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The sharp bounds on general sum-connectivity index of four operations on graphs

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Abstract

The general sum-connectivity index $\chi_\alpha(G)$, for a (molecular) graph G, is defined as the sum of the weights $(d_G(a_1)+d_G(a_2))^\alpha$ of all $a_1a_2 \in E(G)$, where $d_G(a_1)$ (or $d_G(a_2)$) denotes the degree of a vertex a_1 (or a_2) in the graph G; $E(G)$ denotes the set of edges of G, and α is an arbitrary real number. Eliasi and Taeri (Discrete Appl. Math. 157:794-803, 2009) introduced four new operations based on the graphs $S(G)$, $R(G)$, $Q(G)$, and $T(G)$, and they also computed the Wiener index of these graph operations in terms of $W(F(G))$ and $W(H)$, where F is one of the symbols S, R, Q, T. The aim of this paper is to obtain sharp bounds on the general sum-connectivity index of the four operations on graphs.

MSC: 05C12; 05C90

Keywords: general sum-connectivity index; operation on graphs; cartesian product; total graph

1 Introduction

Let $G=(V,E)$ be a simple connected graph having vertex set $V(G)=\{a_1,a_2,a_3,\dots,a_n\}$ and edge set $E(G)=\{e_1,e_2,e_3,\dots,e_m\}$. The order and size of graph G are denoted by n and m, respectively. The degree of a vertex $a\in V(G)$ is the number of vertices whose distance from a is exactly one and denoted by $d_G(a)$. The minimum and maximum degrees of graph G are denoted by δ_G and $\triangle_G$, respectively. We will use the notations of P_n, C_n, and K_n for path, cycle, and complete graph with order n, respectively.

A topological index is a mathematical measure which correlates to the chemical structures of any simple finite graph. They are invariant under the graph isomorphism. They play an important role in the study of QSAR/QSPR. There are numerous topological descriptors that have some applications in theoretical chemistry. Among these topological descriptors the degree-based topological indices are of great importance.

The first degree-based topological indices that were defined by Gutman and Trinajstić [2] in 1972, are the first and second Zagreb indices. These indices were originally defined as follows:

$$M_1(G)=\sum_{a_1\in V(G)}\big(d_G(a_1)\big)^2,\qquad M_2(G)=\sum_{a_1a_2\in E(G)}d_G(a_1)d_G(a_2).$$

Here $M_1(G)$ and $M_2(G)$ denote the first and second Zagreb indices, respectively. The Randić connectivity index, proposed by Randić in 1975 [3], is the most used molecular descriptor. It is defined as the sum over all the edges of the graph of the terms $(d_G(a_1)d_G(a_2))^{-\frac{1}{2}}$. It has been extended to the general Randić connectivity index (product-connectivity index) by Li and Gutman [4], which is defined as follows:

$$R_\alpha(G) = \sum_{a_1a_2\in E(G)} \bigl(d_G(a_1)d_G(a_2)\bigr)^\alpha,$$

where α is a real number. The sum-connectivity index was proposed by Zhou and Trinajstić [5] in 2009, which is defined as the sum over all the edges of the graph of the terms $(d_G(a_1) + d_G(a_2))^{-\frac{1}{2}}$. This concept was extended to the general sum-connectivity index in 2010 [6], which is defined as follows:

$$\chi_\alpha(G) = \sum_{a_1a_2\in E(G)} \bigl(d_G(a_1) + d_G(a_2)\bigr)^\alpha,$$

where α is a real number. Then $\chi_{-1/2}(G)$ is the classical sum-connectivity index. The sum-connectivity index and product-connectivity index correlate well with the π-electron energy of benzenoid hydrocarbons [7]. Another variant of the Randić index of G is the harmonic index, denoted by $H(G)$ and defined as follows:

$$H(G) = \sum_{a_1a_2\in E(G)} \frac{2}{d_G(a_1) + d_G(a_2)} = 2\chi_{-1}(G).$$

We have $H(G) \le R(G)$ by the inequality between arithmetic means and geometric means, with equality if and only if G is a regular graph. For more details of these topological indices we refer the reader to [8–10].

