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The ratio log-concavity of the Cohen numbers

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Abstract
Let U_n denote the nth Cohen number. Some combinatorial properties for U_n have been discovered. In this paper, we prove the ratio log-concavity of U_n by establishing the lower and upper bounds for $\frac{U_n}{U_{n-1}}$.

MSC: 05A20; 11B83

Keywords: the Cohen number; log-concavity; ratio log-concavity

1 Introduction

An infinite sequence $\{a_n\}_{n=0}^{\infty}$ is said to be log-concave (respectively, log-convex) if for any positive integer n,

$$a_n^2 \geq a_{n+1}a_{n-1} \quad \bigl(\text{respectively}, a_n^2 \leq a_{n+1}a_{n-1}\bigr).$$

Furthermore, a positive sequence $\{a_n\}_{n=0}^{\infty}$ is said to be ratio log-concave if the sequence $\{\frac{a_{n+1}}{a_n}\}_{n=0}^{\infty}$ is log-concave. The aim of this paper is to prove the ratio log-concavity of the Cohen numbers. The nth Cohen number was first introduced by Cohen [1] which is defined by

$$U_n = h(n)U_{n-1} + g(n)U_{n-2} \quad (n \geq 2) \tag{1.1}$$

with $U_0 = 1$ and $U_1 = 12$, where

$$h(n) = \frac{3(2n-1)(3n^2-3n+1)(15n^2-15n+4)}{n^5} \tag{1.2}$$

and

$$g(n) = \frac{3(n-1)^3(3n-4)(3n-2)}{n^5}. \tag{1.3}$$

In [2], Zudilin proved that D_nU_n is an integer where D_n is the least common multiple of $1, 2, \dots, n$. Moreover, he conjectured some stronger inclusions that were finally proved by

Krattenthaler and Rivoal [3]. In particular, they proved that

$$U_n = \sum_{i,j} \binom{n}{i}^2 \binom{n}{j}^2 \binom{n+j}{n} \binom{n+j-i}{n} \binom{2n-i}{i},$$

where the binomial coefficients $\binom{a}{b}$ are zero if $b < 0$ or $a < b$; see also [4].

Recently, the combinatorial properties of U_n were considered. Employing a criterion due to Xia and Yao [5], it is easy to prove the log-convexity of U_n. Chen and Xia [6] proved the 2-log-convexity of U_n, that is,

$$\left(U_{n-1}U_{n+1} - U_n^2\right)\left(U_{n+1}U_{n+3} - U_{n+2}^2\right) > \left(U_n U_{n+2} - U_{n+1}^2\right)^2.$$

In this paper, we prove the ratio log-concavity of U_n. The main results of the paper can be stated as follows.

Theorem 1.1 *The sequence $\{U_n\}_{n=0}^{\infty}$ is ratio log-concave, namely, for $n \geq 2$,*

$$\frac{U_n^2}{U_{n-1}^2} > \frac{U_{n+1}}{U_n}\frac{U_{n-1}}{U_{n-2}}. \tag{1.4}$$

2 Lower and upper bounds for $\frac{U_n}{U_{n-1}}$

In order to prove Theorem 1.1, we first establish the lower and upper bounds for $\frac{U_n}{U_{n-1}}$.

Lemma 2.1 *For $n \geq 5$,*

$$l(n) < \frac{U_n}{U_{n-1}}, \tag{2.1}$$

where

$$l(n) = 135 + 78\sqrt{3} - \frac{675 + 390\sqrt{3}}{2n} + \frac{9{,}737\sqrt{3} + 16{,}848}{48n^2} - \frac{3{,}497\sqrt{3} + 6{,}045}{32n^3}. \tag{2.2}$$

