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A note on Cauchy-Lipschitz-Picard theorem

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Abstract
In this note, we try to generalize the classical Cauchy-Lipschitz-Picard theorem on the global existence and uniqueness for the Cauchy initial value problem of the ordinary differential equation with global Lipschitz condition, and we try to weaken the global Lipschitz condition. We can also get the global existence and uniqueness.

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Keywords: initial value problems; generalized Lipschitz condition; global existence and uniqueness

1 Introduction

In his famous book [1], Brezis gave a very sketchy and interesting proof on the classical Cauchy-Lipschitz-Picard theorem.

Theorem 1.1 *Let E be a Banach space and let $F: E \to E$ be a Lipschitz map, i.e., there is a constant L such that*

$$\left\|F(u) - F(v)\right\| \leq L\|u - v\|, \quad \forall u, v \in E. \tag{1.1}$$

Then, for any given $u_0 \in E$, there exists a unique solution $u \in C^1([0, +\infty), E)$ of the problem:

$$\begin{cases} \frac{du(t)}{dt} = F(u(t)), & \forall t \in (0, +\infty), \\ u(0) = u_0. \end{cases} \tag{1.2}$$

It is well known that if we only assume the local Lipschitz condition, we can only get the local existence and uniqueness for Cauchy initial value problems.

In this paper, we try to weaken the global Lipschitz condition, but we also want to get the global existence and uniqueness; we have the following theorem.

Theorem 1.2 *Let E be a Banach space (with norm $\|\cdot\|$) and let $F: [0, +\infty) \times E \to E$ be a map satisfying*

$$\left\|F(t, u) - F(t, v)\right\| \leq L(t) \cdot p\big(\|u - v\|\big), \quad \forall t \in [0, +\infty), u, v \in E,$$

where $L : [0, +\infty) \to [0, +\infty)$, $p : (0, +\infty) \to [0, +\infty)$ *are continuous and there are* $0 \leq a < +\infty$ *such that*

$$L = L(t) \leq a,$$

$$p = p(s) \leq s,$$

and $p(s)$ *is an increasing function. Then, for any given* $u_0 \in E$, *there exists a unique solution* $u \in C^1([0, +\infty); E)$ *for the problem*

$$\begin{cases} \frac{du(t)}{dt} = F(t, u(t)), & \forall t \in (0, +\infty), \\ u(0) = u_0. \end{cases} \tag{1.3}$$

Corollary 1.3 *In Theorem* 1.2, *if we take* Ł$(s) = a > 0$, $p(s) = s$ *or* $p(s) = \ln s$, *then the conditions and the results of Theorem* 1.2 *hold.*

2 The proof of Theorem 1.2

Lemma 2.1 ([1], Banach contraction mapping principle) *Let* X *be a nonempty complete metric space and let* $T : X \to X$ *be a strict contraction, i.e., there is* $0 < k < 1$ *such that* $d(T(x), T(y)) \leq kd(x, y)$, $\forall x, y \in X$, *then* S *has a unique fixed point* $u = T(u)$.

Lemma 2.2 (Gronwall [2]) *Let* $x \in C^1[a, b]$. *If* $R[a, b]$ *denotes the set of Riemann integrable functional on* $[a, b]$; $\beta \in R[a, b]$, *and*

$$x'(t) \leq \beta(t) \cdot x(t), \quad \forall t \in [a, b],$$

then

$$x(t) \leq x(a) \cdot e^{\int_a^t \beta(s)\,ds}, \quad \forall t \in [a, b].$$

The following lemma can be regarded as a natural generalization of the Gronwall inequality.

Lemma 2.3 *Let* $x \in C^1[a, b]$, $\alpha, \beta \in R[a, b]$. *If*

$$x'(t) \leq \alpha(t) + \beta(t)x(t), \quad \forall t \in [a, b],$$

then

$$x(t) \leq \left[x(a) + \int_a^t e^{-\int_0^s \beta(\tau)\,d\tau} \alpha(s)\,ds\right] \cdot e^{\int_a^t \beta(s)\,ds}, \quad \forall t \in [a, b].$$

Proof Let $v = e^{\int_a^t \beta(s)\,ds}$, $w = \frac{x}{v}$, then $v'(t) = \beta(t)v(t)$,

$$w'(t) = \frac{x'v - xv'}{v^2} \leq \frac{(\alpha + \beta x)v - \beta xv}{v^2} = \frac{\alpha v}{v^2} = \frac{\alpha}{v}.$$

