Zhou and He *Journal of Inequalities and Applications* (2016) 2016:257
DOI 10.1186/s13660-016-1198-6

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

A geometrical interpretation of the inverse matrix

Yanping Zhou* and Binwu He

*Correspondence:
zhouyp@i.shu.edu.cn
Department of Mathematics,
Shanghai University, Shanghai,
200444, China

Abstract
Utilizing a new method to structure parallellotopes, a geometrical interpretation of the inverse matrix is given, which includes the generalized inverse of full column rank or a full row rank matrices. Further, some relational volume formulas of parallellotopes are established.

MSC: 15A15; 52A20

Keywords: parallellotope; inverse matrix; generalized inverse

1 Introduction and notations

Let $\mathbb{R}^n$ denote an n-dimensional real Euclidean vector space, for a nonzero $n \times 1$ vector $x \in \mathbb{R}^n$, the generalized inverse of x, denoted by x^+, has the geometrical interpretation that x^T is divided by $\|x\|^2$, that is, $x^+ = x^T/\|x\|^2$, where x^T is the transpose of x (see [1]). A natural question is whether a similar geometrical interpretation holds for the inverse of a matrix.

In this paper, using a new method to structure a m-dimensional parallellotope, the geometrical interpretation of the inverse matrix and the generalized inverse of a matrix with full column rank or full row rank are given.

Let $[z_1, z_2, \dots, z_m]$ be the m-dimensional parallellotope with m linearly independent vectors $z_1, z_2, \dots, z_m$ as its edge vectors, *i.e.*,

$$[z_1, z_2, \dots, z_m] = \{z \in \mathbb{R}^n \mid t_1 z_1 + \cdots + t_m z_m, t_i \in [0,1], i = 1, 2, \dots, m\};$$

$[z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_m]$ denotes the facets of the m-parallellotope $[z_1, z_2, \dots, z_m]$ for an $(m-1)$-hyperplane,

$$\mathcal{H}_i = \operatorname{span}\{z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_m\}.$$

z_i is the altitude vector on facet $[z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_m]$ (see [2, 3]) with the orthogonal component of z_i with respect to $\mathcal{H}_i$. If $[z_1, z_2, \dots, z_m]^*$ denotes the m-parallellotope constructed by m linearly independent vectors $z_1, z_2, \dots, z_m$ as its altitude vectors, then we will show that there exist $z_1^*, z_2^*, \dots, z_m^*$, exclusive such that

$$[z_1, z_2, \dots, z_m]^* = [z_1^*, z_2^*, \dots, z_m^*].$$

2 Main results

Our main results are the following theorems.

Theorem 2.1 *If M is a matrix with full row (column) rank and $z_1, z_2, \ldots, z_m$ is its row (column) vectors, then the right (left) inverse of the matrix M is the matrix whose column (row) vectors are*

$$\frac{z_1^*}{\|z_1\|^2}, \frac{z_2^*}{\|z_2\|^2}, \ldots, \frac{z_m^*}{\|z_m\|^2},$$

where $z_1^*, z_2^*, \ldots, z_m^*$ are m edge vectors of the m-parallellotope $[z_1, z_2, \ldots, z_m]^*$.

Corollary 2.2 *If M is nonsingular $n \times n$ matrix and $z_1, z_2, \ldots, z_n$ is its row (column) vectors, then the inverse of the matrix M is the matrix whose column (row) vectors are*

$$\frac{z_1^*}{\|z_1\|^2}, \frac{z_1^*}{\|z_1\|^2}, \ldots, \frac{z_n^*}{\|z_n\|^2},$$

where $z_1^*, z_2^*, \ldots, z_n^*$ are n edge vectors of the n-parallellotope $[z_1, z_2, \ldots, z_n]^*$.

We may say roughly if the $[z_1, z_2, \ldots, z_m]$ ($z_1, z_2, \ldots, z_m$ as edge vectors) is the geometrical interpretation of the matrix M, then $[z_1, z_2, \ldots, z_m]^*$ ($z_1, z_2, \ldots, z_m$ as altitude vectors) is one of the M^{-1}.

We list some basic facts to state the following theorems.

We write $L(i)$, for the linear subspace spanned by $z_1, z_2, \ldots, z_i, z_i \in \mathbb{R}^n$ $(1 \leq i \leq n)$. Let $\langle z \hat{,} L\rangle$ be the angle between vector z and linear subspace L, where if $z \notin L$, then $\langle z \hat{,} L\rangle$ is the angle between z and the orthogonal projection of z on L, denoted by $z|_L$, *i.e.*, $z|_L = ((L^\perp + x) \cap L)$. If $z \in L$, then $\langle z \hat{,} L\rangle = 0$.