Proof We are ready to prove Lemma 2.1 by induction on n. It is easy to check that (2.1) is true when $n = 5$ and $n = 6$. Suppose that Lemma 2.1 holds when $n = m \geq 5$, that is,

$$l(m) < \frac{U_m}{U_{m-1}}. \tag{2.3}$$

In order to prove Lemma 2.1, it suffices to prove that this lemma holds when $n = m + 2$, that is,

$$l(m+2) < \frac{U_{m+2}}{U_{m+1}}. \tag{2.4}$$

Based on (1.1) and (2.3),

$$\begin{aligned}\frac{U_{m+2}}{U_{m+1}} &= h(m+2) + g(m+2)\frac{1}{\frac{U_{m+1}}{U_m}} = h(m+2) + g(m+2)\frac{1}{h(m+1) + g(m+1)\frac{U_{m-1}}{U_m}} \\ &> h(m+2) + g(m+2)\frac{1}{h(m+1) + \frac{g(m+1)}{l(m)}},\end{aligned} \tag{2.5}$$

where $h(n)$, $g(n)$, and $l(n)$ are defined by (1.2), (1.3), and (2.2), respectively. Thanks to (2.5),

$$\begin{aligned}
&\frac{U_{m+2}}{U_{m+1}} - l(m+2) \\
&\quad > h(m+2) + g(m+2)\frac{1}{h(m+1) + \frac{g(m+1)}{l(m)}} - l(m+2) \\
&\quad = \frac{13(542{,}921 - 313{,}428\sqrt{3})\alpha(m)}{830{,}059{,}024(m+2)^5\beta(m)},
\end{aligned} \tag{2.6}$$

where $\alpha(m)$ and $\beta(m)$ are defined by

$$\begin{aligned}
\alpha(m) = {} & 121{,}396{,}132{,}260m^9 - 257{,}880{,}671{,}236\sqrt{3}m^8 + 675{,}703{,}429{,}830m^8 \\
& + 2{,}176{,}536{,}150{,}666m^7 - 1{,}330{,}058{,}753{,}240\sqrt{3}m^7 + 4{,}927{,}389{,}297{,}804m^6 \\
& - 2{,}983{,}584{,}697{,}467\sqrt{3}m^6 - 4{,}066{,}074{,}230{,}366\sqrt{3}m^5 \\
& + 7{,}060{,}751{,}181{,}826m^5 - 3{,}674{,}684{,}488{,}924\sqrt{3}m^4 \\
& + 6{,}286{,}428{,}416{,}954m^4 + 3{,}481{,}214{,}050{,}452m^3 \\
& - 2{,}206{,}793{,}212{,}277\sqrt{3}m^3 + 1{,}169{,}733{,}232{,}808m^2 - 845{,}639{,}850{,}544\sqrt{3}m^2 \\
& - 187{,}272{,}537{,}764\sqrt{3}m + 218{,}478{,}614{,}224m \\
& - 18{,}263{,}322{,}480\sqrt{3} + 17{,}360{,}076{,}864
\end{aligned}$$

and

$$\begin{aligned}
\beta(m) = {} & 864m^8 - 96m^6 - 468\sqrt{3}m^6 + 468\sqrt{3}m^5 - 810m^5 - 2{,}206{,}269\sqrt{3}m^4 \\
& + 3{,}821{,}499m^4 + 6{,}665{,}166m^3 - 3{,}848{,}598\sqrt{3}m^3 - 2{,}680{,}756\sqrt{3}m^2 \\
& + 4{,}642{,}290m^2 - 875{,}459\sqrt{3}m + 1{,}515{,}969m + 194{,}220 - 112{,}164\sqrt{3}.
\end{aligned}$$

By (2.6) and the fact that $\alpha(m)\beta(m) > 0$ for $m \geq 5$, we obtain (2.4). This completes the proof of Lemma 2.1 by induction. □

Lemma 2.2 *For* $n \geq 5$,

$$\frac{U_n}{U_{n-1}} < u(n), \tag{2.7}$$

where

$$u(n) = 135 + 78\sqrt{3} - \frac{675 + 390\sqrt{3}}{2n} + \frac{9{,}737\sqrt{3} + 16{,}848}{48n^2} - \frac{6{,}994\sqrt{3} + 6{,}045}{64n^3}. \tag{2.8}$$

