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Anisotropic interpolation theorems of Musielak-Orlicz type

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Abstract

Anisotropy is a common attribute of Nature, which shows different characterizations in different directions of all or part of the physical or chemical properties of an object. The anisotropic property, in mathematics, can be expressed by a fairly general discrete group of dilations $\{A^k : k \in \mathbb{Z}\}$, where A is a real $n \times n$ matrix with all its eigenvalues λ satisfy $|\lambda| > 1$. Let $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ be an anisotropic Musielak-Orlicz function such that $\varphi(x,\cdot)$ is an Orlicz function and $\varphi(\cdot,t)$ is a Muckenhoupt $\mathbb{A}_\infty(A)$ weight. The aim of this article is to obtain two anisotropic interpolation theorems of Musielak-Orlicz type, which are weighted anisotropic extension of Marcinkiewicz interpolation theorems. The above results are new even for the isotropic weighted settings.

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Keywords: anisotropic expansive dilation; Muckenhoupt weight; Musielak-Orlicz function; weighted Hardy space; interpolation

1 Introduction

The aim of this article is to obtain two anisotropic interpolation theorems of Musielak-Orlicz type. Anisotropy is a common attribute of Nature, which shows different characterizations in different directions of all or part of the physical or chemical properties of an object. For example, the elastic modulus, hardness or fracture strength of a crystal is different in different directions, which shows the anisotropic property of the crystal. The anisotropic property, in mathematics, can be expressed by a general discrete group of dilations $\{A^k : k \in \mathbb{Z}\}$, where A is a real $n \times n$ matrix with all its eigenvalues λ satisfy $|\lambda| > 1$.

As a natural generalization of $L^p(\mathbb{R}^n)$, the Orlicz space was introduced by Birnbaum-Orlicz [1] and Orlicz [2], which is defined via an Orlicz function. Recall that a Musielak-Orlicz space is defined via a Musielak-Orlicz function (see, for example, [3]) and a Musielak-Orlicz function is a natural generalization of an Orlicz function. Observe that, different from Orlicz functions, Musielak-Orlicz functions may also vary in the spatial variables. Musielak-Orlicz spaces include many function spaces far beyond $L^p(\mathbb{R}^n)$, and the motivation to study function spaces of Musielak-Orlicz type comes from various applications in mathematics and physics (see, for example, [4–9] and the references therein).

On the other hand, there were several efforts of extending classical function spaces arising in harmonic analysis from Euclidean spaces to other domains and non-isotropic settings; see [10–18]. Calderón and Torchinsky initiated the study of Hardy spaces on $\mathbb{R}^n$ with anisotropic dilations [11–13]. The theory of Hardy spaces associated to expansive dilations was recently developed in [10, 19]. Another research direction of extending classical function spaces is the study of weighted function spaces associated with general Muckenhoupt weights; see [20–25]. García-Cuerva [25] and Strömberg-Torchinsky [26] established a theory of weighted Hardy spaces on $\mathbb{R}^n$.

Moreover, many problems in Fourier analysis concern the boundedness of operators on Lebesgue spaces, and interpolation provides a framework that often simplifies this study. For instance, in order to show that a linear operator maps L^p to itself for all $1 < p < \infty$, it is sufficient to show that it maps the (smaller) Lorentz space $L^{p,1}$ into the (larger) Lorentz space $L^{p,\infty}$ for the same range of p. Moreover, some further reductions can be made in terms of the Lorentz space $L^{p,1}$. This and other considerations indicate that interpolation is a powerful tool in the study of boundedness of operators. Therefore, many kinds of interpolation theorems were established. For example, Ding and Lan [27], Theorems 3.2 and 3.4, obtained two interpolation theorems associated with anisotropic (weak) Hardy spaces and Lebesgue spaces. Liang *et al.* [28], Theorem 2.7, obtained an interpolation theorem associated with weighted (weak) Lebesgue spaces. Cao *et al.* [29], Proposition 2.21, also obtained an interpolation theorem associated with weighted Hardy spaces and weighted (weak) Lebesgue spaces.

Let $\mathbb{A}_q(A)$ with $q \in [1,\infty]$ denote the class of anisotropic Muckenhoupt weights (see, for example, [21, 30] for their definitions and properties) and $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ be an anisotropic Musielak-Orlicz function such that $\varphi(x,\cdot)$ is an Orlicz function and $\varphi(\cdot,t) \in \mathbb{A}_\infty(A)$. The main goal of this article is to give two anisotropic interpolation theorems of Musielak-Orlicz type. Precisely, by the boundedness of sublinear operator T on weighted anisotropic Hardy spaces and weighted (weak) Lebesgue spaces, we can further obtain the boundedness of T on anisotropic Musielak-Orlicz function spaces (see Theorems 2.5 and 2.9 below). The interpolation theorems mentioned above are weighted anisotropic extension of classical Marcinkiewicz interpolation theorems, see, for example, [31], and they are also a complement of Liang *et al.* [28], Theorem 2.7. And it is worth mentioning that the classical cubes are not suitable for the anisotropic settings, so we introduce a class of general dyadic cubes of Christ [32] (see Lemma 3.11 below), which plays an important role in the proof of Theorem 2.9 mentioned above.

This article is organized as follows.

In Section 2, we first recall some notation and definitions concerning expansive dilations, anisotropic Muckenhoupt weights and anisotropic Musielak-Orlicz functions. Then we give two anisotropic interpolation theorems of Musielak-Orlicz type, the proofs of which are given in Section 3.

Finally, we make some conventions on notation. Let $\mathbb{Z}_+ := \{1,2,\ldots\}$ and $\mathbb{N} := \{0\} \cup \mathbb{Z}_+$. For any $\alpha := (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$, and $\partial^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$. Throughout the whole paper, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. The *symbol* $D \lesssim F$ means that $D \le CF$. If $D \lesssim F$ and $F \lesssim D$, we then write $D \sim F$. If E is a subset of $\mathbb{R}^n$, we denote by χ_E its *characteristic function*. For any $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the *maximal integer* not larger than a. If there are no special instructions, any space $\mathcal{X}(\mathbb{R}^n)$ is denoted simply by $\mathcal{X}$ and any space $\mathcal{X}(\mathbb{R}^n;A)$ is denoted simply by $\mathcal{X}(A)$. Denote by $\mathcal{S}$ the *space of all Schwartz functions*, $\mathcal{S}'$ the *space of all tempered distributions*. For any set $E \subset \mathbb{R}^n$ and $t \in (0,\infty)$, let $\varphi(E,t) := \int_E \varphi(x,t)\,dx$ and, for any measurable function f and $t \in (0,\infty)$, let $\{|f| > t\} := \{x \in \mathbb{R}^n : |f(x)| > t\}$.

