij} = \begin{cases} 0, & i = j, \\ \sqrt{-1}(C_iD_j - C_jD_i), & i > j, \end{cases}$$

where $A_i \in \mathbf{C}^m$, $C_i \in \mathbf{C}^p$ *is the ith column vector of matrix A and C, meanwhile,* $B_j \in \mathbf{C}^s$, $D_j \in \mathbf{C}^q$ *is the jth row vector of matrix B and D, respectively. Then*

$$\begin{aligned}
[AXB, CXD] = \Biggl[&(F_{11}, F_{21}, \dots, F_{n1}, F_{22}, F_{32}, \dots, F_{n2}, \dots, F_{(n-1)(n-1)}, F_{n(n-1)}, F_{nn}, \\
&G_{21}, G_{31}, \dots, G_{n1}, G_{32}, \dots, G_{n2}, \dots, G_{n(n-1)}) \circ \begin{bmatrix} \operatorname{vec}_S(\operatorname{Re} X) \\ \operatorname{vec}_A(\operatorname{Im} X) \end{bmatrix}, \\
&(H_{11}, H_{21}, \dots, H_{n1}, H_{22}, H_{32}, \dots, H_{n2}, \dots, H_{(n-1)(n-1)}, H_{n(n-1)}, H_{nn}, \\
&K_{21}, K_{31}, \dots, K_{n1}, K_{32}, \dots, K_{n2}, \dots, K_{n(n-1)}) \circ \begin{bmatrix} \operatorname{vec}_S(\operatorname{Re} X) \\ \operatorname{vec}_A(\operatorname{Im} X) \end{bmatrix} \Biggr].
\end{aligned}$$

Proof By Lemma 2, we can get

$$\begin{aligned}
AXB &= A\bigl[K_S \circ \operatorname{vec}_S(\operatorname{Re} X) + \sqrt{-1}\bigl(K_A \circ \operatorname{vec}_A(\operatorname{Im} X)\bigr)\bigr]B \\
&= \bigl[(AK_S) \circ \operatorname{vec}_S(\operatorname{Re} X) + \sqrt{-1}(AK_A) \circ \operatorname{vec}_A(\operatorname{Im} X)\bigr]B \\
&= \bigl[(AK_S) \circ \operatorname{vec}_S(\operatorname{Re} X)\bigr]B + \sqrt{-1}\bigl[(AK_A) \circ \operatorname{vec}_A(\operatorname{Im} X)\bigr]B \\
&= \bigl[\bigl(A(E_{11}, E_{21} + E_{12}, \dots, E_{n(n-1)} + E_{(n-1)n}, E_{nn})\bigr) \circ \operatorname{vec}_S(\operatorname{Re} X)\bigr]B \\
&\quad + \sqrt{-1}\bigl[A(E_{21} - E_{12}, \dots, E_{n1} - E_{1n}, \dots, E_{n(n-1)} - E_{(n-1)n}) \circ \operatorname{vec}_A(\operatorname{Im} X)\bigr]B \\
&= \bigl(AE_{11}B, A(E_{21} + E_{12})B, \dots, A(E_{n(n-1)} + E_{(n-1)n})B, AE_{nn}B\bigr) \circ \operatorname{vec}_S(\operatorname{Re} X) \\
&\quad + \sqrt{-1}\bigl(A(E_{21} - E_{12})B, \dots, A(E_{n1} - E_{1n})B, \dots, A(E_{n(n-1)} - E_{(n-1)n})B\bigr) \circ \operatorname{vec}_A(\operatorname{Im} X) \\
&= (A_1B_1, A_2B_1 + A_1B_2, \dots, A_nB_{n-1} + A_{n-1}B_n, A_nB_n) \circ \operatorname{vec}_S(\operatorname{Re} X) \\
&\quad + \sqrt{-1}(A_2B_1 - A_1B_2, \dots, A_nB_1 - A_1B_n, \dots, A_nB_{n-1} - A_{n-1}B_n) \circ \operatorname{vec}_A(\operatorname{Im} X) \\
&= (F_{11}, F_{21}, \dots, F_{n(n-1)}, F_{nn}) \circ \operatorname{vec}_S(\operatorname{Re} X) + (G_{21}, G_{31}, \dots, G_{n(n-1)}) \circ \operatorname{vec}_A(\operatorname{Im} X) \\
&= (F_{11}, F_{21}, \dots, F_{(n-1)(n-1)}, F_{n(n-1)}, F_{nn}, G_{21}, G_{31}, \dots, G_{n(n-1)}) \circ \begin{bmatrix} \operatorname{vec}_S(\operatorname{Re} X) \\ \operatorname{vec}_A(\operatorname{Im} X) \end{bmatrix}.
\end{aligned}$$