Proof We also prove Lemma 2.2 by induction on n. It is easy to verify that (2.7) holds for $n = 5$ and $n = 6$. Assume that Lemma 2.2 is true for $n = m \geq 5$, that is,

$$\frac{U_m}{U_{m-1}} < u(m), \tag{2.9}$$

where $u(m)$ is defined by (2.8). In order to prove Lemma 2.2, it suffices to prove that Lemma 2.2 is true when $n = m + 2$, namely,

$$\frac{U_{m+2}}{U_{m+1}} < u(m+2). \tag{2.10}$$

Based on (1.1) and (2.9),

$$\begin{aligned}\frac{U_{m+2}}{U_{m+1}} &= h(m+2) + g(m+2)\frac{1}{\frac{U_{m+1}}{U_m}} \\ &= h(m+2) + g(m+2)\frac{1}{h(m+1) + g(m+1)\frac{U_{m-1}}{U_m}} \\ &< h(m+2) + g(m+2)\frac{1}{h(m+1) + \frac{g(m+1)}{u(m)}},\end{aligned} \tag{2.11}$$

where $h(n)$, $g(n)$, and $u(n)$ are defined by (1.2), (1.3), and (2.8), respectively. Thanks to (2.11),

$$\begin{aligned}&\frac{U_{m+2}}{U_{m+1}} - u(m+2) \\ &\quad< h(m+2) + g(m+2)\frac{1}{h(m+1) + \frac{g(m+1)}{u(m)}} - u(m+2) \\ &\quad= \frac{13(2{,}340 - 1{,}351\sqrt{3})\varphi(m)}{192(m+2)^5\psi(m)} < 0,\end{aligned} \tag{2.12}$$

where $\varphi(m)$ and $\psi(m)$ are defined by

$$\begin{aligned}\varphi(m) = {}& 3{,}760{,}473{,}600m^{10} + 24{,}236{,}858{,}880m^9 - 7{,}725{,}471{,}840\sqrt{3}m^9 \\ & + 84{,}297{,}090{,}576m^8 - 46{,}822{,}426{,}128\sqrt{3}m^8 - 123{,}822{,}624{,}402\sqrt{3}m^7 \\ & + 204{,}386{,}453{,}088m^7 - 194{,}450{,}024{,}349\sqrt{3}m^6 + 336{,}953{,}792{,}124m^6 \\ & - 203{,}475{,}797{,}950\sqrt{3}m^5 + 365{,}977{,}131{,}864m^5 + 260{,}362{,}891{,}056m^4 \\ & - 147{,}457{,}384{,}610\sqrt{3}m^4 - 73{,}568{,}487{,}135\sqrt{3}m^3 + 119{,}994{,}653{,}508m^3 \\ & - 24{,}169{,}674{,}728\sqrt{3}m^2 + 34{,}436{,}526{,}528m^2 + 5{,}563{,}246{,}416m \\ & - 4{,}710{,}672{,}460\sqrt{3}m + 382{,}180{,}032 - 412{,}780{,}368\sqrt{3}\end{aligned}$$

and

$$\begin{aligned}\psi(m) = {}& 1{,}728m^8 - 936\sqrt{3}m^6 - 192m^6 + 2{,}205{,}033{,}030m^5 - 1{,}273{,}076{,}064\sqrt{3}m^5 \\ & + 5{,}520{,}229{,}623m^4 - 3{,}187{,}105{,}038\sqrt{3}m^4 - 3{,}317{,}697{,}396\sqrt{3}m^3 \\ & + 5{,}746{,}420{,}422m^3 - 1{,}787{,}669{,}312\sqrt{3}m^2 + 3{,}096{,}333{,}090m^2 \\ & + 860{,}545{,}413m - 496{,}836{,}418\sqrt{3}m - 56{,}805{,}528\sqrt{3} + 98{,}389{,}980.\end{aligned}$$

By (2.12) and the fact that $\varphi(m)\psi(m) > 0$ for $m \geq 5$, we arrive at (2.10). This completes the proof of Lemma 2.2 by induction. □

3 Proof of Theorem 1.1

In this section, we present a proof of Theorem 1.1.