2 Preliminaries and main results

In this section, let us first recall the notion of expansive dilations on $\mathbb{R}^n$; see [10], p.5. A real $n \times n$ matrix A is called an *expansive dilation*, shortly a *dilation*, if $\min_{\lambda\in\sigma(A)} |\lambda| > 1$, where $\sigma(A)$ denotes the set of all *eigenvalues* of A. Let λ_- and λ_+ be two *positive numbers* such that

$$1 < \lambda_- < \min\{|\lambda| : \lambda \in \sigma(A)\} \leq \max\{|\lambda| : \lambda \in \sigma(A)\} < \lambda_+.$$

In the case when A is diagonalizable over $\mathbb{C}$, we can even take $\lambda_- := \min\{|\lambda| : \lambda \in \sigma(A)\}$ and $\lambda_+ := \max\{|\lambda| : \lambda \in \sigma(A)\}$. Otherwise, we need to choose them sufficiently close to these equalities according to what we need in our arguments.

It was proved in [10], p.5, Lemma 2.2, that, for a given dilation A, there exist a number $r \in (1,\infty)$ and a set $\Delta := \{x \in \mathbb{R}^n : |Px| < 1\}$, where P is some non-degenerate $n \times n$ matrix, such that $\Delta \subset r\Delta \subset A\Delta$ and, by a scaling, one can additionally assume that $|\Delta| = 1$, where $|\Delta|$ denotes the n-dimensional Lebesgue measure of the set Δ. Let $B_k := A^k\Delta$ for $k \in \mathbb{Z}$. Then B_k is open, $B_k \subset rB_k \subset B_{k+1}$, and $|B_k| = b^k$, here and hereafter $b := |\det A|$. Throughout the whole paper, let σ be the *minimum positive integer* such that

$$r^\sigma \geq 2$$

and, for any subset E of $\mathbb{R}^n$, let $E^{\complement} := \mathbb{R}^n \setminus E$. Then, for all $k, j \in \mathbb{Z}$ with $k \leq j$, it holds true that

$$B_k + B_j \subset B_{j+\sigma}, \tag{2.1}$$

$$B_k + (B_{j+\sigma})^{\complement} \subset (B_j)^{\complement}, \tag{2.2}$$

where $E + F$ denotes the *algebraic sum* $\{x + y : x \in E, y \in F\}$ of sets $E, F \subset \mathbb{R}^n$.

Definition 2.1 A *quasi-norm*, associated with an expansive matrix A, is a Borel measurable mapping $\rho_A : \mathbb{R}^n \to [0,\infty)$, for simplicity, denoted by ρ, satisfying

(i) $\rho(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, here and hereafter, $\vec{0}_n := (\overbrace{0,\ldots,0}^{n})$;
(ii) $\rho(Ax) = b\rho(x)$ for all $x \in \mathbb{R}^n$, where, as above, $b := |\det A|$;
(iii) $\rho(x + y) \leq H[\rho(x) + \rho(y)]$ for all $x, y \in \mathbb{R}^n$, where $H \in [1,\infty)$ is a constant independent of x and y.

In the standard dyadic case $A := 2I_{n\times n}$, $\rho(x) := |x|^n$ for all $x \in \mathbb{R}^n$ is an example of quasi-norms associated with A, here and hereafter, $|\cdot|$ always denotes the *Euclidean norm* in $\mathbb{R}^n$.

It was proved, in [10], p.6, Lemma 2.4, that all quasi-norms associated with a given dilation A are equivalent. Therefore, for a given expansive dilation A, in the following, for simplicity, we always use the *step quasi-norm* ρ defined by setting, for all $x \in \mathbb{R}^n$,

$$\rho(x) := \sum_{k\in\mathbb{Z}} b^k \chi_{B_{k+1}\setminus B_k}(x) \quad \text{if } x \neq \vec{0}_n, \quad \text{or else} \quad \rho(\vec{0}_n) := 0.$$

By (2.1) and (2.2), we know that, for all $x, y \in \mathbb{R}^n$,

$$\rho(x + y) \leq b^\sigma \left(\max\{\rho(x), \rho(y)\}\right) \leq b^\sigma \left[\rho(x) + \rho(y)\right].$$

Moreover, $(\mathbb{R}^n, \rho, dx)$ is a space of homogeneous type in the sense of Coifman and Weiss [33, 34], where dx denotes the *n-dimensional Lebesgue measure.*

Definition 2.2 Let $q \in [1,\infty)$. A function $w : \mathbb{R}^n \to [0,\infty)$ is said to satisfy the *anisotropic Muckenhoupt condition* $\mathbb{A}_q(A)$, denoted by $w \in \mathbb{A}_q(A)$, if there exists a positive constant C such that, when $q \in (1,\infty)$,

$$\sup_{x\in\mathbb{R}^n} \sup_{k\in\mathbb{Z}} \left\{ b^{-k} \int_{x+B_k} w(y)\,dy \right\} \left\{ b^{-k} \int_{x+B_k} \left[w(y)\right]^{-\frac{1}{q-1}}\,dy \right\}^{q-1} \le [w]_{\mathbb{A}_q(A)} < \infty,$$

and, when $q = 1$,

$$\sup_{x\in\mathbb{R}^n} \sup_{k\in\mathbb{Z}} \left\{ b^{-k} \int_{x+B_k} w(y)\,dy \right\} \left\{ \operatorname*{esssup}_{y\in x+B_k} \left[w(y)\right]^{-1} \right\} \le [w]_{\mathbb{A}_1(A)} < \infty.$$

Define $\mathbb{A}_\infty(A) := \bigcup_{1\le q<\infty} \mathbb{A}_q(A)$ and, for any $w \in \mathbb{A}_\infty(A)$, let

$$q(w) := \inf\left\{ q \in [1,\infty) : w \in \mathbb{A}_q(A) \right\}. \tag{2.3}$$

Obviously, $q(w) \in [1,\infty)$. Moreover, it is known (see [35]) that, if $q(w) \in (1,\infty)$, then $w \notin \mathbb{A}_{q(w)}(A)$ and there exists a $w \in (\bigcap_{q>1} \mathbb{A}_q(A)) \setminus \mathbb{A}_1(A)$ such that $q(w) = 1$.