Similarly, we have

$$CXD = (H_{11}, H_{21}, \dots, H_{(n-1)(n-1)}, H_{n(n-1)}, H_{nn}, K_{21}, K_{31}, \dots, K_{n(n-1)}) \circ \begin{bmatrix} \operatorname{vec}_S(\operatorname{Re} X) \\ \operatorname{vec}_A(\operatorname{Im} X) \end{bmatrix}.$$

Thus we can get the structure of $[AXB, CXD]$ and complete the proof. □

4 The solution of Problem 1

Based on the above results, in this section, we will deduce the solution of Problem 1. From [17], the least-squares problem

$$\|AXB - E\|^2 + \|CXD - F\|^2 = \min$$

with respect to the Hermitian matrix X is equivalent to

$$\left\| \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \begin{bmatrix} \operatorname{vec}_S(\operatorname{Re} X) \\ \operatorname{vec}_A(\operatorname{Im} X) \\ \operatorname{vec}_S(\operatorname{Re} X) \\ \operatorname{vec}_A(\operatorname{Im} X) \end{bmatrix} - \begin{bmatrix} \operatorname{vec}(\operatorname{Re} E) \\ \operatorname{vec}(\operatorname{Im} E) \\ \operatorname{vec}(\operatorname{Re} F) \\ \operatorname{vec}(\operatorname{Im} F) \end{bmatrix} \right\|^2 = \min, \tag{15}$$

where

$$P = \begin{bmatrix} (B^T \otimes A)L_S \\ \sqrt{-1}(B^T \otimes A)L_A \\ (D^T \otimes C)L_S \\ \sqrt{-1}(D^T \otimes C)L_A \end{bmatrix}, \qquad P_1 = \operatorname{Re}(P), \qquad P_2 = \operatorname{Im}(P).$$

It should be noticed that (15) is an unconstrained problem over the real field and can easily be solved by existing methods, therefore, the original complex constrained Problem 1 is translated into an equivalent real unconstrained problem (15). Since the process has been expressed in [17], we omit it here. Based on Theorems 1, 2, and Lemma 3, we now turn to the least-squares Hermitian problem for the matrix equation (1). The following lemmas are necessary for our main results.

Lemma 4 ([19]) *Given $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^n$, the solution of equation $Ax = b$ involves two cases*:

(i) *The equation has a solution $x \in \mathbf{R}^n$ and the general solution can be formulated as*

$$x = A^+b + \big(I - A^+A\big)y \tag{16}$$

if and only if $AA^+b = b$, where $y \in \mathbf{R}^n$ is an arbitrary vector.

(ii) *The least-squares solutions of the equation has the same formulation as* (16) *and the least-squares solution with the minimum norm is $x = A^+b$.*

For convenience, we introduce the following notations and lemmas.