Lemma 3.1 *For $n \geq 5$,*

$$\frac{U_{n+1}U_{n-1}}{U_n^2} < f(n), \tag{3.1}$$

where

$$f(n) = \frac{(144n^2 - 216n + 298 - 39\sqrt{3})(2n-1)^2}{(144n^2 - 504n + 658 - 39\sqrt{3})(2n+1)^2}. \tag{3.2}$$

Proof Let $h(n)$ and $g(n)$ be defined by (1.2) and (1.3), respectively. It is easy to verify that, for $n \geq 5$,

$$h^2(n+1) + 4f(n)g(n+1) = \frac{3a(n)}{(144n^2 - 504n + 658 - 39\sqrt{3})(n+1)^{10}(2n+1)^2} > 0, \tag{3.3}$$

where

$$\begin{aligned} a(n) = {} & 14{,}017{,}536n^{14} + 35{,}043{,}840n^{13} - 3{,}796{,}416\sqrt{3}n^{12} + 3{,}796{,}416n^{12} \\ & - 22{,}767{,}264\sqrt{3}n^{11} - 35{,}430{,}048n^{11} - 63{,}335{,}220\sqrt{3}n^{10} + 136{,}740{,}296n^{10} \\ & - 108{,}042{,}324\sqrt{3}n^{9} + 585{,}572{,}912n^{9} + 1{,}001{,}472{,}846n^{8} - 125{,}838{,}297\sqrt{3}n^{8} \\ & - 105{,}379{,}248\sqrt{3}n^{7} + 1{,}047{,}661{,}216n^{7} - 65{,}023{,}608\sqrt{3}n^{6} + 749{,}372{,}512n^{6} \\ & - 29{,}770{,}650\sqrt{3}n^{5} + 381{,}561{,}324n^{5} + 139{,}393{,}232n^{4} - 10{,}033{,}296\sqrt{3}n^{4} \\ & + 35{,}929{,}568n^{3} - 2{,}426{,}892\sqrt{3}n^{3} - 399{,}789\sqrt{3}n^{2} + 6{,}231{,}942n^{2} \\ & + 654{,}864n - 40{,}248\sqrt{3}n + 31{,}584 - 1{,}872\sqrt{3}. \end{aligned}$$

Moreover, it is easy to check that, for $n \geq 0$,

$$\begin{aligned} & 2f(n)l(n) - h(n+1) \\ & \quad = \frac{\sqrt{3}b(n)}{48n^3(144n^2 - 504n + 658 + 39\sqrt{3})(2n+1)^2(n+1)^5} > 0 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} & \bigl(2f(n)l(n) - h(n+1)\bigr)^2 - \bigl(h^2(n+1) + 4f(n)g(n+1)\bigr) \\ & \quad = \frac{(1{,}351 + 780\sqrt{3})(2n-1)^2(144n^2 - 216n + 298 - 39\sqrt{3})c(n)}{384n^6(144n^2 - 504n + 658 + 39\sqrt{3})(2n+1)^4(n+1)^5} > 0, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} b(n) = {} & 4{,}313{,}088n^{12} - 10{,}048{,}896n^{10} - 1{,}168{,}128\sqrt{3}n^{10} + 2{,}134{,}080\sqrt{3}n^{9} \\ & + 18{,}944{,}640n^{9} + 25{,}025{,}104n^{8} - 18{,}411{,}096\sqrt{3}n^{8} - 23{,}385{,}908n^{7} \end{aligned}$$

$$
\begin{aligned}
&- 52{,}175{,}844\sqrt{3}n^7 - 856{,}596n^6 - 24{,}604{,}446\sqrt{3}n^6 + 30{,}202{,}523n^5 \\
&+ 8{,}838{,}489\sqrt{3}n^5 - 10{,}242{,}333\sqrt{3}n^4 - 14{,}907{,}529n^4 - 17{,}776{,}278n^3 \\
&- 10{,}403{,}322\sqrt{3}n^3 + 6{,}462{,}024\sqrt{3}n^2 + 11{,}237{,}434n^2 + 2{,}501{,}109\sqrt{3}n \\
&+ 4{,}336{,}111n - 2{,}419{,}053 - 1{,}392{,}261\sqrt{3}
\end{aligned}
$$