Let G and H be two vertex-disjoint graphs. The cartesian product of G and H, denoted by $G \square H$, is a graph with vertex set $V(G \square H) = V(G) \times V(H)$ and $(a_1, b_1)(a_2, b_2) \in E(G \square H)$ whenever $[a_1 = a_2$ and $b_1b_2 \in E(H)]$ or $[a_1a_2 \in E(G)$ and $b_1 = b_2]$. The order and size of $G \square H$ are n_1n_2 and $m_1n_2 + m_2n_1$, respectively.

For a connected graph G, define four related graphs as follows:

1. $S(G)$ is the graph obtained by inserting an additional vertex in each edge of G. Equivalently, each edge of G is replaced by a path of length 2. The graph $S(G)$ is called the subdivision graph of G.
2. $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge.
3. $Q(G)$ is obtained from G by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G.
4. $T(G)$ has as its vertices, the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G. The graph $T(G)$ is called the total graph of G.

The four operations on graph $S(C_4)$, $R(C_4)$, $Q(C_4)$, $T(C_4)$ are depicted in Figure 1.

Eliasi and Taeri [1] introduced four new operations that are based on $S(G)$, $R(G)$, $Q(G)$, $T(G)$, as follows:

Let F be one of the symbols S, R, Q, T. The F-sum, denoted by $G +_F H$ of graphs G and H, is a graph with the set of vertices $V(G +_F H) = (V(G) \cup E(G)) \times V(H)$ and $(a_1, b_1)(a_2, b_2) \in$

Figure 1 The graphs C_4, $S(C_4)$, $R(C_4)$, $Q(C_4)$, $T(C_4)$.

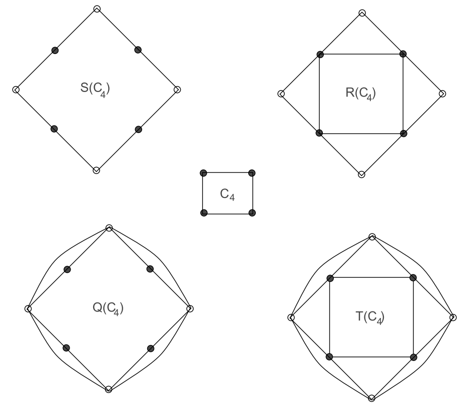
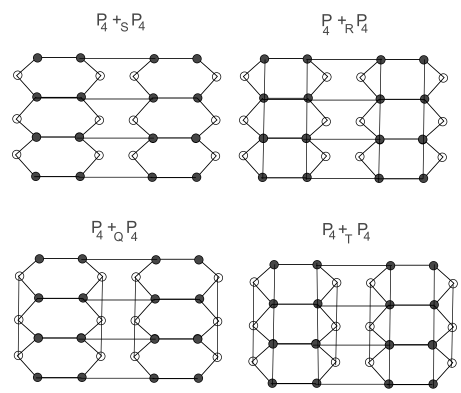


Figure 2 The graphs $P_4 +_S P_4$, $P_4 +_R P_4$, $P_4 +_Q P_4$ and $P_4 +_T P_4$.



$E(G +_F H)$, if and only if [$a_1 = a_2 \in V(G)$ and $b_1b_2 \in E(H)$] or [$b_1 = b_2 \in V(H)$ and $a_1a_2 \in E(F(G))$].

$G +_F H$ is consists of n_2 copies of the graph $F(G)$, and we label these copies by vertices of H. The vertices in each copy have two types, the vertices in $V(G)$ (black vertices) and the vertices in $E(G)$ (white vertices). Now we join only black vertices with the same name in $F(G)$ in which their corresponding labels are adjacent in H. The graphs $P_4 +_F P_4$ are shown in Figure 2.

Several extremal properties of the sum-connectivity index and general sum-connectivity index for trees, unicyclic graphs, 2-connected graphs and bicyclic graphs were given in [11–17]. Eliasi and Taeri [1] computed the expression for the Wiener index of four graph operations which are based on these graphs $S(G)$, $R(G)$, $Q(G)$, and $T(G)$, in terms of $W(F(G))$ and $W(H)$. Deng *et al.* [18] computed the first and second Zagreb indices for the graph operations $S(G)$, $R(G)$, $Q(G)$, and $T(G)$. In this paper, we will compute the sharp bounds on the general sum-connectivity index of F-sums of the graphs.