Let φ be a nonnegative function on $\mathbb{R}^n \times [0,\infty)$. The function φ is called a *Musielak-Orlicz function* if, for any $x \in \mathbb{R}^n$, $\varphi(x,\cdot)$ is an Orlicz function on $[0,\infty)$ and, for any $t \in [0,\infty)$, $\varphi(\cdot,t)$ is measurable on $\mathbb{R}^n$. Here a function $\Phi : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if it is nondecreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for $t \in [0,\infty)$ and $\lim_{t\to\infty} \Phi(t) = \infty$ (see, for example, [3]). Remark that, unlike the usual case, such a Φ may not be convex.

For an Orlicz function Φ, the most useful tool to study its growth property may be the upper and the lower types of Φ. More precisely, for $p \in (-\infty,\infty)$, a function Φ is said to be of *upper* (resp. *lower*) *type p*, if there exists a positive constant C such that, for all $s \in [1,\infty)$ (resp. $s \in [0,1]$) and $t \in (0,\infty)$,

$$\Phi(st) \le Cs^p \Phi(t). \tag{2.4}$$

Let φ be a Musielak-Orlicz function. The *Musielak-Orlicz space* L^φ, which was first introduced by Musielak [3], is defined to be the set of all measurable functions f such that $\int_{\mathbb{R}^n} \varphi(x, |f(x)|)\,dx < \infty$ with the *Luxembourg-Nakano* (*quasi-*)*norm*:

$$\|f\|_{L^\varphi} := \inf\left\{ \lambda \in (0,\infty) : \int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\lambda}\right) dx \le 1 \right\}.$$

Remark 2.3 Let $x \in \mathbb{R}^n$, $t \in [0,\infty)$, w be a classical or an anisotropic Muckenhoupt $\mathbb{A}_\infty$ weight with $q(w)$ being as in (2.3), and $\Phi(x)$ an Orlicz function on $\mathbb{R}^n$. If $\varphi(x,t) := t^p w(x)$, then $L^\varphi = L^p_w$; if $\varphi(x,t) := \Phi(t)$, then $L^\varphi = L^\Phi$; if $\varphi(x,t) := \Phi(t)w(x)$, then $L^\varphi = L^\Phi_w$.

Moreover, throughout the whole article, we always assume that the Musielak-Orlicz functions satisfy the following growth assumptions.

Assumption (φ) Let $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ be a Musielak-Orlicz function satisfying the following two conditions:

(i) for any $t \in (0,\infty)$, $\varphi(\cdot,t) \in \mathbb{A}_\infty(A)$;
(ii) there exists $p_\varphi^-, p_\varphi^+ \in (0,\infty)$ such that, for every $x \in \mathbb{R}^n$, $\varphi(x,\cdot)$ is of uniformly upper type p_φ^+ and of uniformly lower type p_φ^-.

When $p_\varphi^- = p \in (0,1]$ and $p_\varphi^+ = 1$, Assumption (φ) of φ coincides with that of [35], Definition 3.

For a Musielak-Orlicz function φ satisfying Assumption (φ), the following critical indices are useful. Let

$$\begin{aligned} i(\varphi) := \sup\bigl\{&p \in (-\infty,\infty) : \text{for any } x \in \mathbb{R}^n, \varphi(x,\cdot) \text{ is of uniformly} \\ &\text{lower type } p \text{ with } C \text{ being as in (2.4) independent of } x\bigr\} \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} q(\varphi) := \inf\bigl\{&q \in [1,\infty) : \text{for any } t \in (0,\infty), \varphi(\cdot,t) \in \mathbb{A}_q(A) \\ &\text{with } \bigl[\varphi(\cdot,t)\bigr]_{\mathbb{A}_q(A)} \text{ independent of } t\bigr\}. \end{aligned} \tag{2.6}$$

Observe that $i(\varphi)$ may be not attainable, namely, φ may be not of uniformly lower type $i(\varphi)$; see, for example, [28] for some examples. Clearly,

$$\varphi(x,t) := w(x)\Phi(t) \quad \text{for all } x \in \mathbb{R}^n \text{ and } t \in [0,\infty)$$

satisfies Assumption (φ) if w is a classical or an anisotropic $\mathcal{A}_\infty(A)$ Muckenhoupt weight (see, for example, [21]) and Φ is of lower type p_φ^- for some $p_\varphi^- \in (0,\infty)$ and of upper type p_φ^+ with $p_\varphi^+ \in (0,\infty)$. More examples of growth functions can be found in [9, 28, 36, 37].

Now, let us recall the notion of a *weighted Lebesgue space*. For any nonnegative locally integrable function w on $\mathbb{R}^n$ and $p \in (0,\infty]$, the space L^p_w is defined to be the space of all measurable functions f such that, when $p \in (0,\infty)$,

$$\|f\|_{L^p_w} := \left\{\int_{\mathbb{R}^n} \bigl|f(x)\bigr|^p w(x)\,dx\right\}^{\frac{1}{p}} < \infty$$

and, when $p = \infty$,

$$\|f\|_{L^\infty_w} := \inf_{w(E)=0} \sup_{x\in\mathbb{R}^n\setminus E} \bigl|f(x)\bigr| < \infty.$$

The space *weak* L^p_w denotes the set of all measurable functions f such that

$$\|f\|_{\text{weak } L^p_w} := \sup_{\lambda>0} \lambda\bigl[w\bigl(\bigl\{|f| > \lambda\bigr\}\bigr)\bigr]^{\frac{1}{p}} < \infty.$$

Next, let us introduce the notion of a weighted anisotropic Hardy space. For $m \in \mathbb{N}$, let

$$\mathcal{S}_m := \left\{\phi \in \mathcal{S} : \sup_{x\in\mathbb{R}^n} \sup_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\le m+1}} \bigl[1 + \rho(x)\bigr]^{m+2} \bigl|\partial^\alpha \phi(x)\bigr| \le 1\right\}$$

and, for $\phi \in \mathcal{S}$, $k \in \mathbb{Z}$, and $x \in \mathbb{R}^n$, let $\phi_k(x) := b^k \phi(A^k x)$.