and

$$
\begin{aligned}
c(n) = {}& 59{,}719{,}680n^{12} - 106{,}074{,}695{,}808n^{11} + 61{,}088{,}601{,}600\sqrt{3}n^{11} \\
&- 46{,}198{,}518{,}912\sqrt{3}n^{10} + 80{,}016{,}457{,}536n^{10} + 143{,}774{,}162{,}864n^9 \\
&- 82{,}704{,}126{,}024\sqrt{3}n^9 + 283{,}349{,}090{,}856\sqrt{3}n^8 - 491{,}163{,}569{,}992n^8 \\
&- 1{,}030{,}144{,}421{,}232n^7 + 594{,}612{,}735{,}990\sqrt{3}n^7 + 478{,}140{,}044{,}616\sqrt{3}n^6 \\
&- 827{,}484{,}760{,}322n^6 + 199{,}910{,}945{,}130\sqrt{3}n^5 - 346{,}327{,}459{,}907n^5 \\
&- 74{,}608{,}485{,}009n^4 + 42{,}928{,}060{,}188\sqrt{3}n^4 - 6{,}144{,}385{,}390n^3 \\
&+ 3{,}678{,}965{,}082\sqrt{3}n^3 - 499{,}917{,}028n^2 + 293{,}762{,}352\sqrt{3}n^2 - 191{,}636{,}367n \\
&+ 71{,}139{,}198\sqrt{3}n - 55{,}991{,}052\sqrt{3} + 115{,}716{,}159.
\end{aligned}
$$

It follows from (3.3)-(3.5) that, for $n \geq 0$,

$$2f(n)l(n) - h(n+1) > \sqrt{h^2(n+1) + 4f(n)g(n+1)}$$

and thus

$$l(n) > \frac{h(n+1) + \sqrt{h^2(n+1) + 4f(n)g(n+1)}}{2f(n)}. \tag{3.6}$$

In view of (2.1) and (3.6),

$$\frac{U_n}{U_{n-1}} > \frac{h(n+1) + \sqrt{h^2(n+1) + 4f(n)g(n+1)}}{2f(n)}, \tag{3.7}$$

which implies that, for $n \geq 5$,

$$f(n)\left(\frac{U_n}{U_{n-1}}\right)^2 - h(n+1)\frac{U_n}{U_{n-1}} - g(n+1) > 0. \tag{3.8}$$

Thanks to (1.1),

$$f(n)U_n^2 - U_{n-1}U_{n+1} = U_n^2\left(f(n)\left(\frac{U_n}{U_{n-1}}\right)^2 - h(n+1)\frac{U_n}{U_{n-1}} - g(n+1)\right). \tag{3.9}$$

Lemma 3.1 follows from (3.8) and (3.9). This completes the proof. □

Lemma 3.2 *For* $n \geq 5$,

$$\frac{U_{n+1}U_{n-1}}{U_n^2} > f(n+1), \tag{3.10}$$

where $f(n)$ *is defined by* (3.2).