Given $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times s}$, $C \in \mathbf{C}^{p\times n}$, $D \in \mathbf{C}^{n\times q}$, $E \in \mathbf{C}^{m\times s}$, and $F \in \mathbf{C}^{p\times q}$, let

$$\widehat{\Gamma}_{ij} = \begin{bmatrix} \mathrm{Re}(F_{ij}) \\ \mathrm{Im}(F_{ij}) \\ \mathrm{Re}(H_{ij}) \\ \mathrm{Im}(H_{ij}) \end{bmatrix}, \qquad \widehat{\Upsilon}_{ij} = \begin{bmatrix} \mathrm{Re}(G_{ij}) \\ \mathrm{Im}(G_{ij}) \\ \mathrm{Re}(K_{ij}) \\ \mathrm{Im}(K_{ij}) \end{bmatrix}, \qquad \Omega_0 = \begin{bmatrix} \mathrm{Re}(E) \\ \mathrm{Im}(E) \\ \mathrm{Re}(F) \\ \mathrm{Im}(F) \end{bmatrix},$$

$$W = \begin{bmatrix} P & U \\ U^T & V \end{bmatrix}, \qquad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{17}$$

where $n \ge i \ge j \ge 1$,

$$P = \begin{bmatrix} \langle \widehat{\Gamma}_{11}, \widehat{\Gamma}_{11}\rangle & \langle \widehat{\Gamma}_{11}, \widehat{\Gamma}_{21}\rangle & \cdots & \langle \widehat{\Gamma}_{11}, \widehat{\Gamma}_{nn}\rangle \\ \langle \widehat{\Gamma}_{21}, \widehat{\Gamma}_{11}\rangle & \langle \widehat{\Gamma}_{21}, \widehat{\Gamma}_{22}\rangle & \cdots & \langle \widehat{\Gamma}_{21}, \widehat{\Gamma}_{nn}\rangle \\ \vdots & \vdots & & \vdots \\ \langle \widehat{\Gamma}_{nn}, \widehat{\Gamma}_{11}\rangle & \langle \widehat{\Gamma}_{nn}, \widehat{\Gamma}_{21}\rangle & \cdots & \langle \widehat{\Gamma}_{nn}, \widehat{\Gamma}_{nn}\rangle \end{bmatrix},$$

$$U = \begin{bmatrix} \langle \widehat{\Gamma}_{11}, \widehat{\Upsilon}_{21}\rangle & \langle \widehat{\Gamma}_{11}, \widehat{\Upsilon}_{31}\rangle & \cdots & \langle \widehat{\Gamma}_{11}, \widehat{\Upsilon}_{n(n-1)}\rangle \\ \langle \widehat{\Gamma}_{21}, \widehat{\Upsilon}_{21}\rangle & \langle \widehat{\Gamma}_{21}, \widehat{\Upsilon}_{31}\rangle & \cdots & \langle \widehat{\Gamma}_{21}, \widehat{\Upsilon}_{n(n-1)}) \\ \vdots & \vdots & & \vdots \\ \langle \widehat{\Gamma}_{nn}, \widehat{\Upsilon}_{21}\rangle & \langle \widehat{\Gamma}_{nn}, \widehat{\Upsilon}_{31}\rangle & \cdots & \langle \widehat{\Gamma}_{nn}, \widehat{\Upsilon}_{n(n-1)}\rangle \end{bmatrix},$$

$$V = \begin{bmatrix} \langle \widehat{\Upsilon}_{21}, \widehat{\Upsilon}_{21}\rangle & \langle \widehat{\Upsilon}_{21}, \widehat{\Upsilon}_{31}\rangle & \cdots & \langle \widehat{\Upsilon}_{21}, \widehat{\Upsilon}_{n(n-1)}\rangle \\ \langle \widehat{\Upsilon}_{31}, \widehat{\Upsilon}_{21}\rangle & \langle \widehat{\Upsilon}_{31}, \widehat{\Upsilon}_{31}\rangle & \cdots & \langle \widehat{\Upsilon}_{31}, \widehat{\Upsilon}_{n(n-1)}\rangle \\ \vdots & \vdots & & \vdots \\ \langle \widehat{\Upsilon}_{n(n-1)}, \widehat{\Upsilon}_{21}\rangle & \langle \widehat{\Upsilon}_{n(n-1)}, \widehat{\Upsilon}_{31}\rangle & \cdots & \langle \widehat{\Upsilon}_{n(n-1)}, \widehat{\Upsilon}_{n(n-1)}\rangle \end{bmatrix},$$