Then

$$w(t) \leq \int_a^t e^{-\int_0^s \beta(\tau)\,d\tau} \alpha(s)\,ds + w(a),$$

$$\frac{x(t)}{v(t)} \le x(a) + \int_a^t e^{-\int_0^s \beta(\tau)\,d\tau} \alpha(s)\,ds,$$
$$x(t) \le e^{\int_a^t \beta(s)\,ds} \cdot \left[x(a) + \int_a^t e^{-\int_0^s \beta(\tau)\,d\tau} \alpha(s)\,ds\right].$$

Now to prove Theorem 1.2, we use some similar arguments to Brezis [1].

Let $k > 0$, which is to be determined, and assume

$$X = \left\{u \in C\bigl([0,+\infty);E\bigr) \Bigm| \sup_{t\ge 0} e^{-kt} \cdot \bigl\|u(t)\bigr\| < +\infty\right\}.$$

Then it is easy to see that X is a Banach space for the norm

$$\|u\|_X = \sup_{t\ge 0} e^{-kt} \cdot \bigl\|u(t)\bigr\|.$$

For $\forall u \in X$, we define

$$\Phi(u)(t) = u_0 + \int_0^t F\bigl(s,u(s)\bigr)\,ds.$$

Then u is a solution of (1.3) if and only if $\Phi(u) = u$, that is, u is a fixed point of Φ.

(1) We now show that, for every $u \in X$, $\Phi(u)$ also belongs to X.

In fact,

$$\begin{aligned}\|\Phi u\|_X &= \sup_{t\ge 0} e^{-kt} \bigl\|(\Phi u)(t)\bigr\| \\ &\le \sup_{t\ge 0} e^{-kt} \cdot \|u_0\| + \sup_{t\ge 0} e^{-kt} \cdot \left\|\int_0^t F\bigl(s,u(s)\bigr)\,ds\right\|.\end{aligned}$$

We only need to prove

$$\sup_{t\ge 0} e^{-kt} \cdot \int_0^t \bigl\|F\bigl(s,u(s)\bigr)\bigr\|\,ds < +\infty.$$

Notice that

$$\bigl\|F\bigl(s,u(s)\bigr) - F(0,u_0)\bigr\| \le L(s) \cdot p\bigl(\bigl\|u(s)-u_0\bigr\|\bigr),$$
$$\bigl\|F\bigl(s,u(s)\bigr)\bigr\| \le L(s) \cdot p\bigl(\bigl\|u(s)-u_0\bigr\|\bigr) + \bigl\|F(0,u_0)\bigr\|.$$

Since

$$\sup_{t\ge 0} e^{-kt} \cdot \int_0^t \bigl\|F(0,u_0)\bigr\|\,ds = \bigl\|F(0,u_0)\bigr\| \sup_{t\ge 0} e^{-kt} \cdot t < +\infty.$$

Hence we only need to prove

$$\sup_{t\ge 0} e^{-kt} \cdot \int_0^t L(s)p\bigl(\bigl\|u(s)-u_0\bigr\|\bigr)\,ds < +\infty.$$

Let

$$\varphi(t) = e^{-kt}\int_0^t L(s)p\big(\|u(s) - u_0\|\big)\,ds, \qquad \varphi(0) = 0,$$

then

$$\begin{aligned}\varphi(t) &\le \int_0^t \big[L(s)e^{ks}\big]e^{-ks}\big(\|u(s)\| + \|u_0\|\big)\,ds e^{-kt}\\ &\le \Big[\sup_{t\ge 0} e^{-ks}\|u(s)\| + \sup_{t\ge 0} e^{-ks}\|u_0\|\Big]\sup_{t\ge 0}\Big[\int_0^t L(s)e^{ks}\,ds e^{-kt}\Big]\\ &\le C_1C_2 = C_3 < +\infty.\end{aligned}$$

Hence we have proved Φ is a self-mapping from X to X.

