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Applications of anticipated BSDEs driven by time-changing Lévy noises

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Abstract
We study a very particular anticipated BSDEs when the driver is time-changing Lévy noise. We give an estimate of the solutions in the system satisfying some non-Lipschitz conditions. Also, we state an useful comparison theorem for the solutions. At last, we establish another specific Feynman-Kac formula for a quasilinear PDE.

Keywords: anticipated BSDE; an estimate of the solutions; comparison theorem; nonlinear Feynman-Kac formula

1 Introduction

The theory of nonlinear BSDEs under a standard Lipschitz condition was pioneered by Pardoux and Peng in 1990 [1]. From then on it became very popular and has attracted many researchers' attention for theoretical research. It plays an important role in many fields such as mathematical finance, the solution of PDEs, stochastic control, and pricing theory for imperfect markets. See, for example, Baños *et al.* [2], Cordoni *et al.* [3–6], Delong and Imkeller [7], Ekren *et al.* [8], Mohammed [9], Soner *et al.* [10] and the references therein for applications in different areas. In particular, as with applications, different noises are also gradually becoming a direction of research. Liu and Ren [11], Lu and Ren [12], Peng and Wang [13], El Karoui *et al.* [14] and the references therein are very useful to our study.

Especially, Peng and Yang [15] studied a special type of BSDEs, named anticipated BSDEs, such that

$$\begin{cases} dY_t = -f_t(Y_t, Z_t, Y_{t+u(t)}, Z_{t+v(t)})\,dt + Z_t\,dW_t, & t \in [0,T]; \\ Y_t = \xi_t, \quad t \in [T, T+K]; \\ Z_t = \eta_t, \quad t \in [T, T+K], \end{cases} \tag{1}$$

where both $u(\cdot)$ and $v(\cdot)$ are positive measurable functions on $[0,T]$ and they satisfy:

(a1) there is some positive constant K and for all $r \in [0,T]$,

$$r + u(r) \leq T + K, \qquad r + v(r) \leq T + K;$$

(a2) there is some positive constant L, for all $t \in [0, T]$ and every non-negative integrable $g(\cdot)$,

$$\int_t^T g\big(r + u(r)\big)\,dr \le L \int_t^{T+K} g(r)\,dr, \qquad \int_t^T g\big(r + v(r)\big)\,dr \le L \int_t^{T+K} g(r)\,dr.$$

In equations (1), the generator $f(\cdot)$ includes the values of the solution of (1) of this moment and later time. The authors studied in depth the questions of existence and uniqueness, they also gave a useful comparison theorem. On this basis, they developed the connection between SDDEs and anticipated BSDEs, which is a practical idea for solving stochastic control problems (also see Cohen and Elliott [16], Li and Wu [17]). Furthermore, Lu and Ren [12] studied a class of anticipated BSDEs when the driver is Markov chains with finite state. Liu and Ren [11] addressed the classical problem of the solutions when the anticipated BSDEs' driver is time-changing Lévy noises and provided the connection between the two kinds of equations.

On the other hand, Lévy processes are rich mathematical objects and have potential applications in many respects. The class of Lévy processes includes the Brownian case and the Poisson case. Winkel [18] dealt with the recovery problem for time-changing Lévy processes. We also refer to the literature on Lévy processes by Carr and Wu [19], David and Wim [20, 21]. Di Nunno and Sjursen [22] studied the following particular BSDEs with a time-changing Lévy noises driver, which is a framework for the independent time-change of the Brownian motion and Poisson random field:

$$Y_t = \xi + \int_t^T g_s(\lambda_s, Y_s, \phi_s)\,ds - \int_t^T \int_{\mathbb{R}} \phi_s(x)\mu(ds, dx), \quad t \in [0, T], \tag{2}$$

where μ is the mixture of measure between a Wiener process measure B on $[0, T] \times \{0\}$ and a doubly stochastic Poisson measure $\tilde{H}$ on $[0, T] \times \mathbb{R}_0$ with

$$\mu(\Delta) := B\big(\Delta \cap [0, T] \times \{0\}\big) + \tilde{H}\big(\Delta \cap [0, T] \times \mathbb{R}_0\big), \quad \Delta \subseteq X = [0, T] \times \mathbb{R}.$$

The authors proved a classical problem of the solution and addressed an application of the control problems of the BSDE. They also studied the sufficient conditions of an optimal equation.

Liu and Ren [11] discussed the following type of anticipated BSDEs when the driver is time-changing Lévy noises:

$$\begin{cases} dY_t = -f(t, \lambda_t, Y_t, Z_t, Y_{t+u(t)}, Z_{t+v(t)})\,dt + Z_t(x)\mu(dt, dx), & t \in [0, T]; \\ Y_t = \xi_t, & t \in [T, T + K]; \\ Z_t = \eta_t, & t \in [T, T + K]. \end{cases} \tag{3}$$

They addressed the classical problem of existence and uniqueness of (3). In addition, the duality between SDDEs and anticipated BSDEs was established.

The aim of this paper is to discuss some applications of anticipated BSDEs (3). As the first step, we aim to give an estimate of the solutions in the system satisfying some non-Lipschitz conditions. The comparison theorem of the solutions will be established. Also, we solve the solution of a class of quasilinear PDEs. In our following study, we will establish some other applications of mathematical finance, including utility maximization and hedging.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries of time-changing Lévy noises. Section 3 is devoted to giving an estimate of the solutions. Section 4 proves a comparison theorem of the solution. In the last section, we have included an application of anticipated BSDEs to the nonlinear Feynman-Kac formula.

2 Preliminaries of time-changing Lévy noises

In this section, we review some concepts and assumptions of time-changing Lévy noises. we refer to Di Nunno and Sjursen [22], Liu and Ren [11], Peng and Yang [15], and the references therein for details.

In the following, we need the following notations. Given T, a fixed finite horizon. X can be seen as a union of $[0,T]\times\{0\}$ and $[0,T]\times\mathbb{R}_0$, where $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$. Let $\mathcal{F}_t = \sigma\{X_s, 0\le s\le t\}$ be the natural filtration of X, and we assume that the $\mathcal{F}_0$ are the P-null sets of $\mathcal{F}$.