$$e_1 = \begin{bmatrix} \langle \widehat{\Gamma}_{11}, \Omega_0\rangle \\ \langle \widehat{\Gamma}_{21}, \Omega_0\rangle \\ \vdots \\ \langle \widehat{\Gamma}_{nn}, \Omega_0\rangle \end{bmatrix}, \qquad e_2 = \begin{bmatrix} \langle \widehat{\Upsilon}_{21}, \Omega_0\rangle \\ \langle \widehat{\Upsilon}_{31}, \Omega_0\rangle \\ \vdots \\ \langle \widehat{\Upsilon}_{n(n-1)}, \Omega_0\rangle \end{bmatrix}.$$

Theorem 3 *Given $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times s}$, $C \in \mathbf{C}^{m\times n}$, $D \in \mathbf{C}^{n\times t}$, $E \in \mathbf{C}^{m\times s}$, and $F \in \mathbf{C}^{m\times t}$, let W, e be as in* (17). *Then*

$$H_L = \{X \mid X = (K_S, \sqrt{-1}K_A) \circ [W^+ e + (I - W^+ W)y]\}, \tag{18}$$

where $y \in \mathbf{R}^{n^2}$ is an arbitrary vector. Problem 1 *has a unique solution $X_H \in H_L$. This solution satisfies*

$$X_H = (K_S, \sqrt{-1}K_A) \circ (W^+ e). \tag{19}$$

Proof By Lemma 3 and Theorem 2, the least-squares problem

$$\|AXB - E\|^2 + \|CXD - F\|^2 = \min$$

with respect to the Hermitian matrix X can be translated into an equivalent consistent linear equations over the real field

$$W \begin{bmatrix} \operatorname{vec}_S(\operatorname{Re} X) \\ \operatorname{vec}_A(\operatorname{Im} X) \end{bmatrix} = e.$$

It follows by Lemma 4 that

$$\begin{bmatrix} \operatorname{vec}_S(\operatorname{Re} X) \\ \operatorname{vec}_A(\operatorname{Im} X) \end{bmatrix} = W^+ e + \bigl(I - W^+ W\bigr) y.$$

Thus

$$X = (K_S, \sqrt{-1} K_A) \circ \bigl(W^+ e + \bigl(I - W^+ W\bigr) y\bigr).$$

The proof is completed. □

We now turn to the consistency of the matrix equation (1). Denote

$$N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}, \qquad \hat{e} = \begin{bmatrix} \operatorname{vec}(\operatorname{Re} E) \\ \operatorname{vec}(\operatorname{Im} E) \\ \operatorname{vec}(\operatorname{Re} F) \\ \operatorname{vec}(\operatorname{Im} F) \end{bmatrix}, \tag{20}$$