For $f \in \mathcal{S}'$, the *non-tangential grand maximal function* f_m^* of f is defined by setting, for all $x \in \mathbb{R}^n$,

$$f_m^*(x) := \sup_{\phi \in \mathcal{S}_m} \sup_{k \in \mathbb{Z}, y \in x+B_k} \left| f * \phi_k(y) \right|.$$

For $p \in (0,1]$, $w \in \mathbb{A}_\infty(A)$, and $q(w)$ being as in (2.3), let

$$m_{p,w} := \left\lfloor \left(\frac{q(w)}{p} - 1 \right) \frac{\ln b}{\ln(\lambda_-)} \right\rfloor. \tag{2.7}$$

The definition of the following weighted anisotropic Hardy space comes from [30], Definition 3.1.

Definition 2.4 For any $p \in (0,1]$, $w \in \mathbb{A}_\infty(A)$, $m_{p,w}$ being as in (2.7) and $m \in \mathbb{Z}_+$ satisfying $m \geq m_{p,w}$, the *weighted anisotropic Hardy space* $H^p_{w,m}(A)$ is defined as the set of all $f \in \mathcal{S}'$ such that $f_m^* \in L^p_w$ with the *(quasi-)norm* $\|f\|_{H^p_{w,m}(A)} := \|f_m^*\|_{L^p_w}$.

For any $m_1, m_2 \geq m_{p,w}$, since $H^p_{w,m_1}(A) = H^p_{w,m_2}(A)$ with equivalent norms (see [30], Theorem 5.5), then from now on, we denote simply by $H^p_w(A)$ the weighted anisotropic Hardy space $H^p_{w,m}(A)$ with $m \geq m_{p,w}$.

Finally, we need to make several further explanations for the BMO(A) functions. For any $f \in L^1_{\mathrm{loc}}$ and $E \subset \mathbb{R}^n$, set $f_E := \frac{1}{|E|} \int_E f(x)\,dx$, and define the *sharp maximal function* associated with dilation A by setting, for all $x \in \mathbb{R}^n$,

$$\mathcal{M}^\sharp f(x) := \sup_{x \in B \in \mathcal{B}} \frac{1}{|B|} \int_B \left| f(y) - f_B \right| dy,$$

here and hereafter, let $\mathcal{B} := \{B := x + B_k : x \in \mathbb{R}^n, k \in \mathbb{Z}\}$ be the *collection of all dilated balls*.

Moreover, we also define *anisotropic BMO space* by setting

$$\mathrm{BMO}(A) := \left\{ f \in L^1_{\mathrm{loc}} : \mathcal{M}^\sharp f \in L^\infty \right\}$$

and

$$\|f\|_{\mathrm{BMO}(A)} := \left\| \mathcal{M}^\sharp f \right\|_\infty.$$

Now, we introduce the interpolation theorem associated with a Musielak-Orlicz function, which may have independent interest.

Theorem 2.5 *Let $p_1 \in (0,1]$, φ be a Musielak-Orlicz function satisfying Assumption (φ), $q(\varphi)$ as in (2.6) with $[q(\varphi)]^2 < p_\varphi^- \leq p_\varphi^+ < \infty$ and $p_2 \in (p_\varphi^+, \infty]$. Assume that T is a sublinear operator defined on $H^{p_1}_{\varphi(\cdot,t)}(A) + L^{p_2}_{\varphi(\cdot,t)}$ satisfying the requirement that there exist positive constants C_1 and C_2 such that, for all $\alpha \in (0,\infty)$ and $t \in (0,\infty)$,*

$$\varphi\left(\left\{|Tf| > \alpha\right\}, t\right) \leq C_1 \alpha^{-p_1} \|f\|^{p_1}_{H^{p_1}_{\varphi(\cdot,t)}(A)} \tag{2.8}$$

and

$$\varphi\left(\left\{|Tf| > \alpha\right\}, t\right) \leq C_2 \alpha^{-p_2} \|f\|^{p_2}_{L^{p_2}_{\varphi(\cdot,t)}} \tag{2.9}$$

with the usual modification when $p_2 = \infty$. *Then* T *is bounded on* L^φ *and, moreover, there exists a positive constant* C *such that, for any* $f \in L^\varphi$,

$$\int_{\mathbb{R}^n} \varphi\bigl(x, \bigl|Tf(x)\bigr|\bigr)\,dx \le C \int_{\mathbb{R}^n} \varphi\bigl(x, \bigl|f(x)\bigr|\bigr)\,dx.$$

By Remark 2.3 and Theorem 2.5, we have the following three corollaries.

Corollary 2.6 *Let* w *be a classical or an anisotropic Muckenhoupt* $\mathbb{A}_\infty$ *weight with* $q(w)$ *being as in* (2.3) *and* $p_1, p, p_2 \in \mathbb{R}$ *with* $0 < p_1 \le 1 \le [q(w)]^2 < p < p_2 \le \infty$. *If a sublinear operator* T *is bounded from* $H^{p_1}_w$ *to weak* $L^{p_1}_w$ *and from* $L^{p_2}_w$ *to weak* $L^{p_2}_w$, *then* T *is bounded on* L^p_w.

Particularly, when $w(x) = 1$, the above consequence coincides with a generalization of Marcinkiewicz interpolation theorem.

Corollary 2.7 *Let* $p_1, p_2 \in \mathbb{R}$ *and* Φ *be an Orlicz function with uniformly lower type* p^-_Φ *and uniformly upper type* p^+_Φ, *where* $0 < p_1 \le 1 < p^-_\Phi \le p^+_\Phi < p_2 \le \infty$. *If a sublinear operator* T *is bounded from* H^{p_1} *to weak* L^{p_1} *and from* L^{p_2} *to weak* L^{p_2}, *then* T *is bounded on* L^Φ.

Corollary 2.8 *Let* w *be a classical or an anisotropic Muckenhoupt* $\mathbb{A}_\infty$ *weight with* $q(w)$ *being as in* (2.3), Φ *an Orlicz function with uniformly lower type* p^-_Φ *and uniformly upper type* p^+_Φ *with* $[q(w)]^2 < p^-_\Phi$, *and* $p_1, p_2 \in \mathbb{R}$ *with* $0 < p_1 \le 1$ *and* $p^+_\Phi < p_2 \le \infty$. *If a sublinear operator* T *is bounded from* $H^{p_1}_w$ *to weak* $L^{p_1}_w$ *and from* $L^{p_2}_w$ *to weak* $L^{p_2}_w$, *then* T *is bounded on* L^Φ_w.