Given a 2-dimensional stochastic process $\lambda_t = (\lambda_t^B, \lambda_t^H)$, for each component λ_t^i $(i = B, H)$, we set

(b1) for each $t\in[0,T+K]$, $\lambda_t^i \ge 0$ a.s. a.e.;

(b2) for $\forall\epsilon > 0$, and $t\in[0,T+K]$, $\lim_{h\to 0} P(|\lambda_{t+h}^i - \lambda_h^i| \ge \epsilon) = 0$;

(b3) $\mathbb{E}[\int_0^{T+K}\lambda_s^i\,ds] < +\infty$.

For each $\Delta\subseteq X$, we denote the mixture of a measure Λ by

$$\Lambda(\Delta) := \int_0^T \mathbf{1}_{\{(t,0)\in\Delta\}}(t)\lambda_t^B\,dt + \int_0^T\int_{\mathbb{R}_0}\mathbf{1}_{\{(t,x)\in\Delta\}}(t,x)q(dx)\lambda_t^H\,dt,$$

and q is a deterministic, σ-finite measure on the Borel sets of $\mathbb{R}_0$ and it satisfies $\int_{\mathbb{R}_0} z^2 q(dx) < +\infty$. Here, we denote by $\mathcal{F}^\Lambda$ the σ-algebra generated by the values of Λ. Λ^B denotes the restriction of Λ to $[0,T]\times\{0\}$ and Λ^H denotes the restriction of Λ to $[0,T]\times\mathbb{R}_0$.

Now, we recall the noises driving (3); the following characterization is mainly due to Di Nunno and Sjursen [22] (see also Liu and Ren [11]).

Definition 1 B is a signed random measure on $\mathcal{B}\{[0,T]\times\{0\}\}$ and satisfies:

(c1) $P(B(\Delta)\le y|\mathcal{F}^\Lambda) = P(B(\Delta)\le y|\Lambda^B(\Delta)) = \Phi(\frac{y}{\sqrt{\Lambda^B(\Delta)}})$, $y\in\mathbb{R}$, $\Delta\subseteq[0,T]\times\{0\}$, where $\Phi(y) = \int_{-\infty}^y \frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}\,ds$.

(c2) $P(B(\Delta_1)\le x, B(\Delta_2)\le y|\mathcal{F}^\Lambda) = P(B(\Delta_1)\le x|\mathcal{F}^\Lambda)P(B(\Delta_2)\le y|\mathcal{F}^\Lambda)$, $x,y\in\mathbb{R}$, $\Delta_1,\Delta_2\subseteq[0,T]\times\{0\}$, when $\Delta_1\cap\Delta_2 = \phi$.

H is a signed random measure on $\mathcal{B}\{[0,T]\times\mathbb{R}_0\}$ and satisfies:

(c3) $P(H(\Delta) = n|\mathcal{F}^\Lambda) = P(H(\Delta) = n|\Lambda^H(\Lambda)) = \frac{[\Lambda^H(\Delta)]^n}{n!}e^{-\Lambda^H(\Delta)}$, $n\in\mathbb{N}$, $\Delta\subseteq[0,T]\times\mathbb{R}_0$.

(c4) $P(H(\Delta_1)\le x, H(\Delta_2)\le y|\mathcal{F}^\Lambda) = P(H(\Delta_1)\le x|\mathcal{F}^\Lambda)P(H(\Delta_2)\le y|\mathcal{F}^\Lambda)$, $x,y\in\mathbb{R}$, $\Delta_1,\Delta_2\subseteq[0,T]\times\mathbb{R}_0$, when $\Delta_1\cap\Delta_2 = \phi$.

And let us suppose that

(c5) $P(B(\Delta_1)\le y, H(\Delta_2)\le y|\mathcal{F}^\Lambda) = P(B(\Delta_1)\le x|\mathcal{F}^\Lambda)P(H(\Delta_2)\le y|\mathcal{F}^\Lambda)$, $y\in\mathbb{R}$, $\Delta_1\subseteq[0,T]\times\{0\}$, $\Delta_2\subseteq[0,T]\times\mathbb{R}_0$, when $\Delta_1\cap\Delta_2 = \phi$.

For convenience, let $\tilde{H} := H - \Lambda^H$ be the signed random measure with the following form:

$$\tilde{H}(\Delta) = H(\Delta) - \Lambda^H(\Delta), \quad \Delta \subset [0,T] \times \mathbb{R}_0.$$

From (c1)-(c5), we have

$$\mathbb{E}\big[\mu(\Delta)|\mathcal{F}^\Lambda\big] = 0, \qquad \mathbb{E}\big[B(\Delta)^2|\mathcal{F}^\Lambda\big] = \Lambda^B(\Delta),$$
$$\mathbb{E}\big[\tilde{H}(\Delta)^2|\mathcal{F}^\Lambda\big] = \Lambda^H(\Delta), \qquad \mathbb{E}\big[\mu(\Delta)^2|\mathcal{F}^\Lambda\big] = \Lambda(\Delta),$$

and

$$\mathbb{E}\big[\mu(\Delta_1)\mu(\Delta_2)|\mathcal{F}^\Lambda\big] = \mathbb{E}\big[\mu(\Delta_1)|\mathcal{F}^\Lambda\big]\mathbb{E}\big[\mu(\Delta_2)|\mathcal{F}^\Lambda\big] = 0,$$

where $\Delta_1, \Delta_2 \subseteq X$, $\Delta_1 \cap \Delta_2 = \phi$. Hence Δ_1 and Δ_2 are conditionally orthogonal. The random measures B and H are regarded as a mixture of time-changing processes. Particularly, we also define $\mathbb{F} = \{\mathcal{F}_t, t \in [0,T]\}$, where $\mathcal{F}_t = \bigcap_{r>t} \mathcal{F}_r^\mu$ and $\mathcal{F}_t^\mu = \mathcal{F}_t^B \vee \mathcal{F}_t^H \vee \mathcal{F}_t^\Lambda$, $\mathcal{F}_t^B$ is generated by $B(\Delta) \cap [0,T]$. $\mathcal{F}_t^H$ by $H(\Delta) \cap [0,T]$, $\mathcal{F}_t^\Lambda$ by $\Lambda(\Delta)$. We set $\mathbb{G} = \{\mathcal{G}_t, t \in [0,T]\}$ with $\mathcal{G}_t = \mathcal{F}_t^\mu \vee \mathcal{F}^\Lambda$, which includes time-changing noises' information of the future values of Λ^B and Λ^H.

For convenience, we list the following spaces of this paper as follows.