where

$$\begin{aligned}
N_1 = \bigl[&\operatorname{vec}\bigl(\operatorname{Re}(F_{11})\bigr), \operatorname{vec}\bigl(\operatorname{Re}(F_{21})\bigr), \ldots, \operatorname{vec}\bigl(\operatorname{Re}(F_{n1})\bigr), \operatorname{vec}\bigl(\operatorname{Re}(F_{22})\bigr), \ldots, \\
&\operatorname{vec}\bigl(\operatorname{Re}(F_{n2})\bigr), \ldots, \operatorname{vec}\bigl(\operatorname{Re}(F_{(n-1)(n-1)})\bigr), \operatorname{vec}\bigl(\operatorname{Re}(F_{n(n-1)})\bigr), \operatorname{vec}\bigl(\operatorname{Re}(F_{nn})\bigr), \\
&\operatorname{vec}\bigl(\operatorname{Re}(G_{21})\bigr), \ldots, \operatorname{vec}\bigl(\operatorname{Re}(G_{n1})\bigr), \operatorname{vec}\bigl(\operatorname{Re}(G_{32})\bigr), \ldots, \\
&\operatorname{vec}\bigl(\operatorname{Re}(G_{n2})\bigr), \ldots, \operatorname{vec}\bigl(\operatorname{Re}(G_{n(n-1)})\bigr)\bigr], \\
N_2 = \bigl[&\operatorname{vec}\bigl(\operatorname{Im}(F_{11})\bigr), \operatorname{vec}\bigl(\operatorname{Im}(F_{21})\bigr), \ldots, \operatorname{vec}\bigl(\operatorname{Im}(F_{n1})\bigr), \operatorname{vec}\bigl(\operatorname{Im}(F_{22})\bigr), \ldots, \\
&\operatorname{vec}\bigl(\operatorname{Im}(F_{n2})\bigr), \ldots, \operatorname{vec}\bigl(\operatorname{Im}(F_{(n-1)(n-1)})\bigr), \operatorname{vec}\bigl(\operatorname{Im}(F_{n(n-1)})\bigr), \operatorname{vec}\bigl(\operatorname{Im}(F_{nn})\bigr), \\
&\operatorname{vec}\bigl(\operatorname{Im}(G_{21})\bigr), \ldots, \operatorname{vec}\bigl(\operatorname{Im}(G_{n1})\bigr), \operatorname{vec}\bigl(\operatorname{Im}(G_{32})\bigr), \ldots, \\
&\operatorname{vec}\bigl(\operatorname{Im}(G_{n2})\bigr), \ldots, \operatorname{vec}\bigl(\operatorname{Im}(G_{n(n-1)})\bigr)\bigr], \\
N_3 = \bigl[&\operatorname{vec}\bigl(\operatorname{Re}(H_{11})\bigr), \operatorname{vec}\bigl(\operatorname{Re}(H_{21})\bigr), \ldots, \operatorname{vec}\bigl(\operatorname{Re}(H_{n1})\bigr), \operatorname{vec}\bigl(\operatorname{Re}(H_{22})\bigr), \ldots, \\
&\operatorname{vec}\bigl(\operatorname{Re}(H_{n2})\bigr), \ldots, \operatorname{vec}\bigl(\operatorname{Re}(H_{(n-1)(n-1)})\bigr), \operatorname{vec}\bigl(\operatorname{Re}(H_{n(n-1)})\bigr), \operatorname{vec}\bigl(\operatorname{Re}(H_{nn})\bigr), \\
&\operatorname{vec}\bigl(\operatorname{Re}(K_{21})\bigr), \ldots, \operatorname{vec}\bigl(\operatorname{Re}(K_{n1})\bigr), \operatorname{vec}\bigl(\operatorname{Re}(K_{32})\bigr), \ldots, \\
&\operatorname{vec}\bigl(\operatorname{Re}(K_{n2})\bigr), \ldots, \operatorname{vec}\bigl(\operatorname{Re}(K_{n(n-1)})\bigr)\bigr],
\end{aligned}$$

$$\begin{aligned}N_4 = \big[&\mathrm{vec}\big(\mathrm{Im}(H_{11})\big), \mathrm{vec}\big(\mathrm{Im}(H_{21})\big), \ldots, \mathrm{vec}\big(\mathrm{Im}(H_{n1})\big), \mathrm{vec}\big(\mathrm{Im}(H_{22})\big), \ldots,\\ &\mathrm{vec}\big(\mathrm{Im}(H_{n2})\big), \ldots, \mathrm{vec}\big(\mathrm{Im}(H_{(n-1)(n-1)})\big), \mathrm{vec}\big(\mathrm{Im}(H_{n(n-1)})\big), \mathrm{vec}\big(\mathrm{Im}(H_{nn})\big),\\ &\mathrm{vec}\big(\mathrm{Im}(K_{21})\big), \ldots, \mathrm{vec}\big(\mathrm{Im}(K_{n1})\big), \mathrm{vec}\big(\mathrm{Im}(K_{32})\big), \ldots,\\ &\mathrm{vec}\big(\mathrm{Im}(K_{n2})\big), \ldots, \mathrm{vec}\big(\mathrm{Im}(K_{n(n-1)})\big)\big].\end{aligned}$$

By Lemma 3, we have

$$(AXB, CXD) = (E, F) \quad \Longleftrightarrow \quad N \begin{bmatrix} \mathrm{vec}_S(\mathrm{Re}\, X) \\ \mathrm{vec}_A(\mathrm{Im}\, X) \end{bmatrix} = \hat{e}. \tag{21}$$

Thus we can get the following conclusions by Lemma 4 and Theorem 3.