2 The general sum-connectivity index of *F*-sum of graphs

In this section, we derive the sharp bounds on the general sum-connectivity index of four operations on graphs. First we compute the case $F = S$.

Theorem 2.1 *If $\alpha < 0$, then the lower and upper bounds on the general sum-connectivity index are $\gamma_1 \leq \chi_\alpha(G +_S H) \leq \gamma_2$, where*

$$\gamma_1 = 2^\alpha n_1 m_2(\triangle_G + \triangle_H)^\alpha + 2^\alpha n_2 m_1(2\triangle_G + \triangle_H)^\alpha,$$
$$\gamma_2 = 2^\alpha n_1 m_2(\delta_G + \delta_H)^\alpha + 2^\alpha n_2 m_1(2\delta_G + \delta_H)^\alpha.$$

Equality holds if and only if G and H are regular graphs.

Proof By the definition of the general sum-connectivity index, we have

$$\begin{aligned}\chi_\alpha(G +_S H) &= \sum_{(a_1,b_1)(a_2,b_2)\in E(G+_S H)} \big[d_{G+_S H}(a_1,b_1) + d_{G+_S H}(a_2,b_2)\big]^\alpha \\ &= \sum_{a_1\in V(G)} \sum_{b_1 b_2 \in E(H)} \big[d_{G+_S H}(a_1,b_1) + d_{G+_S H}(a_1,b_2)\big]^\alpha \\ &\quad + \sum_{b_1\in V(H)} \sum_{a_1 a_2 \in E(S(G))} \big[d_{G+_S H}(a_1,b_1) + d_{G+_S H}(a_2,b_1)\big]^\alpha. \end{aligned} \tag{1}$$

Note that $d_G(a) \leq \triangle_G$ and $d_G(a) \geq \delta_G$, equality holds if and only if G is a regular graph, and similarly $d_H(b) \leq \triangle_H$ and $d_H(b) \leq \delta_G$, equality holds if and only if H is a regular graph. We have

$$\begin{aligned}&\sum_{a_1\in V(G)} \sum_{b_1 b_2 \in E(H)} \big[d_{G+_S H}(a_1,b_1) + d_{G+_S H}(a_1,b_2)\big]^\alpha \\ &\quad= \sum_{a_1\in V(G)} \sum_{b_1 b_2 \in E(H)} \big[\big(d_G(a_1) + d_H(b_1)\big) + \big(d_G(a_1) + d_H(b_2)\big)\big]^\alpha \\ &\quad= \sum_{a_1\in V(G)} \sum_{b_1 b_2 \in E(H)} \big[2d_G(a_1) + d_H(b_1) + d_H(b_2)\big]^\alpha \\ &\quad\geq 2^\alpha n_1 m_2(\triangle_G + \triangle_H)^\alpha. \end{aligned} \tag{2}$$

Since $|E(S(G))| = 2|E(G)|$ and $\triangle_{S(G)} = \triangle_G$, we have

$$\begin{aligned}&\sum_{b_1\in V(H)} \sum_{a_1 a_2 \in E(S(G))} \big[d_{G+_S H}(a_1,b_1) + d_{G+_S H}(a_2,b_1)\big]^\alpha \\ &\quad= \sum_{b_1\in V(H)} \sum_{a_1 a_2 \in E(S(G))} \big[d_{S(G)}(a_1) + d_H(b_1) + d_{S(G)}(a_2)\big]^\alpha \\ &\quad\geq n_2\big|E\big(S(G)\big)\big|(2\triangle_{S(G)} + \triangle_H) \\ &\quad= 2n_2 m_1(2\triangle_G + \triangle_H). \end{aligned} \tag{3}$$

Using equations (2) and (3) in equation (1), we get

$$\chi_\alpha(G +_S H) \geq 2^\alpha n_1 m_2(\triangle_G + \triangle_H)^\alpha + 2n_2 m_1(2\triangle_G + \triangle_H).$$

Similarly we can compute

$$\chi_\alpha(G +_S H) \leq 2^\alpha n_1 m_2(\delta_G + \delta_H)^\alpha + 2n_2 m_1(2\delta_G + \delta_H).$$

Equality holds if and only if G and H are regular graphs. This completes the proof. □