Theorem 2.3 *Suppose $y_1, y_2, \ldots, y_n$ are n row vectors of the matrix M, and $z_1, z_2, \ldots, z_n$ are column vectors of the matrix M^{-1},*

(1) *if $\|y_i\| \to 0$, then $\|z_i\| \to +\infty$;*
(2) *if $\langle y_i \hat{,} L(i-1)\rangle \to 0$, then there is k $(1 \leq k \leq n)$ such that $\|z_k\| \to +\infty$.*

Theorem 2.3 will be required in the study of matrix disturbances (see [4–6]).

Utilizing the geometrical interpretation of the inverse matrix, we have the following relational volume formulas of parallellotopes for the $n \times n$ real matrices M, N.

Theorem 2.4 *Let $[z_1, z_2, \ldots, z_n]^{**}$ be the parallellotope structured by the edge vectors of $[z_1, z_2, \ldots, z_n]^*$ as altitude vectors. Then*

$$\operatorname{vol}\bigl([z_1, z_2, \ldots, z_n]^*\bigr) \cdot \operatorname{vol}\bigl([z_1, z_2, \ldots, z_n]\bigr) = \left(\prod_{i=1}^{n} \|z_i\|\right)^2, \tag{2.1}$$

$$\operatorname{vol}\bigl([z_1, z_2, \ldots, z_n]^{**}\bigr) / \operatorname{vol}\bigl([z_1, x_2, \ldots, z_n]\bigr) = \left(\prod_{i=1}^{n} \|z_i^*\| / \|z_i\|\right)^2, \tag{2.2}$$

where $\operatorname{vol}([z_1, \ldots, z_n])$ denotes the volume of the parallellotope $[z_1, \ldots, z_n]$.

The proofs of the theorems will be given in Section 3.

3 Proofs of the theorems

Given m linearly independent vectors $z_1, z_2, \dots, z_m$ in $\mathbb{R}^n$, if we structure an m-parallellotope $[z_1, z_2, \dots, z_m]$ by them as edge vectors, then $[z_1, z_2, \dots, z_m]$ has m linearly independent altitude vectors. Conversely, for any given m linearly independent vectors $z_1, z_2, \dots, z_m$, can we structure an m-parallellotope by them as m altitude vectors? The following lemma gives an affirmative answer.

Lemma 3.1 *If* $\{z_1, z_2, \dots, z_m\}$ $(m \geq 2)$ *is a given set of linearly independent vectors in* $\mathbb{R}^n$, *then there is an m-parallellotope* $[z_1, z_2, \dots, z_m]^*$ *whose m altitude vectors are* $z_1, z_2, \dots, z_m$.

Proof If $z_1, z_2, \dots, z_m$ are linearly independent, then we have m linear functionals $g_1, g_2, \dots, g_m$ such that

$$g_j(z_i) = \delta_{ij} \|z_i\|^2, \quad i, j = 1, 2, \dots, m,$$

where δ_{ij} is the Kronecker delta symbol.

From Riesz's representation theorem for the linear functional, we get $z_1^*, z_2^*, \dots, z_m^*$ such that

$$\langle z_i, z_j^* \rangle = \delta_{ij} \|z_i\|^2, \quad i, j = 1, 2, \dots, m, \tag{3.1}$$

where $\langle , \rangle$ is the ordinary inner product in $\mathbb{R}^n$.

Further, let

$$\sum_{j=1}^{m} \alpha_j z_j^* = 0, \quad \alpha_j \in \mathbb{R},$$

by

$$0 = \left\langle z_i, \sum_{j=1}^{m} \alpha_j z_j^* \right\rangle = \alpha_i \|z_i\|^2,$$

we have $\alpha_i = 0, i = 1, 2, \dots, m$. This shows that $z_1^*, z_2^*, \dots, z_m^*$ are linearly independent.

Now, we prove that $z_1, z_2, \dots, z_m$ are altitude vectors of the m-parallellotope $[z_1^*, z_2^*, \dots, z_m^*]$ (the edge vectors of $[z_1^*, z_2^*, \dots, z_m^*]$ are $z_1^*, z_2^*, \dots, z_m^*$).

Suppose that $[z_1^*, z_2^*, \dots, z_{i-1}^*, z_{i+1}^*, \dots, z_m^*]$ are the facets of $[z_1^*, z_2^*, \dots, z_m^*]$. From $z_i \perp z_j^*$ $(j \neq i)$, we have

$$z_i \perp \left[z_1^*, z_2^*, \dots, z_{i-1}^*, z_{i+1}^*, \dots, z_m^*\right]. \tag{3.2}$$