Corollary 1 *The matrix equation* (1) *has a solution $X \in \mathbf{HC}^{n\times n}$ if and only if*

$$NN^{+}\hat{e} = \hat{e}. \tag{22}$$

In this case, denote by H_E the solution set of (1). *Then*

$$H_E = \big\{X | X = (K_S, \sqrt{-1}K_A) \circ \big[M^{+}e + \big(I - W^{+}W\big)y\big]\big\}, \tag{23}$$

where $y \in \mathbf{R}^{n^2}$ is an arbitrary vector.

Furthermore, if (22) *holds, then the matrix equation* (1) *has a unique solution $X \in H_E$ if and only if*

$$\mathrm{rank}(N) = n^2. \tag{24}$$

In this case,

$$H_E = \big\{X | X = (K_S, \sqrt{-1}K_A) \circ \big(W^{+}e\big)\big\}. \tag{25}$$

The least norm problem

$$\|X_H\| = \min_{X \in H_E} \|X\|$$

has a unique solution $X_H \in H_E$ and X_H can be expressed as (19).

5 Numerical experiments

In this section, based on the results in the above sections, we first give the numerical algorithm to find the solution of Problem 1. Then numerical experiments are proposed to demonstrate the efficiency of the algorithm. The following algorithm provides the main steps to find the solutions of Problem 1.

Algorithm 4

(1) Input $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times s}$, $C \in \mathbf{C}^{m\times n}$, $D \in \mathbf{C}^{n\times t}$, $E \in \mathbf{C}^{m\times s}$, and $F \in \mathbf{C}^{m\times t}$.

(2) Compute F_{ij}, $G_{i,j}$, $H_{i,j}$ and $K_{i,j}$ $(i,j = 1, 2, \ldots, n,\ i \geq g)$ by Lemma 3.

(3) Compute P, U, V and e according to (17).
(4) If (22) and (24) hold then calculate X_H ($X_H \in H_E$) according to (25).
(5) If (22) holds then calculate X_H ($X_H \in H_E$) according to (19), otherwise go to the next step.
(6) Calculate X_H ($X_H \in H_L$) according to (19).

For convenience, in the following examples, the random matrix, the Hilbert matrix, the Toeplitz matrix, the matrix whose all elements are one and the magic matrix are all denoted as by the corresponding Matlab function.

Example 1 Let $A_r = 10\,\mathrm{rand}(8,10)$, $B_r = \mathrm{rand}(10,12)$, $C_r = 10\,\mathrm{rand}(8,10)$, $D_r = \mathrm{rand}(10,12)$, $A_i = \mathrm{rand}(8,10)$, $B_i = 10\,\mathrm{rand}(10,12)$, $C_i = \mathrm{rand}(8,10)$, $D_i = 10\,\mathrm{rand}(10,12)$. Let $\tilde{X} = \mathrm{rand}(10,10)$, $X_r = \tilde{X} + \tilde{X}^T$; $\hat{X} = \mathrm{hilb}(10)$, $X_i = \hat{X} - \hat{X}^T$; $A = A_r + \sqrt{-1}A_i$, $B = B_r + \sqrt{-1}B_i$, $C = C_r + \sqrt{-1}C_i$, $D = D_r + \sqrt{-1}D_i$, $X = X_r + \sqrt{-1}X_i$, $E = AXB$, $F = CXD$. By using Matlab 7 and Algorithm 4, we obtain $\mathrm{rank}(W) = 100$, $\|W\| = 2.1069e{+}07$, $\|WW^+e - e\| = 6.1091e{-}06$. $\mathrm{rank}(N) = 100$, $\|N\| = 5.9155e{+}03$, $\|NN^+\hat{e} - \hat{e}\| = 4.0346e{-}11$. From Algorithm 4(4), it can be concluded that the complex matrix equation $[AXB, CXD] = [E, F]$ is consistent, and it has a unique solution $X_H \in H_E$; further, $\|X_H - X\| = 1.4536e{-}11$ can easily be tested.