Example 1 The lower and upper bounds on the general sum-connectivity index of $P_n +_S P_m$ are

$$\gamma_1 = mn\big(8^\alpha + 2\times 6^\alpha\big) - 8^\alpha n - 2\times 6^\alpha m,$$
$$\gamma_2 = mn\big(4^\alpha + 2\times 3^\alpha\big) - 4^\alpha n - 2\times 3^\alpha m.$$

Example 2 The general sum-connectivity index of $C_n +_S C_m$ and $K_n +_S K_m$ is

$$\chi_\alpha(C_n +_S C_m) = mn\big(8^\alpha + 2\times 6^\alpha\big),$$
$$\chi_\alpha(K_n +_S K_m) = mn\big[2^{\alpha-1}(m-1)(m+n-2)^\alpha + (n-1)(2n+m-3)^\alpha\big].$$

Theorem 2.2 *If $\alpha < 0$, then the lower and upper bounds on the general sum-connectivity index are $\gamma_1 \le \chi_\alpha(G +_R H) \le \gamma_2$, where*

$$\gamma_1 = 2^\alpha(n_1m_2 + n_2m_1)(2\triangle_G + \triangle_H)^\alpha + 2n_2m_1(2\triangle_G + \triangle_H + 2)^\alpha,$$
$$\gamma_2 = 2^\alpha(n_1m_2 + n_2m_1)(2\delta_G + \delta_H)^\alpha + 2n_2m_1(2\delta_G + \delta_H + 2)^\alpha.$$

Equality holds if and only if G and H are regular graphs.

Proof By the definition of the general sum-connectivity index, we have

$$\begin{aligned}\chi_\alpha(G +_R H) &= \sum_{(a_1,b_1)(a_2,b_2)\in E(G+_RH)} \big[d_{G+_RH}(a_1,b_1) + d_{G+_RH}(a_2,b_2)\big]^\alpha \\ &= \sum_{a_1\in V(G)}\sum_{b_1b_2\in E(H)} \big[d_{G+_RH}(a_1,b_1) + d_{G+_RH}(a_1,b_2)\big]^\alpha \\ &\quad + \sum_{b_1\in V(H)}\sum_{a_1a_2\in E(R(G))} \big[d_{G+_RH}(a_1,b_1) + d_{G+_RH}(a_2,b_1)\big]^\alpha. \end{aligned} \tag{4}$$

Note that $d_G(a) \le \triangle_G$ and $d_G(a) \ge \delta_G$, equality holds if and only if G is a regular graph, and similarly $d_H(b) \le \triangle_H$ and $d_H(b) \le \delta_G$, equality holds if and only if H is a regular graph. We have

$$\begin{aligned}&\sum_{a_1\in V(G)}\sum_{b_1b_2\in E(H)} \big[d_{G+_RH}(a_1,b_1) + d_{G+_RH}(a_1,b_2)\big]^\alpha \\ &\quad = \sum_{a_1\in V(G)}\sum_{b_1b_2\in E(H)} \big[d_{R(G)}(a_1) + d_H(b_1) + d_{R(G)}(a_1) + d_H(b_2)\big]^\alpha \\ &\quad = \sum_{a_1\in V(G)}\sum_{b_1b_2\in E(H)} \big[2d_{R(G)}(a_1) + \big(d_H(b_1) + d_H(b_2)\big)\big]^\alpha \\ &\quad = \sum_{a_1\in V(G)}\sum_{b_1b_2\in E(H)} \big[4d_G(a_1) + \big(d_H(b_1) + d_H(b_2)\big)\big]^\alpha \\ &\quad \ge 2^\alpha n_1m_2(2\triangle_G + \triangle_H), \end{aligned} \tag{5}$$

$$\sum_{b_1\in V(H)}\sum_{a_1a_2\in E(R(G))} \big[d_{G+_RH}(a_1,b_1) + d_{G+_RH}(a_2,b_1)\big]^\alpha$$

$$= \sum_{b_1 \in V(H)} \sum_{\substack{a_1a_2 \in E(R(G)) \\ a_1,a_2 \in V(G)}} \left[d_{G+_RH}(a_1,b_1) + d_{G+_RH}(a_2,b_1)\right]^\alpha$$
$$+ \sum_{b_1 \in V(H)} \sum_{\substack{a_1a_2 \in E(R(G)) \\ a_1 \in V(G), a_2 \in V(R(G))-V(G)}} \left[d_{G+_RH}(a_1,b_1) + d_{G+_RH}(a_2,b_1)\right]^\alpha. \tag{6}$$