Proof It is easy to check that, for $n \geq 5$,

$$\begin{aligned} &h^2(n+1) + 4f(n+1)g(n+1) \\ &\quad = \frac{3(2n+1)^2 d(n)}{(n+1)^{10}(144n^2 - 216n + 298 - 39\sqrt{3})(2n+3)^2} > 0 \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &2f(n+1)l(n) - h(n+1) \\ &\quad = \frac{\sqrt{3}(2n+1)e(n)}{48n^3(144n^2 - 216n + 298 - 39\sqrt{3})(2n+3)^2(n+1)^5} > 0, \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} d(n) = {} & 3{,}504{,}384n^{12} + 19{,}274{,}112n^{11} + 45{,}630{,}000n^{10} - 949{,}104\sqrt{3}n^{10} \\ & - 6{,}640{,}920\sqrt{3}n^9 + 71{,}887{,}752n^9 + 109{,}395{,}878n^8 - 20{,}342{,}049\sqrt{3}n^8 \\ & + 166{,}770{,}736n^7 - 36{,}215{,}400\sqrt{3}n^7 - 41{,}834{,}832\sqrt{3}n^6 + 203{,}641{,}520n^6 \\ & - 32{,}969{,}274\sqrt{3}n^5 + 177{,}081{,}116n^5 + 106{,}446{,}904n^4 - 18{,}035{,}004\sqrt{3}n^4 \\ & - 6{,}785{,}376\sqrt{3}n^3 + 43{,}438{,}424n^3 + 11{,}552{,}574n^2 - 1{,}684{,}917\sqrt{3}n^2 \\ & - 249{,}912\sqrt{3}n + 1{,}816{,}272n + 128{,}736 - 16{,}848\sqrt{3}, \\ e(n) = {} & 2{,}156{,}544n^{11} + 7{,}547{,}904n^{10} + 9{,}532{,}224n^9 - 584{,}064\sqrt{3}n^9 \\ & - 6{,}029{,}856\sqrt{3}n^8 + 6{,}413{,}472n^8 - 17{,}305{,}116\sqrt{3}n^7 + 3{,}738{,}488n^7 \\ & - 849{,}238n^6 - 23{,}508{,}972\sqrt{3}n^6 - 7{,}884{,}539n^5 - 19{,}078{,}677\sqrt{3}n^5 \\ & - 15{,}177{,}045n^4 - 13{,}428{,}969\sqrt{3}n^4 - 21{,}962{,}902n^3 - 13{,}234{,}830\sqrt{3}n^3 \\ & - 11{,}506{,}974\sqrt{3}n^2 - 20{,}028{,}320n^2 - 5{,}355{,}441\sqrt{3}n - 9{,}314{,}279n \\ & - 1{,}663{,}701 - 957{,}021\sqrt{3}. \end{aligned}$$

By (3.11) and (3.12),

$$-\sqrt{h^2(n+1) + 4f(n+1)g(n+1)} < 2f(n+1)l(n) - h(n+1)$$

and thus

$$\frac{h(n+1) - \sqrt{h^2(n+1) + 4f(n+1)g(n+1)}}{2f(n+1)} < l(n). \tag{3.13}$$

Furthermore, it is easy to check that, for $n \geq 0$,

$$\begin{aligned} &2f(n+1)u(n) - h(n+1) \\ &\quad = \frac{\sqrt{3}(2n+1)r(n)}{96n^3(144n^2 - 216n + 298 - 39\sqrt{3})(2n+3)^2(n+1)^5} > 0 \end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
&\bigl(h^2(n+1)+4f(n+1)g(n+1)\bigr)-\bigl(2f(n+1)u(n)-h(n+1)\bigr)^2\\
&\quad=\frac{(32{,}424+13{,}481\sqrt{3})(144n^2+72n+226-39\sqrt{3})(2n+1)^2 s(n)}{1{,}554{,}750{,}544{,}896n^6(2n+3)^4(n+1)^5(144n^2-216n+298-39\sqrt{3})^2}>0, \qquad (3.15)
\end{aligned}$$