- $S^2_{\mathbb{G}}(0,T+K;\mathbb{R}) = \{Y_t : \Omega \times [0,T+K] | Y(t,\omega)$ is a $\mathbb{G}$-adapted stochastic processes satisfying that $\mathbb{E}[\sup_{0\le t\le T+K} |Y_t|^2] < \infty\}$.
- $L^2_{\mathbb{G}}(0,T+K;\mathbb{R}) = \{Z_t : \Omega \times [0,T+K] \times \mathbb{R} | Z(t,x,\omega)$ is a $\mathbb{G}$-adapted stochastic processes satisfying $\|Z\| = \mathbb{E}[\int_0^{T+K} |Z_s(0)|^2 \lambda_s^B\, ds + \int_0^{T+K}\int_{\mathbb{R}_0} |Z_s(x)|^2 q(dx)\lambda_s^H\, ds] < \infty\}$.

For the proof of results, we give the following assumptions.

(H1) $f(s,\lambda,y,z,\xi,\eta) : [0,T] \times [0,+\infty)^2 \times \Omega \times \mathbb{R} \times L^2_{\mathbb{G}}(s,T;\mathbb{R}) \times S^2_{\mathbb{G}}(s,T+K;\mathbb{R}) \times L^2_{\mathbb{G}}(s,T+K;\mathbb{R}) \to L^2(\mathcal{G}_s;\mathbb{R})$ is $\mathcal{G}_s$-measurable for all $s \in [0,T]$;

(H2) $\exists C > 0$, all $s \in [0,T]$, $y^1, y^2 \in \mathbb{R}$, $z^1, z^2 \in L^2_{\mathbb{G}}(s,T;\mathbb{R})$, $\xi^1, \xi^2 \in S^2_{\mathbb{G}}(s,T+K;\mathbb{R})$, $\eta^1, \eta^2 \in L^2_{\mathbb{G}}(s,T+K;\mathbb{R})$, $r^1, r^2 \in [s,T+K]$, we have

$$\begin{aligned}
&\big|f_s\big(\lambda,y^1,z^1,\xi^1_{r1},\eta^1_{r2}\big) - f_s\big(\lambda,y^2,z^2,\xi^2_{r1},\eta^2_{r2}\big)\big| \\
&\quad\le C\Bigg[\big|y^1-y^2\big| + \big|z^1(0)-z^2(0)\big|\sqrt{\lambda_s^B} + \sqrt{\int_{\mathbb{R}_0}\big|z^1(x)-z^2(x)\big|^2 q(dx)}\sqrt{\lambda_s^H} \\
&\qquad + \mathbb{E}^{\mathcal{G}_s}\Bigg(\big|\xi^1_{r1}-\xi^2_{r1}\big| + \big|\eta^1_{r2}(0)-\eta^2_{r2}(0)\big|\sqrt{\lambda^B_{r2}} \\
&\qquad + \sqrt{\int_{\mathbb{R}_0}\big|\eta^1_{r2}(x)-\eta^2_{r2}(x)\big|^2 q(dx)}\sqrt{\lambda^H_{r2}}\Bigg)\Bigg];
\end{aligned}$$

(H3) $f_s(\lambda,0,0,0,0) \in L^2_{\mathbb{G}}(0,T+K;\mathbb{R})$.

3 Estimate of the solutions to anticipated BSDEs

Liu and Ren [11] established in general existence and uniqueness of solution of (3). Now, we study these equations in depth. Our main question is to study an estimate of the solutions.

Theorem 2 *Suppose* (H1), (H2), *and* (H3) *hold*, $u(t)$ *and* $v(t)$ *satisfying* (a1) *and* (a2). *Then, for any* $\xi_t \in S^2_{\mathbb{G}}(T, T+K;\mathbb{R})$ *and* $\eta_t \in L^2_{\mathbb{G}}(T, T+K;\mathbb{R})$, *the solution* $(Y, Z) \in S^2_{\mathbb{G}}(0, T+K;\mathbb{R}) \times L^2_{\mathbb{G}}(0, T+K;\mathbb{R})$ *of the anticipated BSDEs* (3) *satisfies*

$$\begin{aligned}
&\mathbb{E}^{\mathcal{G}_t}\Big[\sup_{t\le s\le T}|Y_s|^2\,ds + \int_t^T\int_{\mathbb{R}}\big|Z_s(x)\big|^2\Lambda(dx,ds)\Big]\\
&\quad\le C_0\mathbb{E}^{\mathcal{G}_t}\Big[|\xi_T|^2 + \int_T^{T+K}|\xi_s|^2\,ds + \int_T^{T+K}\int_{\mathbb{R}}|\eta_s|^2\Lambda(dx,ds)\\
&\qquad + \Big(\int_t^T\big|f(0,0,0,0)\big|\,ds\Big)^2\Big],
\end{aligned}$$

for $t \in [0, T]$, *and* $C_0 > 0$ *only depends on* C, L, *and* K.

Proof By the Itô formula on $e^{\beta r}|Y_r|^2$, we get

$$\begin{aligned}
de^{\beta r}|Y_r|^2 &= \big(\beta e^{\beta r}|Y_r|^2 + 2e^{\beta r}Y_rf_r(Y_r, Z_r, Y_{r+u(r)}, Z_{r+v(r)})\big)\,dr\\
&\quad + e^{\beta r}\int_{\mathbb{R}}|Z_r|^2(x)\Lambda(dr,dx)\\
&\quad + 2e^{\beta r}Y_r\int_{\mathbb{R}}Z_r(x)\mu(dr,dx).
\end{aligned}$$

Iterating each side from s to T, $s \in [0, T]$, we obtain

$$\begin{aligned}
&e^{\beta T}|\xi_T|^2 + 2\int_s^T e^{\beta r}Y_rf_r(Y_r, Z_r, Y_{r+u(r)}, Z_{r+v(r)})\,dr\\
&\quad = e^{\beta s}|Y_s|^2 + \int_s^T\beta e^{\beta r}|Y_r|^2\,dr + \int_s^T\int_{\mathbb{R}}e^{\beta r}\big|Z_r(x)\big|^2\Lambda(dr,dx)\\
&\qquad + 2\int_s^T\int_{\mathbb{R}}e^{\beta r}Y_rZ_r(x)\mu(dr,dx).
\end{aligned}\tag{4}$$