(i) $a_1a_2 \in E(R(G))$ and $a_1, a_2 \in V(G)$ if and only if $a_1a_2 \in E(G)$, (ii) $d_{R(G)}(a_1) = 2d_G(a_1)$, we have

$$\sum_{b_1 \in V(H)} \sum_{\substack{a_1a_2 \in E(R(G)) \\ a_1,a_2 \in V(G)}} \left[d_{G+_RH}(a_1,b_1) + d_{G+_RH}(a_2,b_1)\right]^\alpha$$
$$= \sum_{b_1 \in V(H)} \sum_{a_1a_2 \in E(G)} \left[d_{G+_RH}(a_1,b_1) + d_{G+_RH}(a_2,b_1)\right]^\alpha$$
$$= \sum_{b_1 \in V(H)} \sum_{a_1a_2 \in E(G)} \left[\left(d_{R(G)}(a_1) + d_H(b_1)\right) + \left(d_{R(G)}(a_2) + d_H(b_1)\right)\right]^\alpha$$
$$= \sum_{b_1 \in V(H)} \sum_{a_1a_2 \in E(G)} \left[2\left(d_G(a_1) + d_G(a_2)\right) + 2d_H(b_1)\right]^\alpha$$
$$\geq 2^\alpha n_2 m_1 (2 \triangle_G + \triangle_H)^\alpha. \tag{7}$$

Note that $|E(R(G))| = 2|E(G)|$, and if $a_1 \in V(G)$ then $d_{R(G)}(a_1) = 2d_G(a_1)$ and if $a_2 \in V(R(G)) - V(G)$ then $d_{R(G)}(a_2) = 2$, we have

$$\sum_{b_1 \in V(H)} \sum_{\substack{a_1a_2 \in E(R(G)) \\ a_1 \in V(G), a_2 \in V(R(G))-V(G)}} \left[d_{G+_RH}(a_1,b_1) + d_{G+_RH}(a_2,b_1)\right]^\alpha$$
$$= \sum_{b_1 \in V(H)} \sum_{\substack{a_1a_2 \in E(R(G)) \\ a_1 \in V(G), a_2 \in V(R(G))-V(G)}} \left[\left(d_{R(G)}(a_1) + d_H(b_1)\right) + d_{R(G)}(a_2)\right]^\alpha$$
$$= \sum_{b_1 \in V(H)} \sum_{\substack{a_1a_2 \in E(R(G)) \\ a_1 \in V(G), a_2 \in V(R(G))-V(G)}} \left[\left(d_{R(G)}(a_1) + d_{R(G)}(a_2)\right) + d_H(b_1)\right]^\alpha$$
$$= \sum_{b_1 \in V(H)} \sum_{\substack{a_1a_2 \in E(R(G)) \\ a_1 \in V(G), a_2 \in V(R(G))-V(G)}} \left[2\left(d_G(a_1) + 1\right) + d_H(b_1)\right]^\alpha$$
$$\geq n_2 \left|E\left(R(G)\right)\right| (2 \triangle_G + \triangle_H + 2)^\alpha$$
$$= 2n_2 m_1 (2 \triangle_G + \triangle_H + 2)^\alpha. \tag{8}$$

Using equations (5)-(8) in equation (4), we get the required result,

$$\chi_\alpha(G +_R H) \geq 2^\alpha (n_1 m_2 + n_2 m_1)(2 \triangle_G + \triangle_H)^\alpha + 2n_2 m_1 (2 \triangle_G + \triangle_H + 2)^\alpha.$$

Similarly, we can compute

$$\chi_\alpha(G +_R H) \leq 2^\alpha (n_1 m_2 + n_2 m_1)(2\delta_G + \delta_H)^\alpha + 2n_2 m_1 (2\delta_G + \delta_H + 2)^\alpha.$$

Equality holds if and only if G and H are regular graphs. This completes the proof. □

Example 3 The lower and upper bounds on the general sum-connectivity index of $P_n +_R P_m$ are

$$\gamma_1 = 2^{2\alpha+1}mn\big(3^\alpha + 2^\alpha\big) - 12^\alpha n - 2^{2\alpha}m\big(3^\alpha + 2^{\alpha+1}\big),$$
$$\gamma_2 = 2mn\big(6^\alpha + 5^\alpha\big) - 6^\alpha n - m\big(6^\alpha + 2\times 5^\alpha\big).$$