where

$$\begin{aligned}
r(n)={}&4{,}313{,}088n^{11}+15{,}095{,}808n^{10}+19{,}064{,}448n^9-1{,}168{,}128\sqrt{3}n^9\\
&-10{,}318{,}752\sqrt{3}n^8+12{,}826{,}944n^8-24{,}164{,}472\sqrt{3}n^7+7{,}476{,}976n^7\\
&-3{,}113{,}006n^6-17{,}735{,}964\sqrt{3}n^6-23{,}548{,}993n^5+13{,}865{,}916\sqrt{3}n^5\\
&-48{,}035{,}715n^4+37{,}763{,}112\sqrt{3}n^4-65{,}143{,}754n^3+29{,}313{,}600\sqrt{3}n^3\\
&+8{,}226{,}612\sqrt{3}n^2-54{,}201{,}940n^2-712{,}452\sqrt{3}n-23{,}579{,}413n\\
&-4{,}034{,}667-547{,}872\sqrt{3},\\
s(n)={}&10{,}074{,}783{,}530{,}926{,}080n^{12}-7{,}734{,}966{,}108{,}060{,}672\sqrt{3}n^{11}\\
&+26{,}936{,}680{,}137{,}670{,}656n^{11}+26{,}619{,}191{,}006{,}206{,}464n^{10}\\
&-23{,}315{,}086{,}010{,}211{,}840\sqrt{3}n^{10}-33{,}744{,}534{,}523{,}380{,}928\sqrt{3}n^9\\
&+41{,}557{,}278{,}922{,}739{,}904n^9-43{,}703{,}007{,}806{,}729{,}856\sqrt{3}n^8\\
&+98{,}524{,}492{,}003{,}096{,}704n^8+111{,}734{,}337{,}079{,}096{,}644n^7\\
&-47{,}522{,}128{,}660{,}057{,}480\sqrt{3}n^7-24{,}419{,}890{,}979{,}639{,}584\sqrt{3}n^6\\
&+24{,}142{,}153{,}127{,}323{,}080n^6-69{,}518{,}699{,}851{,}789{,}311n^5\\
&+24{,}355{,}619{,}096{,}164{,}226\sqrt{3}n^5-90{,}260{,}471{,}288{,}625{,}639n^4\\
&+57{,}503{,}789{,}690{,}970{,}194\sqrt{3}n^4-66{,}508{,}795{,}463{,}791{,}122n^3\\
&+46{,}318{,}086{,}975{,}242{,}316\sqrt{3}n^3+17{,}867{,}770{,}385{,}080{,}772\sqrt{3}n^2\\
&-35{,}327{,}233{,}535{,}891{,}622n^2+2{,}975{,}211{,}060{,}929{,}562\sqrt{3}n\\
&-11{,}558{,}904{,}059{,}737{,}827n+111{,}646{,}193{,}915{,}178\sqrt{3}\\
&-1{,}615{,}722{,}383{,}317{,}419.
\end{aligned}$$

Combining (3.11), (3.14), and (3.15) yields

$$u(n)<\frac{h(n+1)+\sqrt{h^2(n+1)+4f(n+1)g(n+1)}}{2f(n+1)}. \qquad (3.16)$$

It follows from (2.1), (2.7), (3.13), and (3.16) that, for $n\geq 5$,

$$\begin{aligned}
&\frac{h(n+1)-\sqrt{h^2(n+1)+4f(n+1)g(n+1)}}{2f(n+1)}\\
&\quad<l(n)<\frac{P_n}{P_{n-1}}<u(n)<\frac{h(n+1)+\sqrt{h^2(n+1)+4f(n+1)g(n+1)}}{2f(n+1)},
\end{aligned}$$

which yields

$$f(n+1)\left(\frac{U_n}{U_{n-1}}\right)^2 - h(n+1)\frac{U_n}{U_{n-1}} - g(n+1) < 0. \tag{3.17}$$

In view of (1.1),

$$f(n+1)U_n^2 - U_{n-1}U_{n+1} = P_n^2\left(f(n+1)\left(\frac{U_n}{U_{n-1}}\right)^2 - h(n+1)\frac{U_n}{U_{n-1}} - g(n+1)\right). \tag{3.18}$$

Lemma 3.2 follows from (3.17) and (3.18). This completes the proof. □

Now, we turn to the proof of Theorem 1.1.

Proof of Theorem 1.1 Replacing n by $n-1$ in (3.10), we deduce that, for $n \geq 6$,

$$\frac{U_n U_{n-2}}{U_{n-1}^2} > f(n). \tag{3.19}$$

In view of (3.1) and (3.19), we deduce that, for $n \geq 6$,

$$\frac{U_n^2}{U_{n-1}^2} > \frac{U_{n+1}}{U_n}\frac{U_{n-1}}{U_{n-2}}. \tag{3.20}$$

It is easy to verify that (3.20) also holds for $2 \leq n \leq 5$. This completes the proof of Theorem 1.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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