Under condition (H2), we get

$$\begin{aligned}
&2\int_s^T e^{\beta r}Y_rf_r(Y_r, Z_r, Y_{r+u(r)}, Z_{r+v(r)})\,dr\\
&\quad = 2\int_s^T e^{\beta r}Y_r\big(f_r(Y_r, Z_r, Y_{r+u(r)}, Z_{r+v(r)}) - f_r(Y_r, Z_r, Y_{r+u(r)}, 0)\big)\,dr\\
&\qquad + 2\int_s^T e^{\beta r}Y_r\big(f_r(Y_r, Z_r, Y_{r+u(r)}, 0) - f_r(Y_r, Z_r, 0, 0)\big)\,dr\\
&\qquad + 2\int_s^T e^{\beta r}Y_r\big(f_r(Y_r, Z_r, 0, 0) - f_r(Y_r, 0, 0, 0)\big)\,dr\\
&\qquad + 2\int_s^T e^{\beta r}Y_r\big(f_r(Y_r, 0, 0, 0) - f_r(0, 0, 0, 0)\big)\,dr + 2\int_s^T e^{\beta r}Y_rf_r(0, 0, 0, 0)\,dr\\
&\quad \le 2C\int_s^T e^{\beta r}\mathbb{E}^{\mathcal{G}_r}\Big[\big|Z_{r+v(r)}(0)\big|\sqrt{\lambda^B_{r+v(r)}} + \sqrt{\int_{\mathbb{R}_0}\big|Z_{r+v(r)}(x)\big|^2q(dx)}\sqrt{\lambda^H_{r+v(r)}}\Big]|Y_r|\,dr\\
&\qquad + 2C\int_s^T e^{\beta r}\mathbb{E}^{\mathcal{G}_r}\big[|Y_{r+u(r)}|\big]|Y_r|\,dr
\end{aligned}$$

$$
\begin{aligned}
&+ 2C\int_s^T e^{\beta r}\Big[\big|Z(0)\big|\sqrt{\lambda_r^B} + \sqrt{\int_{\mathbb{R}_0}\big|Z_r(x)\big|^2 q(dx)}\sqrt{\lambda_r^H}\Big]|Y_r|\,dr \\
&+ 2C\int_s^T e^{\beta r}|Y_r|^2\,dr + 2\int_s^T e^{\beta r} f_r(0,0,0,0)Y_r\,dr. \\
\le{}& \frac{1}{6L}\int_s^T e^{\beta r}\mathbb{E}^{\mathcal{G}_r}\Big[\big|Z_{r+\nu(r)}(0)\big|\sqrt{\lambda_{r+\nu(r)}^B} + \sqrt{\int_{\mathbb{R}_0}\big|Z_{r+\nu(r)}(x)\big|^2 q(dx)}\sqrt{\lambda_{r+\nu(r)}^H}\Big]^2\,dr \\
&+ \frac{1}{4LT}\int_s^T e^{\beta r}\mathbb{E}^{\mathcal{G}_r}\big[|Y_{r+u(r)}|\big]^2\,dr \\
&+ \frac{1}{6}\int_s^T e^{\beta r}\Big[\big|Z(0)\big|\sqrt{\lambda_r^B} + \sqrt{\int_{\mathbb{R}_0}\big|Z_r(x)\big|^2 q(dx)}\sqrt{\lambda_r^H}\Big]^2\,dr \\
&+ \big(6LC^2 + 4LTC^2 + 6C^2 + 2C\big)\int_s^T e^{\beta r}|Y_r|^2\,dr + 2\int_s^T e^{\beta r} f_r(0,0,0,0)Y_r\,dr. \\
\le{}& \frac{1}{3}\int_s^{T+K}\int_{\mathbb{R}} e^{\beta r}\mathbb{E}^{\mathcal{G}_r}\big[\big|Z_r(x)\big|^2\big]\Lambda(dr,dx) + \frac{1}{4T}\int_s^{T+K} e^{\beta r}\mathbb{E}^{\mathcal{G}_r}\big[|Y_r|^2\big]\,dr \\
&+ \frac{1}{3}\int_s^T\int_{\mathbb{R}} e^{\beta r}\big|Y_r(x)\big|^2\Lambda(dr,dx) + 2\int_s^T e^{\beta r} f_r(0,0,0,0)Y_r\,dr \\
&+ \big(6LC^2 + 4LTC^2 + 6C^2 + 2C\big)\int_s^T e^{\beta r}|Y_r|^2\,dr.
\end{aligned}
$$

From (4) we have

$$
\begin{aligned}
& e^{\beta s}|Y_s|^2 + \frac{2}{3}\int_s^T\int_{\mathbb{R}} e^{\beta r}\big|Z_r(x)\big|^2\Lambda(dr,dx) \\
&\quad + \int_s^T\big(\beta - 6LC^2 - 4LTC^2 - C^2 - 2C\big)e^{\beta r}|Y_r|^2\,dr \\
&\le e^{\beta T}|\xi_T|^2 + 2\int_s^T e^{\beta r}Y_r f_r(0,0,0,0)\,dr - 2\int_s^T\int_{\mathbb{R}} e^{\beta r}Y_r Z_r(x)\mu(dr,dx) \\
&\quad + \frac{1}{3}\int_s^{T+K}\int_{\mathbb{R}} e^{\beta r}\mathbb{E}^{\mathcal{G}_r}\big[\big|Z_r(x)\big|^2\big]\Lambda(dr,dx) + \frac{1}{4T}\int_s^{T+K} e^{\beta r}\mathbb{E}^{\mathcal{G}_r}\big[|Y_r|^2\big]\,dr.
\end{aligned}
\tag{5}
$$

Taking condition expectations under $\mathcal{G}_s$, we have

$$
\begin{aligned}
& e^{\beta s}|Y_s|^2 + \frac{2}{3}\mathbb{E}^{\mathcal{G}_s}\Big[\int_s^T\int_{\mathbb{R}} e^{\beta r}\big|Z_r(x)\big|^2\Lambda(dr,dx)\Big] \\
&\quad + \mathbb{E}^{\mathcal{G}_s}\Big[\int_s^T\big(\beta - 6LC^2 - 4LTC^2 - C^2 - 2C\big)e^{\beta r}|Y_r|^2\,dr\Big] \\
&\le \mathbb{E}^{\mathcal{G}_s}\big[e^{\beta T}|\xi_T|^2\big] + 2\mathbb{E}^{\mathcal{G}_s}\Big[\int_s^T e^{\beta r}Y_r f_r(0,0,0,0)\,dr\Big] \\
&\quad + \frac{1}{3}\mathbb{E}^{\mathcal{G}_s}\Big[\int_s^{T+K}\int_{\mathbb{R}} e^{\beta r}\mathbb{E}^{\mathcal{G}_r}\big[\big|Z_r(x)\big|^2\big]\Lambda(dr,dx)\Big] \\
&\quad + \frac{1}{4T}\mathbb{E}^{\mathcal{G}_s}\Big[\int_s^{T+K} e^{\beta r}\mathbb{E}^{\mathcal{G}_r}\big[|Y_r|^2\big]\Big]\,dr \\
&\le \mathbb{E}^{\mathcal{G}_s}\big[e^{\beta T}|\xi_T|^2\big] + 2\mathbb{E}^{\mathcal{G}_s}\Big[\int_s^T e^{\beta r}Y_r f_r(0,0,0,0)\,dr\Big]
\end{aligned}
$$