Example 4 The general sum-connectivity index of $C_n +_R C_m$ and $K_n +_R K_m$ is

$$\chi_\alpha(C_n +_R C_m) = 2^{2\alpha+1}mn\big(3^\alpha + 2^\alpha\big),$$
$$\chi_\alpha(K_n +_R K_m) = 2^{\alpha-1}mn(m+n-2)(m+2n-3)^\alpha + mn(n-1)(2n+m-1)^\alpha.$$

Theorem 2.3 *If $\alpha < 0$, then the lower and upper bounds on the general sum-connectivity index are $\gamma_1 \leq \chi_\alpha(G +_Q H) \leq \gamma_2$, where*

$$\gamma_1 = 2^\alpha n_1 m_2(\triangle_G + \triangle_H)^\alpha + 2n_2 m_1(3\triangle_G + \triangle_H)^\alpha + 4^\alpha n_2 \triangle_G^\alpha \left(\frac{1}{2}M_1(G) - m_1\right),$$
$$\gamma_2 = 2^\alpha n_1 m_2(\delta_G + \delta_H)^\alpha + 2n_2 m_1(3\delta_G + \delta_H)^\alpha + 4^\alpha n_2 \delta_G^\alpha \left(\frac{1}{2}M_1(G) - m_1\right).$$

Equality holds if and only if G and H are regular graphs.

Proof By the definition of the general sum-connectivity index, we have

$$\begin{aligned}\chi_\alpha(G +_Q H) &= \sum_{(a_1,b_1)(a_2,b_2)\in E(G+_Q H)} \big[d_{G+_Q H}(a_1,b_1) + d_{G+_Q H}(a_2,b_2)\big]^\alpha \\ &= \sum_{a_1\in V(G)}\sum_{b_1b_2\in E(H)} \big[d_{G+_Q H}(a_1,b_1) + d_{G+_Q H}(a_1,b_2)\big]^\alpha \\ &\quad + \sum_{b_1\in V(H)}\sum_{a_1a_2\in E(Q(G))} \big[d_{G+_Q H}(a_1,b_1) + d_{G+_Q H}(a_1,b_2)\big]^\alpha. \end{aligned} \tag{9}$$

Note that $d_G(a) \leq \triangle_G$ and $d_G(a) \geq \delta_G$, equality holds if and only if G is a regular graph, and similarly $d_H(b) \leq \triangle_H$ and $d_H(b) \leq \delta_G$, equality holds if and only if H is a regular graph. We have

$$\begin{aligned}&\sum_{a_1\in V(G)}\sum_{b_1b_2\in E(H)} \big[d_{G+_Q H}(a_1,b_1) + d_{G+_Q H}(a_1,b_2)\big]^\alpha \\ &\quad= \sum_{a_1\in V(G)}\sum_{b_1b_2\in E(H)} \big[\big(d_{Q(G)}(a_1) + d_H(b_1)\big) + \big(d_{Q(G)}(a_1) + d_H(b_2)\big)\big]^\alpha \\ &\quad= \sum_{a_1\in V(G)}\sum_{b_1b_2\in E(H)} \big[2d_{Q(G)}(a_1) + \big(d_H(b_1) + d_H(b_2)\big)\big]^\alpha \\ &\quad= \sum_{a_1\in V(G)}\sum_{b_1b_2\in E(H)} \big[2d_G(a_1) + \big(d_H(b_1) + d_H(b_2)\big)\big]^\alpha \\ &\quad\geq 2^\alpha n_1 m_2(\triangle_G + \triangle_H)^\alpha, \end{aligned} \tag{10}$$

$$\sum_{b_1\in V(H)}\sum_{a_1a_2\in E(Q(G))} \big[d_{G+_Q H}(a_1,b_1) + d_{G+_Q H}(a_2,b_2)\big]^\alpha$$

$$= \sum_{b_1\in V(H)} \sum_{\substack{a_1a_2\in E(Q(G))\\ a_1\in V(G),a_2\in V(Q(G))-V(G)}} \left[d_{G+_QH}(a_1,b_1)+d_{G+_QH}(a_2,b_1)\right]^\alpha$$