It should be pointed that when $p_1 = 1$ and $p_2 = \infty$, there also exists an interpolation theorem which differs from Theorem 2.5. It can be stated as follows.

Theorem 2.9 *Suppose* φ *is a Musielak-Orlicz function satisfying* $\varphi(\cdot, t) \in \mathbb{A}_1(A)$ *for any* $t \in (0, \infty)$ *and Assumption* (φ)(ii) *with* $p^-_\varphi \in (1, \infty)$. *Assume that* T *is a sublinear operator defined on*

$$H^1_{\varphi(\cdot,t)}(A) + \mathrm{BMO}(A)$$

satisfying that there exist positive constants C_1 *and* C_2 *such that, for all* $\alpha \in (0, \infty)$ *and* $t \in (0, \infty)$,

$$\|Tf\|_{L^1_{\varphi(\cdot,t)}} \le C_1 \alpha^{-1} \|f\|^1_{H^1_{\varphi(\cdot,t)}(A)} \tag{2.10}$$

and

$$\|Tf\|_{\mathrm{BMO}(A)} \le C_2 \|f\|_\infty. \tag{2.11}$$

Then T *is bounded on* L^φ *and, moreover, there exists a positive constant* C *such that, for any* $f \in L^\varphi$,

$$\int_{\mathbb{R}^n} \varphi\bigl(x, \bigl|Tf(x)\bigr|\bigr)\,dx \le C \int_{\mathbb{R}^n} \varphi\bigl(x, \bigl|f(x)\bigr|\bigr)\,dx.$$

Remark 2.10 For any $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, if we let $\varphi(x,t) := t^p w(x)$, $\varphi(x,t) := \Phi(t)$ or $\varphi(x,t) := \Phi(t)w(x)$, respectively, then, by Theorem 2.9, we may also obtain three corollaries similar to Corollaries 2.6, 2.7, and 2.8, respectively.

3 Proofs of Theorems 2.5 and 2.9

In order to prove Theorem 2.5, we begin with several lemmas. Lemma 3.1 is from [26], pp.7-8.

Lemma 3.1 *Let $q \in [1,\infty)$ and $w \in \mathbb{A}_q(A)$. Then, for any measurable set $E \subset B \in \mathcal{B}$, there exist two positive constants C_1 and C_2 such that*

$$C_1\left(\frac{|B|}{|E|}\right)^{1/q} \le \frac{w(B)}{w(E)} \le C_2\left(\frac{|B|}{|E|}\right)^q.$$

The following lemma is similar to [9], Lemma 4.1(ii).

Lemma 3.2 *Let φ be a Musielak-Orlicz function with uniformly lower type p_φ^- and uniformly upper type p_φ^+, where $0 < p_\varphi^- \le p_\varphi^+ < \infty$, and, for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$,*

$$\tilde{\varphi}(x,t) := \int_0^t \frac{\varphi(x,s)}{s}\,ds.$$

Then $\tilde{\varphi}$ is a Musielak-Orlicz function, which is equivalent to φ, moreover, $\tilde{\varphi}(x,\cdot)$ is continuous and strictly increasing for all $x \in \mathbb{R}^n$.

Proof Since φ is a Musielak-Orlicz function, it is easy to see that, for all $x \in \mathbb{R}^n$, the function $\tilde{\varphi}(x,\cdot)$ is continuous and strictly increasing. Moreover, φ is of uniformly lower type p_φ^- with $p_\varphi^- \in (0,\infty)$, then we have, for any $x \in \mathbb{R}^n$,

$$\tilde{\varphi}(x,t) = \int_0^t \frac{\varphi(x,s)}{s}\,ds \le C\frac{\varphi(x,t)}{t^{p_\varphi^-}}\int_0^t \frac{1}{s^{1-p_\varphi^-}}\,ds \le C\varphi(x,t),$$

where C is a positive constant as in (2.4).

On the other hand, since φ is of uniformly upper type $p_\varphi^+ \in (0,\infty)$, we get, for any $x \in \mathbb{R}^n$,

$$\tilde{\varphi}(x,t) = \int_0^t \frac{\varphi(x,s)}{s}\,ds \ge C\frac{\varphi(x,t)}{t^{p_\varphi^+}}\int_0^t \frac{1}{s^{1-p_\varphi^+}}\,ds \ge C\varphi(x,t),$$

where C is a positive constant as in (2.4). This finishes the proof of Lemma 3.2. □

Remark 3.3 By Lemma 3.2, in the future, we always consider a Musielak-Orlicz function φ of uniformly lower type p_φ^- and of uniformly upper type p_φ^+, and $\varphi(x,\cdot)$ is continuous and strictly increasing for all $x \in \mathbb{R}^n$.

By using Remark 3.3 and repeating the proofs of [9], Lemma 4.2(i), and [9], Lemma 4.3(i), we obtain the following lemma.

Lemma 3.4 *Let φ be a Musielak-Orlicz function with uniformly lower type p_φ^- and uniformly upper type p_φ^+, where $0 < p_\varphi^- \le p_\varphi^+ < \infty$.*

(i) $\int_{\mathbb{R}^n} \varphi(x, \frac{|f(x)|}{\|f\|_{L^\varphi}})\, dx = 1$ *for all* $f \in L^\varphi \setminus \{\mathbf{0}\}$.
(ii) *Given c is a positive constant. Then there exists a positive constant C such that the inequality, for any* $t \in (0, \infty)$,

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{t}\right) dx \le c$$

holds true implies that $\|f\|_{L^\varphi} \le Ct$.

Lemma 3.5 *Let T be a sublinear operator. If there exists a positive constant C such that, for any* $f \in L^\varphi$,

$$\int_{\mathbb{R}^n} \varphi(x, |Tf(x)|)\, dx \le C \int_{\mathbb{R}^n} \varphi(x, |f(x)|)\, dx, \tag{3.1}$$

then T is bounded on L^φ.