$$+ \frac{1}{3}\mathbb{E}^{\mathcal{G}_s}\left[\int_s^T \int_{\mathbb{R}} e^{\beta r}\left|Z_r(x)\right|^2 \Lambda(dr,dx)\right]$$
$$+ \frac{1}{3}\mathbb{E}^{\mathcal{G}_s}\left[\int_T^{T+K} \int_{\mathbb{R}} e^{\beta r}\left|\eta_r(x)\right|^2 \Lambda(dr,dx)\right] + \frac{1}{4T}\mathbb{E}^{\mathcal{G}_s}\left[\int_s^{T+K} e^{\beta r}|Y_r|^2\right]dr.$$

Set $\beta = 6LC^2 + 4LTC^2 + 6C^2 + \frac{1}{4T} + 2C$, from (4) we have

$$\begin{aligned}
&\mathbb{E}^{\mathcal{G}_s}\int_s^T \int_{\mathbb{R}} e^{\beta r}\left|Z_r(x)\right|^2 \Lambda(dr,dx) \\
&\quad\le 3\mathbb{E}^{\mathcal{G}_s}e^{\beta T}|\xi_T|^2 + \mathbb{E}^{\mathcal{G}_s}\int_T^{T+K}\int_{\mathbb{R}} e^{\beta r}\left|\eta_r(x)\right|^2\Lambda(dr,dx) \\
&\qquad + \frac{3}{4T}\mathbb{E}^{\mathcal{G}_s}\int_T^{T+K} e^{\beta r}|\xi_r|^2\,dr + 6\mathbb{E}^{\mathcal{G}_s}\int_s^T e^{\beta r}f_r(0,0,0,0)Y_r\,dr.
\end{aligned} \tag{6}$$

For $\forall s \in [t,T]$,

$$\begin{aligned}
&\left|\int_s^T \int_{\mathbb{R}} e^{\beta r}Y_rZ_r(x)\mu(dr,dx)\right| \\
&\quad\le \left|\int_t^T \int_{\mathbb{R}} e^{\beta r}Y_rZ_r(x)\mu(dr,dx)\right| + \left|\int_t^s \int_{\mathbb{R}} e^{\beta r}Y_rZ_r(x)\mu(dr,dx)\right|.
\end{aligned}$$

By the Burkholder-Davis-Gundy inequality and the elementary inequality $ab \le \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2$ for $\varepsilon > 0$, we obtain

$$\begin{aligned}
&\mathbb{E}^{\mathcal{G}_t}\left[\left|\sup_{t\le s\le T}\int_s^T \int_{\mathbb{R}} e^{\beta r}Y_rZ_r(x)\mu(dr,dx)\right|\right] \\
&\quad\le 2\mathbb{E}^{\mathcal{G}_t}\left[\sup_{t\le s\le T}\left|\int_t^s \int_{\mathbb{R}} e^{\beta r}Y_rZ_r(x)\mu(dr,dx)\right|\right] \\
&\quad\le 2\sqrt{32}\mathbb{E}^{\mathcal{G}_t}\left[\int_t^s e^{2\beta r}\left(\int_t^T Y_rZ_r(x)\mu(dr,dx)\right)^2\right]^{\frac{1}{2}} \\
&\quad\le \frac{1}{4}\mathbb{E}^{\mathcal{G}_t}\sup_{t\le r\le T}\left[e^{\beta r}|Y_r|^2\right] + 128\mathbb{E}^{\mathcal{G}_t}\left[\int_t^T \int_{\mathbb{R}} e^{\beta r}\left|Z_r(x)\right|^2\Lambda(dr,dx)\right].
\end{aligned} \tag{7}$$

From (5) and (7), we get

$$\begin{aligned}
&\mathbb{E}^{\mathcal{G}_t}\left[\sup_{t\le s\le T} e^{\beta s}|Y_s|\right] \\
&\quad\le \mathbb{E}^{\mathcal{G}_t}\left[e^{\beta T}|\xi_T|^2 + 2\int_s^T e^{\beta r}|Y_r|\left|f_r(0,0,0,0)\right|dr\right. \\
&\qquad + 2\sup_{t\le s\le T}\left.\left|\int_s^T \int_{\mathbb{R}} e^{\beta r}Y_rZ_r(x)\mu(dr,dx)\right|\right] \\
&\qquad + \frac{1}{3}\mathbb{E}^{\mathcal{G}_t}\left\{\int_s^{T+K}\int_{\mathbb{R}} e^{\beta r}\mathbb{E}^{\mathcal{G}_r}\left[\left|Z_r(x)\right|^2\right]\Lambda(dr,dx)\right\} + \frac{1}{4T}\mathbb{E}^{\mathcal{G}_t}\left\{\int_s^{T+K} e^{\beta r}\mathbb{E}^{\mathcal{G}_r}\left[|Y_r|^2\right]dr\right\} \\
&\quad\le \mathbb{E}^{\mathcal{G}_t}\left[e^{\beta T}|\xi_T|^2 + 2\int_s^T e^{\beta r}|Y_r|\left|f_r(0,0,0,0)\right|dr\right] + \frac{1}{2}\mathbb{E}^{\mathcal{G}_t}\sup_{t\le r\le T}\left[e^{\beta r}|Y_r|^2\right]
\end{aligned}$$