Thus, $z_1, z_2, \dots, z_m$ are altitude vectors of $[z_1^*, z_2^*, \dots, z_m^*]$, *i.e.*,

$$[z_1, z_2, \dots, z_m]^* = \left[z_1^*, z_2^*, \dots, z_m^*\right].$$

This yields the desired m-parallellotope $[z_1, z_2, \dots, z_m]^*$. $\square$

Proof of Theorem 2.1 For a given $m \times n$ matrix full row rank $M = (c_{ij})_{m\times n}$, let

$$z_i = (c_{i1}, c_{i2}, \dots, c_{in}), \quad i = 1, 2, \dots, m.$$

By Lemma 3.1, we have an unique vector set $\{z_1^*, z_2^*, \dots, z_m^*\}$ such that

$$\langle z_i, z_j^* \rangle = \delta_{ij} \|z_i\|^2, \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n,$$

i.e.,

$$\left\langle z_i, \frac{z_j^*}{\|z_i\|^2} \right\rangle = \delta_{ij}, \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n, \tag{3.3}$$

and $z_1^*, z_2^*, \dots, z_m^*$ are m edge vectors of the parallellotope $[z_1, z_2, \dots, z_m]^*$.

Suppose

$$d_i = \frac{z_i^*}{\|z_i\|^2}, \quad i = 1, 2, \dots, m,$$

and

$$N = (d_1, d_2, \dots, d_m).$$

It follows from (3.3) that

$$MN = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{pmatrix} (d_1, d_2, \dots, d_m) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

Thus, the matrix N is the inverse of the matrix M, and the column vectors $d_1, d_2, \dots, d_m$ of the matrix N are the edge vectors of $[z_1, z_2, \dots, z_m]^*$ divided by $\|z_1\|^2, \|z_2\|^2, \dots, \|z_m\|^2$, respectively.

Together with Theorem 2.1 and taking M for an $n \times n$ matrix with full rank, we have Corollary 2.2.

Here, we will complete the proof of Theorem 2.3. The following lemma will be required. □

Lemma 3.2 *For $L(i)$ the linear subspace spanned by $z_1, z_2, \dots, z_i, i = 1, 2, \dots, m$ $(\leq n)$, if* $\mathrm{vol}([z_1, z_2, \dots, z_m])$ *is the volume of the parallellotope* $[z_1, z_2, \dots, z_m]$ (*see* [7]), *we have*

$$\mathrm{vol}\big([z_1, z_2, \dots, z_m]\big) = \prod_{i=1}^{m} \|z_i\| \cdot \prod_{i=2}^{m} \sin\big\langle z_i \hat{,} L(i-1)\big\rangle. \tag{3.4}$$

Proof Assume that h_i, p_i are the orthogonal component and orthogonal projection of z_i with respect to $L(i-1)$, respectively ($i = 2, \dots, m, h_1 = z_1, p_1 = 0$). Since $\|z_i\| \cos\langle z_i \hat{,} p_i\rangle = \|p_i\|$, we have

$$\cos\big\langle z_i \hat{,} L(i-1)\big\rangle = \frac{\langle z_i, p_i\rangle}{\|z_i\| \|p_i\|} = \frac{\langle p_i, p_i\rangle}{\|z_i\| \|p_i\|} = \frac{p_i}{\|z_i\|}. \tag{3.5}$$

By $\|z_i\|^2 = \|p_i\|^2 + \|h_i\|^2$, it follows that

$$\|h_i\| = \sqrt{\|z_i\|^2 - \|p_i\|^2} = \|z_i\| \sin\langle z_i, \hat{} L(i-1)\rangle.$$

From the definition of the volume of the parallellotope, we get (see [7–9])

$$\operatorname{vol}([z_1, z_2, \dots, z_m]) = \prod_{i=1}^{m} \|h_i\| = \prod_{i=1}^{m} \|z_i\| \cdot \prod_{i=2}^{m} \sin\langle z_i, \hat{} L(i-1)\rangle. \tag{3.6}$$

The proof of Lemma 3.2 is completed. □

Proof of Theorem 2.3 From Theorem 2.1, it follows that

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} (z_1, z_2, \dots, z_n) = \begin{pmatrix} \langle y_1, z_1\rangle & & 0 \\ & \ddots & \\ 0 & & \langle y_1, z_1\rangle \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \tag{3.7}$$

i.e.,

$$\langle y_i, z_i\rangle = 1, \quad i = 1, 2, \dots, n.$$

It follows from the Cauchy inequality that

$$1 = \left|\langle y_i, z_i\rangle\right| \le \|y_i\| \|z_i\|.$$

Thus the assertion (1) holds.