Proof If $f \in L^\varphi \setminus \{\mathbf{0}\}$, then we have $f\|f\|_{L^\varphi}^{-1} \in L^\varphi$. From this, the homogeneity property of sublinear operator T, (3.1) and Lemma 3.4(i), it follows that

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{|Tf(x)|}{\|f\|_{L^\varphi}}\right) dx = \int_{\mathbb{R}^n} \varphi\left(x, \left|T\left(\frac{f(x)}{\|f\|_{L^\varphi}}\right)\right|\right) dx \lesssim \int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\|f\|_{L^\varphi}}\right) dx \sim 1,$$

which, together with Lemma 3.4(ii), implies that

$$\|Tf\|_{L^\varphi} \lesssim \|f\|_{L^\varphi}.$$

This finishes the proof of Lemma 3.5. □

Lemma 3.6 *Let* φ *be a Musielak-Orlicz function satisfying Assumption* (φ) *and let* $q(\varphi)$ *be as in* (2.6) *with* $q(\varphi) < p_\varphi^- < \infty$. *If* $f \in L^\varphi$, *then* $f \in L^1_{\mathrm{loc}}$.

Proof For any $f \in L^\varphi$, we only need to prove, for any $B \in \mathcal{B}$,

$$\mathrm{I} := \frac{1}{|B|} \int_B |f(x)|\, dx < \infty.$$

Let $q(\varphi)$ be as in (2.6) and $p_\varphi^- \in (q(\varphi), \infty)$. For any $t \in (0, \infty)$, since $\varphi(\cdot, t) \in \mathbb{A}_{p_\varphi^-}(A)$, the Hölder inequality yields

$$\begin{aligned} \mathrm{I} &\lesssim \left(\frac{1}{|B|} \int_B |f(x)|^{p_\varphi^-} \varphi(x,t)\, dx\right)^{\frac{1}{p_\varphi^-}} \left(\frac{1}{|B|} \int_B [\varphi(x,t)]^{-\frac{1}{p_\varphi^- - 1}}\, dx\right)^{p_\varphi^- - 1} \\ &\lesssim \left(\frac{1}{\varphi(B,t)} \int_B |f(x)|^{p_\varphi^-} \varphi(x,t)\, dx\right)^{\frac{1}{p_\varphi^-}} \end{aligned}$$

and hence, it suffices to prove $\int_B |f(x)|^{p_\varphi^-} \varphi(x,t)\, dx < \infty$. Notice that

$$\begin{aligned} \int_B |f(x)|^{p_\varphi^-} \varphi(x,t)\, dx &= \int_{\{x \in B: |f(x)| > t\}} |f(x)|^{p_\varphi^-} \varphi(x,t)\, dx + \int_{\{x \in B: |f(x)| \le t\}} |f(x)|^{p_\varphi^-} \varphi(x,t)\, dx \\ &=: \mathrm{I}_1 + \mathrm{I}_2. \end{aligned}$$

For I_1, by $t/|f(x)| < 1$, the uniformly lower type p_φ^- property of φ and $f \in L^\varphi$, we obtain, for any $t \in (0,\infty)$,

$$\begin{aligned}\mathrm{I}_1 &\lesssim \int_{\{x\in B:|f(x)|>t\}} |f(x)|^{p_\varphi^-}\left(\frac{t}{|f(x)|}\right)^{p_\varphi^-}\varphi(x,|f(x)|)\,dx \\ &\lesssim t^{p_\varphi^-}\int_{\mathbb{R}^n}\varphi(x,|f(x)|)\,dx < \infty.\end{aligned}$$

For I_2, by $|f(x)| \le t$ and $\varphi(\cdot,t) \in L^1_{\mathrm{loc}}$, we have $\mathrm{I}_2 \lesssim t^{p_\varphi^-}\varphi(B,t) < \infty$, which, together with $\mathrm{I}_1 < \infty$, we finally obtain $\mathrm{I} < \infty$ and finish the proof of Lemma 3.6. □

For any locally integrable function f, the *anisotropic Hardy-Littlewood maximal function* $\mathcal{M}_A f$ is defined by

$$\mathcal{M}_A f(x) := \sup_{x\in B\in\mathcal{B}} \frac{1}{|B|}\int_B |f(y)|\,dy, \quad x \in \mathbb{R}^n.$$

Let $w \in \mathbb{A}_\infty(A)$. For any locally integrable function f with respect to measure $w(x)\,dx$, the *anisotropic weighted Hardy-Littlewood maximal function* $\mathcal{M}_w f$ is defined by

$$\mathcal{M}_w f(x) := \sup_{x\in B\in\mathcal{B}} \frac{1}{w(B)}\int_B |f(y)|w(y)\,dy, \quad x \in \mathbb{R}^n.$$

For any $q \in (q(w),\infty)$ with $q(w)$ as in (2.3), by [19], Theorem 2.8, we have the boundedness of $\mathcal{M}_w$ from L^q_w to L^q_w. From this and Lemma 3.2, with an argument similar to that of [28], Theorem 2.7 and Corollary 2.8, we deduce the following lemma, the details being omitted.

Lemma 3.7 *Let φ be a Musielak-Orlicz function satisfying Assumption (φ) with $q(\varphi) < p_\varphi^- \le p_\varphi^+ < \infty$, where $q(\varphi)$ is as in* (2.6). *Then, for any $t \in (0,\infty)$, the weighted Hardy-Littlewood maximal operator $\mathcal{M}_{\varphi(\cdot,t)}$ is bounded on L^φ and, moreover, there exists a positive constant C such that, for all $f \in L^\varphi$,*

$$\int_{\mathbb{R}^n}\varphi(x,\mathcal{M}_{\varphi(\cdot,t)}f(x))\,dx \le C\int_{\mathbb{R}^n}\varphi(x,|f(x)|)\,dx.$$

The definition of weighted atomic anisotropic Hardy spaces is from [30], Definition 3.2.

Definition 3.8 Let $w \in \mathbb{A}_\infty(A)$ and $q(w)$ be as in (2.3). A triplet $(p,q,s)_w$ is called admissible, if $p \in (0,1]$, $q \in (q(w),\infty]$, and $s \in \mathbb{N}$ with $s \ge \lfloor (q(w)/p - 1)\ln b/\ln(\lambda_-)\rfloor$. A function a on $\mathbb{R}^n$ is said to be a $(p,q,s)_w$-atom if it satisfies the following three conditions:

(i) $\operatorname{supp} a \subset x_0 + B_j$ for some $j \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^n$;

(ii) $\|a\|_{L^q_w} \le [w(x_0 + B_j)]^{\frac{1}{q}-\frac{1}{p}}$;

(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0$ for any multi-index α satisfying $|\alpha| \le s$.