$$
\begin{aligned}
&+ 256\mathbb{E}^{\mathcal{G}_t}\left[\int_t^T \int_{\mathbb{R}} e^{\beta r}\left|Z_r(x)\right|^2 \Lambda(dr,dx)\right] + \frac{1}{3}\mathbb{E}^{\mathcal{G}_t}\left[\int_s^{T+K} \int_{\mathbb{R}} e^{\beta r}\left|Z_r(x)\right|^2 \Lambda(dr,dx)\right] \\
&+ \frac{1}{4T}\mathbb{E}^{\mathcal{G}_t}\left[\int_s^{T+K} e^{\beta r}|Y_r|^2\,dr\right] \\
\le{}& \mathbb{E}^{\mathcal{G}_t}\left[e^{\beta T}|\xi_T|^2 + 2\int_s^T e^{\beta r}|Y_r|\left|f_r(0,0,0,0)\right|dr\right] + \frac{3}{4}\mathbb{E}^{\mathcal{G}_t}\sup_{t\le r\le T}\left[e^{\beta r}|Y_r|^2\right] \\
&+ \frac{1}{3}\mathbb{E}^{\mathcal{G}_t}\left[\int_T^{T+K} \int_{\mathbb{R}} e^{\beta r}\left|\eta_r(x)\right|^2 \Lambda(dr,dx)\right] \\
&+ \left(256 + \frac{1}{3}\right)\mathbb{E}^{\mathcal{G}_t}\left[\int_t^T \int_{\mathbb{R}} e^{\beta r}\left|Z_r(x)\right|^2 \Lambda(dr,dx)\right] \\
&+ \frac{1}{4T}\mathbb{E}^{\mathcal{G}_t}\left[\int_T^{T+K} e^{\beta r}|\xi_r|^2\,dr\right].
\end{aligned}
\tag{8}
$$

From (6) and (8), let C_0 be a positive number that depends on T, L, and C, we get

$$
\begin{aligned}
&\frac{1}{4}\mathbb{E}^{\mathcal{G}_t}\left[\sup_{t\le s\le T} e^{\beta s}|Y_s|\right] \\
\le{}& C_0\mathbb{E}^{\mathcal{G}_t}\left[e^{\beta T}|\xi_T|^2 + \int_T^{T+K} e^{\beta r}|\xi_r|^2\,dr + \int_T^{T+K}\int_{\mathbb{R}} e^{\beta r}\left|\eta_r(x)\right|^2 \Lambda(dr,dx)\right] \\
&+ C_0\mathbb{E}^{\mathcal{G}_t}\left(\sup_{t\le r\le T} e^{\frac{1}{2}\beta r}|Y_r|\right)\left(\int_s^T e^{\frac{1}{2}\beta r}\left|f_r(0,0,0,0)\right|dr\right) \\
\le{}& C_0\mathbb{E}^{\mathcal{G}_t}\left[|\xi_T|^2 + \int_T^{T+K}|\xi_r|^2\,dr + \int_T^{T+K}\int_{\mathbb{R}}\left|\eta_r(x)\right|^2 \Lambda(dr,dx)\right] \\
&+ \frac{1}{8}\mathbb{E}^{\mathcal{G}_t}\left[\sup_{t\le r\le T} e^{\frac{1}{2}\beta r}|Y_r|\right] + 2C_0^2\mathbb{E}^{\mathcal{G}_t}\left[\left(\int_s^T\left|f_r(0,0,0,0)\right|dr\right)^2\right].
\end{aligned}
\tag{9}
$$

From (8) and (9), we get

$$
\begin{aligned}
&\mathbb{E}^{\mathcal{G}_t}\left[\sup_{t\le s\le T}|Y_s|^2\,ds + \int_t^T\int_{\mathbb{R}}\left|Z_s(x)\right|^2\Lambda(dx,ds)\right] \\
\le{}& C_0\mathbb{E}^{\mathcal{G}_t}\left[|\xi_T|^2 + \int_T^{T+K}|\xi_s|^2\,ds + \int_T^{T+K}\int_{\mathbb{R}}|\eta_s|^2\Lambda(dx,ds)\right. \\
&\left.+ \left(\int_t^T\left|f(0,0,0,0)\right|ds\right)^2\right].
\end{aligned}
$$

The proof is complete. □

4 Comparison theorem

The comparison theorem has important applications in many respects. It is one of the most important theorems in the history of BSDEs. Because of wide application it attracts more and more attention from researchers. Now we introduce the comparison theorem of anticipated BSDEs (3) that the generator $f(\cdot)$ is linear. The linear anticipated BSDE is written in the following form:

$$
\begin{cases}
-dY_t = [A_tY_t + D_tY_{t+u(t)} + C_t + E_t(0)Z_t(0)\sqrt{\lambda_t^B} + \int_{\mathbb{R}_0}E_t(x)Z_t(x)q(dx)\sqrt{\lambda_t^H}]\,dt \\
\qquad\quad - Z_t(0)\,dB_t - \int_{\mathbb{R}_0}Z_t(x)\tilde{H}(dt,dx); \\
Y_t = \xi_T, \quad t\in[T,T+K].
\end{cases}
\tag{10}
$$

Lemma 3 *Assume that we have a linear driver anticipated BSDE of* (10), *and we assume that*:

(d1) *There exists* $C_1 > 0$ *for each* $t \in [0,T]$, *it hold that* $|A_t| < C_1$, *and* $D_t < C_1$.

(d2) $C_t \in S^2_{\mathbb{G}}(0,T+K;\mathbb{R})$, *and* $E_t \in L^2_{\mathbb{G}}(0,T+K;\mathbb{R})$.

(d3) Y_t *is increasing.*

It is then true that $Y_t \geq \mathbb{E}^{\mathcal{G}_t}[\xi_T X_T + \int_t^T C_t X_t\,dt]$, $t \in [0,T]$ *a.e., a.s.*

Here

$$X_s = \begin{cases} \exp[\int_t^s \int_{\mathbb{R}} E_r \mu(dr,dx) - \frac{1}{2}\int_t^s \int_{\mathbb{R}} E_r^2 \Lambda(dr,dx) + \int_t^s (A_r + D_r)\,dr], & s \in [t,T]; \\ 1, & s \in [0,t]. \end{cases}$$

Proof We can easily conclude that the anticipated BSDE of (10) satisfies conditions (H1)-(H3). Following Liu and Ren [11], we find that the anticipated BSDE of (10) has a unique solution.