(2) We prove the contraction property of Φ. We have

$$\|\Phi u - \Phi v\| \le \int_0^t \big\|F\big(s,u(s)\big) - F\big(s,v(s)\big)\big\|\,ds \le \int_0^t L(s)p\big(\|u(s) - v(s)\|\big)\,ds,$$

$$\begin{aligned}\|\Phi u - \Phi v\|_X &\le \sup_{t\ge 0} e^{-kt}\cdot\int_0^t L(s)p\big(\|u(s) - v(s)\|\big)\,ds\\ &\le \sup_{t\ge 0}\int_0^t L(s)e^{ks}\big[e^{-ks}\|u(s) - v(s)\|\big]\,ds e^{-kt}\\ &\le \sup_{t\ge 0}\big[e^{-ks}\|u(s) - v(s)\|\big]\sup_{t\ge 0} e^{-kt}\int_0^t L(s)e^{ks}\,ds\\ &\le \sup_{t\ge 0} e^{-kt}\int_0^t L(s)e^{ks}\,ds\|u - v\|_X.\end{aligned}$$

Hence we have

$$\|\Phi u - \Phi v\|_X \le \frac{L}{k}\|u - v\|_X, \quad \forall u, v \in X.$$

We can choose $k > L$, then we use Banach contraction mapping principle to find that (1.3) has at least one solution on $[0, +\infty)$.

Furthermore, by Gronwall's inequality, we can get the uniqueness.

In fact, let u_1, u_2 be two solutions of (1.3), then

$$\begin{aligned}\big\|u_1(t) - u_2(t)\big\| &\le \int_0^t \big\|F\big(s,u_1(s)\big) - F\big(s,u_2(s)\big)\big\|\\ &\le \int_0^t L(s)p\big(\|u_1(s) - u_2(s)\|\big)\,ds\\ &\le \int_0^t L(s)\big(\|u_1(s) - u_2(s)\|\big)\,ds.\end{aligned}$$

By Gronwall's inequality, we have

$$\big\|u_1(t) - u_2(t)\big\| \le 0.$$

Hence for any $t \in [0, +\infty)$, we have $u_1(t) = u_2(t)$. □

3 Examples

Example 3.1

$$\begin{cases} \frac{du(t)}{dt} = F(t,u(t)) = \frac{1}{1+t+|u|}, \quad \forall t \in (0,+\infty), \\ u(0) = u_0. \end{cases} \tag{3.1}$$

Then

$$\bigl|F\bigl(t,u(t)\bigr) - F\bigl(t,v(t)\bigr)\bigr| = \frac{\||v| - |u|\|}{(1+t+|u|)(1+t+|v|)} \le \||v| - |u|\|.$$

By the triangle inequality, we have

$$\bigl|F\bigl(t,u(t)\bigr) - F\bigl(t,v(t)\bigr)\bigr| \le |u - v|.$$

So $F(t,u(t)) = \frac{1}{1+t+|u|}$ is Lipschitz with $a = 1$ and $p(s) = s$.

Example 3.2 Let $p_1(t) : [0,+\infty) \to [0,+\infty)$ continuous and there is $0 \le a < +\infty$ such that

$$p_1(t) \le a.$$

Let $p_2(t) : [0,+\infty) \to R$ continuous.

We consider

$$\begin{cases} \frac{du(t)}{dt} = F(t,u(t)) = p_1(t)|u - p_2(t)|, \quad \forall t \in (0,+\infty), \\ u(0) = u_0. \end{cases} \tag{3.2}$$

Then

$$\bigl|F\bigl(t,u(t)\bigr) - F\bigl(t,v(t)\bigr)\bigr| = p_1(t)\bigl(\bigl|\bigl|u - p_2(t)\bigr| - \bigl|v - p_2(t)\bigr|\bigr|\bigr).$$

By the triangle inequality, we have

$$\bigl|F\bigl(t,u(t)\bigr) - F\bigl(t,v(t)\bigr)\bigr| \le p_1(t)|u - v|.$$

So $F(t,u(t)) = p_1(t)|u - p_2(t)|$ is Lipschitz with $L(t) = p_1(t)$ and $p(s) = s$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The research and writing of this manuscript was a collaborative effort from all the authors. All authors read and approved the final manuscript.

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References

1. Brezis, H: Functional Analysis, Sobolev Spaces and PDE, pp. 184-185. Springer, New York (2011)
2. Thomas, HG: Note on the derivatives with respect to a parameter of the solutions of a system of differential equations. Ann. Math. **20**, 292-296 (1919)