Definition 3.9 Let $w \in \mathbb{A}_\infty(A)$ and $(p,q,s)_w$ be an admissible triplet as in Definition 3.8. The *weighted atomic anisotropic Hardy space* $H^{p,q,s}_w(A)$ is defined to be the set of all $f \in$

$\mathcal{S}'$ satisfying that $f = \sum_i \lambda_i a_i$ in $\mathcal{S}'$, where $\{\lambda_i\}_i \subset \mathbb{C}$, $\sum_i |\lambda_i|^p < \infty$, and $\{a_i\}_i$ are $(p,q,s)_w$-atoms. Moreover, the *(quasi-)norm* of $f \in H_w^{p,q,s}(A)$ is defined by

$$\|f\|_{H_w^{p,q,s}(A)} := \inf\left\{\left[\sum_i |\lambda_i|^p\right]^{1/p}\right\},$$

where the infimum is taken over all admissible decompositions of f as above.

Proof of Theorem 2.5 Let φ be a Musielak-Orlicz function, $q(\varphi)$ as in (2.6), $p_2 \in (p_\varphi^+, \infty)$, $p_\varphi^- \in ([q(\varphi)]^2, \infty)$, and $q \in (q(\varphi), \infty)$ close to $q(\varphi)$ such that $q(\varphi) < p_\varphi^-/q$. Suppose $f \in L^\varphi$, for any $t \in (0,\infty)$, define

$$\mathcal{M}^q_{\varphi(\cdot,t)} f(x) := \sup_{x\in B\in\mathcal{B}} \left(\frac{1}{\varphi(B,t)} \int_B |f(y)|^q \varphi(y,t)\,dy\right)^{\frac{1}{q}}, \quad x \in \mathbb{R}^n. \tag{3.2}$$

For any $\alpha \in (0,\infty)$, let

$$\Omega_\alpha := \left\{\mathcal{M}^q_{\varphi(\cdot,t)} f > \alpha\right\}.$$

By an anisotropic variant of Whitney covering lemma in [30], Lemma 2.3, we know that there exist a positive constant L depending only on σ, a sequence $\{x_i\}_i \subset \Omega_\alpha$, and a sequence $\{\ell_i\}_i$ of integers such that

$$\Omega_\alpha = \bigcup_i (x_i + B_{\ell_i}), \tag{3.3}$$

$$(x_i + B_{\ell_i - 2\sigma}) \cap (x_j + B_{\ell_j - 2\sigma}) = \emptyset \quad \text{for all } i,j \text{ with } i \neq j, \tag{3.4}$$

$$\sharp\left\{j : (x_i + B_{\ell_i + 2\sigma}) \cap (x_j + B_{\ell_j + 2\sigma}) \neq \emptyset\right\} \le L \quad \text{for all } i, \tag{3.5}$$

where we denote by $\sharp E$ the cardinality of the set E and, for any i,

$$(x_i + B_{\ell_i + 4\sigma}) \cap \Omega_\alpha^\complement = \emptyset \quad \text{and} \quad (x_i + B_{\ell_i + 4\sigma + 1}) \cap \Omega_\alpha^\complement \neq \emptyset. \tag{3.6}$$

For any i and any $x \in \mathbb{R}^n$, let $\chi_i(x) := \chi_{x_i + B_{\ell_i}}(x)$ and

$$\eta_i(x) := \begin{cases} \frac{\chi_i(x)}{\sum_j \chi_j(x)}, & \text{for } x \in \Omega_\alpha, \\ 0, & \text{for } x \in \Omega_\alpha^\complement. \end{cases}$$

Let $s \in \mathbb{N}$ with $s \ge m_{p_1,\varphi(\cdot,t)}$, where $p_1 \in (0,1]$ and $m_{p_1,\varphi(\cdot,t)}$ is as in (2.7), and $\mathcal{P}_s$ denote the linear space of polynomials of degrees not more than s. For any $B \in \mathcal{B}$, let $\pi_B : L^1(B) \to \mathcal{P}_s$ be the natural projection defined, via the Riesz lemma, by setting, for all $\tilde{f} \in L^1(B)$ and $Q \in \mathcal{P}_s$,

$$\int_B \pi_B \tilde{f}(x) Q(x)\,dx = \int_B \tilde{f}(x) Q(x)\,dx.$$

Then, by [10], (8.9), there exists a positive constant C, depending only on s, such that, for all $B \in \mathcal{B}$ and $\tilde{f} \in L^1(B)$,

$$\sup_{x\in B} \left|\pi_B \tilde{f}(x)\right| \le C \frac{1}{|B|} \int_B \left|\tilde{f}(x)\right|\,dx. \tag{3.7}$$

Setting

$$g(x) := \begin{cases} f(x), & \text{for } x \in \Omega_\alpha^\complement, \\ \sum_i (\pi_{x_i+B_{\ell_i}} f\eta_i)(x)\chi_i(x), & \text{for } x \in \Omega_\alpha, \end{cases}$$

and, for any $x \in \mathbb{R}^n$,

$$h(x) := \sum_i h_i(x), \quad \text{where } h_i(x) := f(x)\chi_i(x) - (\pi_{x_i+B_{\ell_i}} f\eta_i)(x)\chi_i(x).$$

Obviously, for any $x \in \mathbb{R}^n$, we have $f(x) = g(x) + h(x)$.

By Hölder's inequality and the definition of $\mathbb{A}_q(A)$ with $\varphi(\cdot,t) \in \mathbb{A}_q(A)$, we have, for any $B \in \mathcal{B}$ and $t \in (0,\infty)$,

$$\frac{1}{|B|}\int_B |f(x)|\,dx \le \left(\frac{1}{\varphi(B,t)}\int_B |f(x)|^q \varphi(x,t)\,dx\right)^{\frac{1}{q}}. \tag{3.8}$$

Moreover, notice that $(x_i + B_{\ell_i+4\sigma+1}) \cap \Omega_\alpha^\complement \neq \emptyset$ (see (3.6)), then there exists some

$$y_0 \in (x_i + B_{\ell_i+4\sigma+1}) \cap \Omega_\alpha^\complement. \tag{3.9}$$