Set

$$dX_t = X_t\left[(A_t + D_t)\,dt + \frac{E_t(0)}{\sqrt{\lambda_t^B}}\,dB_t + \int_{\mathbb{R}_0} \frac{E_t(x)}{\sqrt{\lambda_t^H}}\tilde{H}(dx,dt)\right].$$

We have

$$X_s = \begin{cases} \exp[\int_t^s \int_{\mathbb{R}} E_r \mu(dr,dx) - \frac{1}{2}\int_t^s \int_{\mathbb{R}} E_r^2 \Lambda(dr,dx) + \int_t^s (A_r + D_r)\,dr], & s \in [t,T]; \\ 1, & s \in [0,t]. \end{cases}$$

Applying Itô's formula to $X_s Y_s$ on $s \in [t,T]$,

$$\begin{aligned} dX_s Y_s &= -X_s\left[A_s Y_s + D_s Y_{s+u(s)} + C_s + E_s(0)Z_s(0)\sqrt{\lambda_s^B} + \int_{\mathbb{R}_0} E_s(x)Z_s(x)q(dx)\sqrt{\lambda_s^H}\right]ds \\ &\quad + X_s\left[Z_s(0)\,dB_s + \int_{\mathbb{R}_0} Z_s(x)\tilde{H}(ds,dx)\right] \\ &\quad + X_s Y_s\left[(A_s + D_s)\,ds + \frac{E_s(0)}{\sqrt{\lambda_s^B}}\,dB_s + \int_{\mathbb{R}_0} \frac{E_s(x)}{\sqrt{\lambda_s^H}}\tilde{H}(dx,ds)\right] \\ &\quad + X_s\left[\frac{E_s(0)Z_s(0)}{\sqrt{\lambda_s^B}}\lambda_s^B\,ds + \int_{\mathbb{R}_0} \frac{E_s(x)Z_s(x)}{\sqrt{\lambda_s^H}}q(dx)\lambda_s^H\,ds\right] \\ &= -[X_s D_s Y_{s+u(s)} - X_s D_s Y_s + X_s C_s]\,ds + \left[X_s Z_s + X_s Y_s \frac{E_s(x)}{\sqrt{\lambda_s}}\right]\mu(dx,ds). \end{aligned}$$

Integrating each side from t to T,

$$\begin{aligned} &X_T Y_T - X_t Y_t \\ &\quad = -\int_t^T [X_s D_s Y_{s+u(s)} - X_s D_s Y_t + X_s C_s]\,ds \\ &\qquad + \int_t^T \int_{\mathbb{R}}\left[X_s Z_s + X_s Y_s \frac{E_s(x)}{\sqrt{\lambda_s}}\right]\mu(dx,ds). \end{aligned}$$

Taking conditional expectations, we obtain

$$\begin{aligned} Y_t &= \mathbb{E}^{\mathcal{G}_t}\left[X_T\xi_T + \int_t^T C_sX_s\,ds\right] + \mathbb{E}^{\mathcal{G}_t}\int_t^T \left[X_sD_s(Y_{s+u(s)} - Y_s)\right]ds \\ &\geq \mathbb{E}^{\mathcal{G}_t}\left[X_T\xi_T + \int_t^T C_sX_s\,ds\right]. \end{aligned}$$

□

Theorem 4 *Suppose that*

$$g_2\big(t,y,y',z\big) = f\left(t,\lambda,y,y',z(0)E_t(0)\sqrt{\lambda_t^B},\int_{\mathbb{R}_0} z(x)E_t(x)q(dx)\sqrt{\lambda_t^H}\right).$$

Let f have the form of (7), *satisfying* (H1)-(H3). $(Y_t^{(1)},Z_t^{(1)})$ *and* $(Y_t^{(2)},Z_t^{(2)})$ *are, respectively, solutions to their corresponding anticipated BSDEs* (10). *Moreover, we suppose that*:

(e1) $\xi_s^{(1)} \geq \xi_s^{(2)}$, $s \in [T,T+K]$ *a.s.*

(e2) $g_1(t,\lambda,y,z,y') \geq g_2(t,\lambda,y,z,y')$, $t \in [0,T]$, $y \in \mathbb{R}$, $y' \in S^2_{\mathbb{G}}(0,T+K;\mathbb{R})$, $z \in L^2_{\mathbb{G}}(0,T+K;\mathbb{R})$.

(e3) $g_2(t,\lambda,y,\cdot,y')$ *is increasing, i.e.*, $g_2(t,\lambda,y,z_1,y') \geq g_2(t,\lambda,y,z_2,y')$, *if* $z_1 \geq z_2$.

Then

$$Y_t^{(1)} \geq Y_t^{(2)}, \quad a.e., a.s.$$

Proof Set $\tilde{Y}_t = Y_t^{(1)} - Y_t^{(2)}$, $\tilde{Z}_t = Z_t^{(1)} - Z_t^{(2)}$, $\tilde{\xi}_t = \xi_t^{(1)} - \xi_t^{(2)}$, and $C_t = g_1(t,Y_t^{(1)},Z_t^{(1)},Y_{t+u(t)}^{(1)}) - g_2(t,Y_t^{(1)},Z_t^{(1)},Y_{t+u(t)}^{(1)})$; by the BSDEs (10), we have

$$\begin{cases} -d\tilde{Y}_t = [g_2(t,Y_t^{(1)},Z_t^{(1)},Y_{t+u(t)}^{(1)}) - g_2(t,Y_t^{(2)},Z_t^{(2)},Y_{t+u(t)}^{(2)}) + C_t]\,dt \\ \qquad\qquad - \tilde{Z}_t\mu(dt,dx), \quad t \in [0,T]; \\ \tilde{Y}_T = \tilde{\xi}_T. \end{cases} \tag{11}$$

Note that $g_2(t,Y_t^{(1)},Z_t^{(1)},Y_{t+u(t)}^{(1)}) - g_2(t,Y_t^{(2)},Z_t^{(2)},Y_{t+u(t)}^{(2)}) = A_tY_t + D_tY_{t+u(t)} + C_t + P_tE_t(0) \times \tilde{Z}_t(0)\sqrt{\lambda_t^B} + Q_t\int_{\mathbb{R}_0} E_t(x)\tilde{Z}_t(x)q(dx)\sqrt{\lambda_t^H}$.