$$+ \sum_{b_1\in V(H)} \sum_{\substack{a_1a_2\in E(Q(G))\\ a_1,a_2\in V(Q(G))-V(G)}} \left[d_{G+_QH}(a_1,b_1)+d_{G+_QH}(a_2,b_1)\right]^\alpha, \tag{11}$$

$$\sum_{b_1\in V(H)} \sum_{\substack{a_1a_2\in E(Q(G))\\ a_1\in V(G),a_2\in V(Q(G))-V(G)}} \left[d_{G+_QH}(a_1,b_1)+d_{G+_QH}(a_2,b_1)\right]^\alpha$$

$$= \sum_{b_1\in V(H)} \sum_{\substack{a_1a_2\in E(Q(G))\\ a_1\in V(G),a_2\in V(Q(G))-V(G)}} \left[d_{Q(G)}(a_1)+d_H(b_1)+d_{Q(G)}(a_2)\right]^\alpha$$

$$= \sum_{b_1\in V(H)} \sum_{\substack{a_1a_2\in E(Q(G))\\ a_1\in V(G),a_2\in V(Q(G))-V(G)}} \left[d_G(a_1)+d_H(b_1)+d_{Q(G)}(a_2)\right]^\alpha. \tag{12}$$

Note that $d_{Q(G)}(a_2) = d_G(w_i) + d_G(w_j)$ for $a_2 \in V(Q(G)) - V(G)$, a_2 is the vertex inserted into the edge w_iw_j of G. Then we have

$$\sum_{b_1\in V(H)} \sum_{\substack{a_1a_2\in E(Q(G))\\ a_1\in V(G),a_2\in V(Q(G))-V(G)}} \left[d_G(a_1)+d_H(b_1)+\left(d_G(w_i)+d_G(w_j)\right)\right]^\alpha$$

$$\geq 2n_2m_1(3\bigtriangleup_G+\bigtriangleup_H)^\alpha,$$

$$\sum_{b_1\in V(H)} \sum_{\substack{a_1a_2\in E(Q(G))\\ a_1,a_2\in V(Q(G))-V(G)}} \left[d_{G+_QH}(a_1,b_1)+d_{G+_QH}(a_2,b_1)\right]^\alpha \tag{13}$$

$$= \sum_{b_1\in V(H)} \sum_{\substack{a_1a_2\in E(Q(G))\\ a_1,a_2\in V(Q(G))-V(G)}} \left[d_{Q(G)}(a_1)+d_{Q(G)}(a_2)\right]^\alpha.$$

Since a_1 is the vertex inserted into the edge w_iw_j of G and a_2 is the vertex inserted into the edge w_jw_k of G,

$$\sum_{b_1\in V(H)} \sum_{\substack{w_iw_j\in E(G)\\ w_jw_k\in E(G)}} \left[d_G(w_i)+d_G(w_j)+d_G(w_j)+d_G(w_k)\right]^\alpha$$

$$= \sum_{b_1\in V(H)} \sum_{\substack{w_iw_j\in E(G)\\ w_jw_k\in E(G)}} \left[d_G(w_i)+d_G(w_k)+2d_G(w_j)\right]^\alpha$$

$$\geq 4^\alpha \bigtriangleup_G^\alpha n_2\left(\frac{1}{2}M_1(G)+m_1\right). \tag{14}$$

Therefore, using equations (10)-(14) in equation (9), we get the required result,

$$\chi_\alpha(G+_Q H) \geq 2^\alpha n_1m_2(\bigtriangleup_G+\bigtriangleup_H)^\alpha + 2n_2m_1(3\bigtriangleup_G+\bigtriangleup_H)^\alpha + 4^\alpha n_2 \bigtriangleup_G^\alpha\left(\frac{1}{2}M_1(G)-m_1\right).$$

Similarly, we can compute

$$\chi_\alpha(G+_Q H) \leq 2^\alpha n_1m_2(\delta_G+\delta_H)^\alpha + 2n_2m_1(3\delta_G+\delta_H)^\alpha + 4^\alpha n_2\delta_G^\alpha\left(\frac{1}{2}M_1(G)-m_1\right).$$

Equality holds if and only if G and H are regular graphs. This completes the proof. □