Example 2 Let $m = 8$, $n = 10$, $s = 12$. Take $A_r = [\mathrm{toeplitz}(1:8), 0_{m\times 2}]$, $B_r = \mathrm{ones}(10,12)$, $C_r = [\mathrm{hilb}(8), 0_{8\times 2}]$, $D_r = \mathrm{ones}(10,12)$, $A_i = 0_{8\times 10}$, $B_i = \mathrm{ones}(10,12)$, $C_i = 0_{8\times 10}$, $D_i = 0_{10\times 12}$. Let $\tilde{X} = \mathrm{rand}(10,10)$, $X_r = \tilde{X} + \tilde{X}^T$; $\hat{X} = \mathrm{hilb}(n)$, $X_i = \hat{X} - \hat{X}^T$; $A = A_r + \sqrt{-1}A_i$, $B = B_r + \sqrt{-1}B_i$, $C = C_r + \sqrt{-1}C_i$, $D = D_r + \sqrt{-1}D_i$, $X = X_r + \sqrt{-1}X_i$, $E = AXB$, $F = CXD$. From Algorithm 4(5), we can obtain $\mathrm{rank}(W) = 16$, $\|W\| = 3.6293e{+}05$, $\|WW^+e - e\| = 5.6271e{-}08$. $\mathrm{rank}(N) = 16$, $\|N\| = 699.6486$, $\|NN^+\hat{e} - \hat{e}\| = 7.1024e{-}12$. From Algorithm 4, it can be concluded that the complex matrix equation $[AXB, CXD] = [E, F]$ is consistent, and it has a unique solution $X_H \in H_E$, further, $\|X_H - X\| = 5.7501$ can easily be tested.

Example 3 Let $m = 8$, $n = 10$, $s = 12$. Take $A_r = 0_{8\times 10}$, $B_r = \mathrm{rand}(10,12)$, $C_r = 0_{8\times 10}$, $D_r = \mathrm{rand}(10,12)$, $A_i = \mathrm{rand}(8,10)$, $B_i = 0_{10\times 12}$, $C_i = \mathrm{rand}(8,10)$, $D_i = 0_{10\times 12}$. Let $\tilde{X} = \mathrm{rand}(10,10)$, $X_r = \tilde{X} + \tilde{X}^T$; $\hat{X} = \mathrm{rand}(10,10)$, $X_i = \hat{X} - \hat{X}^T$; $A = A_r + \sqrt{-1}A_i$, $B = B_r + \sqrt{-1}B_i$, $C = C_r + \sqrt{-1}C_i$, $D = D_r + \sqrt{-1}D_i$, $X = X_r + \sqrt{-1}X_i$, $E = AXB + 10\,\mathrm{ones}(m,s)$, $F = CXD + 10\,\mathrm{ones}(m,s)$. By using Matlab 7 and Algorithm 4, we obtain $\mathrm{rank}(W) = 100$, $\|W\| = 2.6274e{+}03$, $\|WW^+e - e\| = 2.4857e{-}09$. $\mathrm{rank}(N) = 100$, $\|N\| = 64.7511$, $\|NN^+\hat{e} - \hat{e}\| = 109.1680$. According to Algorithm 4(6) we can see that the complex matrix equation $[AXB, CXD] = [E, F]$ is inconsistent, and it has a unique solution $X_H \in H_E$ and we can get $\|X_H - X\| = 41.9521$.

6 Conclusions

In this paper, we derive the explicit expressions of least-squares Hermitian solution with the minimum norm for the complex matrix equations $(AXB, CXD) = (E, F)$. A numerical algorithm and examples show the effectiveness of our method.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors read and approved the final manuscript.

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