Let $\{y_1, y_2, \dots, y_n\}$ and $\{z_1, z_2, \dots, z_n\}$ in Lemma 3.2. From (3.7), we get

$$\left(\prod_{i=1}^{n} \|y_i\| \cdot \prod_{i=1}^{n} \sin\langle y_i, \hat{} L(i-1)\rangle\right) \cdot \left(\prod_{j=1}^{n} \|z_j\| \cdot \prod_{j=1}^{n} \sin\langle z_j, \hat{} L(j-1)\rangle\right) = 1. \tag{3.8}$$

From

$$0 \le \left|\prod_{j=1}^{n} \sin\langle y_j, \hat{} L(j-1)\rangle\right| \le 1$$

and

$$\prod_{i=1}^{n} \|y_i\| \le G,$$

the assertion (2) is given. □

Proof of Theorem 2.4 Together with Theorem 2.1, we get

$$\begin{pmatrix} \frac{z_1^*}{\|z_1\|^2} \\ \frac{z_2^*}{\|z_2\|^2} \\ \vdots \\ \frac{z_n^*}{\|z_n\|^2} \end{pmatrix} (z_1, z_2, \ldots, z_n) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}. \tag{3.9}$$

Thus

$$\det\left(\begin{pmatrix} \frac{z_1^*}{\|z_1\|^2} \\ \frac{z_2^*}{\|z_2\|^2} \\ \vdots \\ \frac{z_n^*}{\|z_n\|^2} \end{pmatrix} (z_1, z_2, \ldots, z_n) \right) = 1,$$

$$\det \begin{pmatrix} z_1^* \\ z_2^* \\ \vdots \\ z_n^* \end{pmatrix} \cdot \det(z_1, z_2, \ldots, z_n) = \left(\prod_{i=1}^n \|z_i\| \right)^2.$$

From

$$[x_1, x_2, \ldots, x_n]^* = \left[z_1^*, z_2^*, \ldots, z_n^*\right],$$

and the definition of the volume of parallellotopes, the equality (2.1) holds.

Assume that $\{z_1^{**}, z_2^{**}, \ldots, z_n^{**}\}$ is a set of the edge vectors of $[z_1, z_2, \ldots, z_n]^{**}$. Together with Theorem 2.1, we get

$$\begin{pmatrix} z_1^* \\ z_2^* \\ \vdots \\ z_n^* \end{pmatrix} \left(\frac{z_1^{**}}{\|z_1^*\|^2}, \frac{z_2^{**}}{\|z_2^*\|^2}, \vdots, \frac{z_n^{**}}{\|z_n^*\|^2} \right) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}. \tag{3.10}$$

If follows from (3.10) that

$$\det \begin{pmatrix} z_1^* \\ z_2^* \\ \vdots \\ z_n^* \end{pmatrix} \cdot \det\left(z_1^{**}, z_2^{**}, \ldots, z_n^{**}\right) = \left(\prod_{i=1}^n \|z_i\| \right)^2.$$

Thus

$$\mathrm{vol}\left([z_1, z_2, \ldots, z_n]^*\right) \cdot \mathrm{vol}\left([z_1, z_2, \ldots, z_n]^{**}\right) = \left(\prod_{i=1}^n \left\| z_i^* \right\| \right)^2. \tag{3.11}$$

Taking together (2.1) and (3.11), the equality (2.2) holds. □

For $\{z_1, z_2, \ldots, z_n\}$, from Lemma 3.1, $[z_1, z_2, \ldots, z_n]^*$ is structured by them as altitude vectors. Denote $[z_1, z_2, \ldots, z_n]^*$ by $z_1^*, z_2^*, \ldots, z_n^*$.

Let

$$[z_1, z_2, \ldots, z_n]^{**} = \left[z_1^*, z_2^*, \ldots, z_n^*\right]^*.$$

Thus Theorem 2.4 denotes the relationship of volumes about $[z_1, z_2, \ldots, z_n]$, $[z_1, z_2, \ldots, z_n]^*$, and $[z_1, z_2, \ldots, z_n]^{**}$.

Remark 1 By (3.10), we get

$$\begin{pmatrix} \frac{z_1^*}{\|z_1\|^2} \\ \frac{z_2^*}{\|z_2\|^2} \\ \vdots \\ \frac{z_n^*}{\|z_n\|^2} \end{pmatrix} \left(\frac{\|z_1\|^2}{\|z_1^*\|^2} z_1^{**}, \frac{\|z_2\|^2}{\|z_2^*\|^2} z_2^{**}, \vdots, \frac{\|z_n\|^2}{\|z_n^*\|^2} z_n^{**} \right) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \tag{3.12}$$

From (3.9) and (3.12), we see that

$$z_i^{**} = \frac{\|z_i^*\|^2}{\|z_i\|^2} z_i, \quad i = 1, 2, \ldots, n. \tag{3.13}$$

By (3.13), we can see that $[z_1, z_2, \ldots, z_n]^{**}$ and $[z_1, z_2, \ldots, z_n]$ are two parallellotopes and their edge vectors are of the same direction.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
The authors would like to acknowledge the support from the National Natural Science Foundation of China (11371239).

Received: 26 April 2016 Accepted: 5 October 2016 Published online: 19 October 2016

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