Example 5 The lower and upper bounds on the general sum-connectivity index of $P_n +_Q P_m$ are

$$\gamma_1 = 2^{3\alpha}(4mn - n - 4m), \qquad \gamma_2 = 2^{2\alpha}(4mn - n - 4m).$$

Example 6 The general sum-connectivity index of $C_n +_Q C_m$ and $K_n +_Q K_m$ is

$$\begin{aligned} \chi_\alpha(C_n +_Q C_m) &= 2^{3\alpha+2}mn, \\ \chi_\alpha(K_n +_Q K_m) &= 2^{\alpha-1}mn(m-1)(m+n-2)^\alpha + mn(n-1)(3n+m-4)^\alpha \\ &\quad + 2^{2\alpha-1}mn(n-2)(n-1)^\alpha. \end{aligned}$$

Since $\deg_{G+_TH}(a,b) = \deg_{G+_RH}(a,b)$ for $a \in V(G)$ and $b \in V(H)$, $\deg_{G+_TH}(a,b) = \deg_{G+_QH}(a, b)$ for $a \in V(T(G)) - V(G)$ and $b \in V(H)$, we can get the following result by the proofs of Theorems 2.2 and 2.3.

Theorem 2.4 *If $\alpha < 0$, then the lower and upper bounds on the general sum-connectivity index are $\gamma_1 \leq \chi_\alpha(G +_T H) \leq \gamma_2$, where*

$$\begin{aligned} \gamma_1 &= 2^\alpha(n_1m_2 + n_2m_1)(2\triangle_G + \triangle_H)^\alpha + 2n_2m_1(4\triangle_G + \triangle_H)^\alpha \\ &\quad + 4^\alpha n_2 \triangle_G^\alpha \left(\frac{1}{2}M_1(G) - m_1\right), \\ \gamma_2 &= 2^\alpha(n_1m_2 + n_2m - 1)(2\delta_G + \delta_H)^\alpha + 2n_2m_1(4\delta_G + \delta_H)^\alpha \\ &\quad + 4^\alpha n_2\delta_G^\alpha\left(\frac{1}{2}M_1(G) - m_1\right). \end{aligned}$$

Equality holds if and only if G and H are regular graphs.

Example 7 The lower and upper bounds on the general sum-connectivity index of $P_n +_T P_m$ are

$$\begin{aligned} \gamma_1 &= 2^\alpha mn\left(2\times 6^\alpha + 4^\alpha + 2\times 5^\alpha\right) - 12^\alpha n - 2^\alpha m\left(6^\alpha + 2\times 4^\alpha + 2\times 5^\alpha\right), \\ \gamma_2 &= mn\left(2\times 6^\alpha + 4^\alpha + 2\times 5^\alpha\right) - 6^\alpha n - m\left(6^\alpha + 2\times 4^\alpha + 2\times 5^\alpha\right). \end{aligned}$$

Example 8 The general sum-connectivity index of $C_n +_T C_m$ and $K_n +_T K_m$ is

$$\begin{aligned} \chi_\alpha(C_n +_T C_m) &= 2^\alpha mn\left(2\times 6^\alpha + 4^\alpha + 2\times 5^\alpha\right), \\ \chi_\alpha(K_n +_T K_m) &= 2^{\alpha-1}mn(m+n-2)(m+2n-3)^\alpha + mn(n-1)(4n+m-5)^\alpha \\ &\quad + 2^{2\alpha-1}mn(n-2)(n-1)^{\alpha+1}. \end{aligned}$$

3 Conclusion

The sharp bounds on the general sum-connectivity index of the new four sums of the graphs were computed in this paper, for $\alpha < 0$. However, if $\alpha > 0$ then these bounds will become $\gamma_2 \leq \chi_\alpha(G +_F H) \leq \gamma_1$. These results can be extended for a tenser product and the normal product of the graphs with respect to the general sum-connectivity index for all values of α and this still remains an open and challenging problem for researchers.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The idea to obtain the bounds for the four operations of the graphs for general sum-connectivity index of the graphs was proposed by MI. After several discussions, SA obtained some sharp bounds for four operations. MI checked these bounds and suggested improvements. The first draft was prepared by SA which was verified and improved by MI. The final version was prepared by SA. Both authors read and approved the final manuscript.

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