Here

$$A_t = \begin{cases} \frac{g_2(t,Y_t^{(1)},Z_t^{(1)},Y_{t+u(t)}^{(1)}) - g_2(t,Y_t^{(2)},Z_t^{(1)},Y_{t+u(t)}^{(1)})}{Y_t^{(1)} - Y_t^{(2)}}, & \text{if } Y_t^{(1)} \neq Y_t^{(2)}; \\ 0, & \text{if } Y_t^{(1)} = Y_t^{(2)}, \end{cases}$$

$$D_t = \begin{cases} \frac{g_2(t,Y_t^{(2)},Z_t^{(1)},Y_{t+u(t)}^{(1)}) - g_2(t,Y_t^{(2)},Z_t^{(1)},Y_{t+u(t)}^{(2)})}{Y_{t+u(t)}^{(1)} - Y_{t+u(t)}^{(2)}}, & \text{if } Y_{t+u(t)}^{(1)} \neq Y_{t+u(t)}^{(2)}; \\ 0, & \text{if } Y_{t+u(t)}^{(1)} = Y_{t+u(t)}^{(2)}, \end{cases}$$

$$P_t = \begin{cases} \frac{g_2(t,Y_t^{(2)},Z_t^{(1,B)},Y_{t+u(t)}^{(2)}) - g_2(t,Y_t^{(2)},Z_t^{(2,B)},Y_{t+u(t)}^{(2)})}{E_t(0)\tilde{Z}_t(0)\sqrt{\lambda_t^B}}, & \text{if } E_t(0)\tilde{Z}_t(0)\sqrt{\lambda_t^B} \neq 0; \\ 0, & \text{if } E_t(0)\tilde{Z}_t(0)\sqrt{\lambda_t^B} = 0, \end{cases}$$

$$Q_t = \begin{cases} \frac{g_2(t,Y_t^{(2)},Z_t^{(1,H)},Y_{t+u(t)}^{(2)}) - g_2(t,Y_t^{(2)},Z_t^{(2,H)},Y_{t+u(t)}^{(2)})}{\int_{\mathbb{R}_0} E_t(x)\tilde{Z}_t(x)q(dx)\sqrt{\lambda_t^H}}, & \text{if } \int_{\mathbb{R}_0} E_t(x)\tilde{Z}_t(x)q(dx)\sqrt{\lambda_t^H} \neq 0; \\ 0, & \text{if } \int_{\mathbb{R}_0} E_t(x)\tilde{Z}_t(x)q(dx)\sqrt{\lambda_t^H} = 0. \end{cases}$$

So we get

$$\begin{cases} -d\tilde{Y}_t = [A_t\tilde{Y}_t + D_t\tilde{Y}_{t+u(t)} + C_t + P_tE_t(0)\tilde{Z}_t(0)\sqrt{\lambda_t^B} + Q_t\int_{\mathbb{R}_0} E_t(x)\tilde{Z}_t(x)q(dx)\sqrt{\lambda_t^H}]\,dt \\ \qquad\quad - \tilde{Z}_t(0)\,dB_t - \int_{\mathbb{R}_0} \tilde{Z}_t(x)\tilde{H}(dt,dx), \quad t \in [0,T]; \\ \tilde{Y}_T = \tilde{\xi}_T. \end{cases}$$

Since g_2 satisfies (H1)-(H3), and A_t, D_t, P_t, and Q_t satisfy the conditions of Lemma 3, the solution $\tilde{Y}_t$ of the anticipated BSDE in (9) satisfies

$$\tilde{Y}_t \geq \mathbb{E}^{\mathcal{G}_t}\left[\tilde{\xi}_T X_T + \int_t^T C_t X_t\,dt\right] \geq 0 \quad \text{a.e., a.s.}$$

Hence $Y_t^{(1)} \geq Y_t^{(2)}$ a.e., a.s. □

5 A nonlinear Feynman-Kac formula

In order to solve the quasilinear PDE, by El Karoui *et al.* [14] let us denote by $X_s^{x,t}$ the solution of the SDE of the following form:

$$\begin{cases} dX_s^{x,t} = b(X_s^{x,t})\,ds + \sigma(X_s^{x,t},0)\,dBs + \int_{\mathbb{R}_0}\sigma(X_s^{x,t},\alpha)\tilde{H}(ds,d\alpha); \\ X_t^{x,t} = x. \end{cases} \tag{12}$$

Here $b(x) : \mathbb{R} \to \mathbb{R}$ and $\sigma(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions.

Now we set up a nonlinear Feynman-Kac formula for a special class of quasilinear PDEs in the form

$$\begin{cases} \frac{\partial\Phi}{\partial t} + \mathcal{L}\Phi + f(t,\Phi(x,t),z(x,t),\Phi(x,t+u(t)),z(x,t+v(t))) \\ \quad = 0, \quad (t,x) \in [0,T] \times \mathbb{R}; \\ \Phi(T,x) = g(x), \quad x \in \mathbb{R}. \end{cases} \tag{13}$$

Here $\mathcal{L}$ is a two order elliptic differential operator,

$$\mathcal{L}\Phi = \frac{1}{2}\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2\Phi}{\partial x_i\,\partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial\Phi}{\partial x_i} \quad \left(a_{ij} = \left[\sigma\sigma^T\right]_{ij}\right).$$

It is interesting to notice that we consider coupled BSDE with (12),

$$\begin{cases} -dY_s^{x,t} = f_s(\lambda_s, X_s^{x,t}, Y_s^{x,t}, Z_s^{x,t}, Y_{s+u(s)}^{x,t}, Z_{s+v(s)}^{x,t})\,ds \\ \qquad\qquad - \int_{\mathbb{R}} Z_s^{x,t}(\alpha)\mu(ds,d\alpha), \quad s \in [t,T]; \\ Y_t^{x,t} = g(x), \quad t \in [T,T+K]; \\ Z_t^{x,t} = \eta_t, \quad t \in [T,T+K], \end{cases} \tag{14}$$

where $u(\cdot)$ and $v(\cdot)$ satisfy assumptions (a1) and (a2), f satisfies (H1), (H2), and (H3).

Given fixed $(t,x) \in [0,T] \times \mathbb{R}$, we note that $\Phi(t,x) = Y_t^{t,x}$ is an adapted solution of the anticipated BSDE (14). Applying the Itô formula to $(\Phi(X_s^{x,t},s))$ on $s \in [t,T]$, from (13) we achieve the theorem as follows.

Theorem 5 *Assume that, for a given time duration* $[t,T]$, *the anticipated BSDEs* (14) *has an adapted solution* $(Y_t^{t,x}, Z_t^{t,x})$. *Then the function* $\Phi(t,x) := Y_t^{t,x}$, $(t,x) \in [0,T] \times \mathbb{R}$ *is a viscosity solution of the quasilinear PDE* (13).

Competing interests
The author declares that he has no competing interests.

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