Hence, for any $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, by (3.7) with $f \in L^1(B)$ (see Lemma 3.6 with $f \in L^\varphi$), (3.8), Lemma 3.1 with $\varphi(\cdot,t) \in \mathbb{A}_q(A)$ and (3.9), we obtain

$$\begin{aligned} |h_i(x)| &\le |f(x)| + C\left(\frac{1}{\varphi(x_i + B_{\ell_i},t)}\int_{x_i+B_{\ell_i}} |f(y)|^q \varphi(y,t)\,dy\right)^{\frac{1}{q}} \chi_i(x) \\ &\le |f(x)| + C\left(\frac{1}{\varphi(x_i + B_{\ell_i+4\sigma+1},t)}\int_{x_i+B_{\ell_i+4\sigma+1}} |f(y)|^q \varphi(y,t)\,dx\right)^{\frac{1}{q}} \chi_i(x) \\ &\le |f(x)| + C\mathcal{M}^q_{\varphi(\cdot,t)} f(y_0)\chi_i(x) \\ &\le |f(x)| + C\alpha\chi_i(x). \end{aligned}$$

Therefore, from the above estimate, Minkowski's inequality, Lemma 3.1 with $\varphi(\cdot,t) \in \mathbb{A}_q(A)$, and (3.9) again, it follows that

$$\begin{aligned} &\left(\frac{1}{\varphi(x_i + B_{\ell_i},t)}\int_{x_i+B_{\ell_i}} |h_i(x)|^q \varphi(x,t)\,dx\right)^{\frac{1}{q}} \\ &\quad\le \left(\frac{1}{\varphi(x_i + B_{\ell_i},t)}\int_{x_i+B_{\ell_i}} |f(x)|^q \varphi(x,t)\,dx\right)^{\frac{1}{q}} + C\alpha \\ &\quad\le C\left(\frac{1}{\varphi(x_i + B_{\ell_i+4\sigma+1},t)}\int_{x_i+B_{\ell_i+4\sigma+1}} |f(x)|^q \varphi(x,t)\,dx\right)^{\frac{1}{q}} + C\alpha \\ &\quad\le C\alpha. \end{aligned} \tag{3.10}$$

For any $p_1 \in (0,1]$, $x \in \mathbb{R}^n$, and $t \in (0,\infty)$, set

$$a_i(x) := \frac{h_i(x)}{C\alpha[\varphi(x_i + B_{\ell_i},t)]^{1/p_1}},$$

where C is a constant as in (3.10), then a_i is a $(p_1, q, s)_{\varphi(\cdot,t)}$-atom with integer $s \geq m_{p_1,\varphi(\cdot,t)}$, where $m_{p_1,\varphi(\cdot,t)}$ is as in (2.7). Thus

$$h(x) = C\alpha \sum_i \bigl[\varphi(x_i + B_{\ell_i}, t)\bigr]^{\frac{1}{p_1}} a_i(x) \in H^{p_1,q,s}_{\varphi(\cdot,t)}(A)$$

and, by Lemma 3.1 with $\varphi(\cdot, t) \in \mathbb{A}_q(A)$, (3.3) and (3.4), we obtain

$$\|h\|_{H^{p_1,q,s}_{\varphi(\cdot,t)}(A)} \leq C\alpha \biggl[\sum_i \varphi(x_i + B_{\ell_i - 2\sigma}, t)\biggr]^{\frac{1}{p_1}} \leq C\alpha \bigl[\varphi(\Omega_\alpha, t)\bigr]^{\frac{1}{p_1}}. \tag{3.11}$$

Also, when $x \in \Omega_\alpha$, for any $t \in (0, \infty)$, by (3.5), (3.7), (3.8), Lemma 3.1 with $\varphi(\cdot, t) \in \mathbb{A}_q(A)$, and (3.9) again, we have

$$\bigl|g(x)\bigr| \leq C\alpha. \tag{3.12}$$

Moreover, for a.e. $x \in \Omega_\alpha^{\complement}$,

$$\bigl|g(x)\bigr| = \bigl|f(x)\bigr| \leq \mathcal{M}^q_{\varphi(\cdot,t)} f(x) \leq \alpha.$$

Thus, we obtain, for a.e. $x \in \mathbb{R}^n$,

$$\bigl|g(x)\bigr| \leq C\alpha. \tag{3.13}$$

We prove the theorem by two cases.

Case (i): $p_2 \in (p_\varphi^+, \infty)$. In this case, by Lemma 3.2, Fubini's theorem, and the fact that T is a sublinear operator, we further have

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi\bigl(x, \bigl|Tf(x)\bigr|\bigr)\,dx &\sim \int_0^\infty \frac{1}{\alpha} \int_{\{|Tf|>\alpha\}} \varphi(x, \alpha)\,dx\,d\alpha \\
&\lesssim \int_0^\infty \frac{1}{\alpha} \int_{\{|Th|>\frac{\alpha}{2}\}} \varphi(x, \alpha)\,dx\,d\alpha \\
&\quad + \int_0^\infty \frac{1}{\alpha} \int_{\{|Tg|>\frac{\alpha}{2}\}} \varphi(x, \alpha)\,dx\,d\alpha \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

By (2.8), $H^{p_1}_{\varphi(\cdot,\alpha)}(A) = H^{p_1,q,s}_{\varphi(\cdot,\alpha)}(A)$ with equivalent norms (see [30], Theorem 5.5), (3.11) and Lemma 3.2, we conclude that

$$\begin{aligned}
\mathrm{I} &\lesssim \int_0^\infty \frac{1}{\alpha} \biggl(\frac{\|h\|_{H^{p_1,q,s}_{\varphi(\cdot,\alpha)}(A)}}{\alpha}\biggr)^{p_1} d\alpha \lesssim \int_0^\infty \frac{1}{\alpha} \varphi(\Omega_\alpha, \alpha)\,d\alpha \\
&\sim \int_0^\infty \frac{1}{\alpha} \int_{\{\mathcal{M}^q_{\varphi(\cdot,t)} f>\alpha\}} \varphi(x, \alpha)\,dx\,d\alpha \sim \int_{\mathbb{R}^n} \varphi\bigl(x, \mathcal{M}^q_{\varphi(\cdot,t)} f(x)\bigr)\,dx.
\end{aligned}$$

Let $\ddot{\varphi}(x, t) := \varphi(x, t^{1/q})$. Obviously, $\ddot{\varphi}(x, t)$ is uniformly lower type p_φ^-/q. From this, the definitions of $\mathcal{M}^q_{\varphi(\cdot,t)} f(x)$ and $\mathcal{M}_{\varphi(\cdot,t)}(|f|^q)(x)$, $q(\varphi) < p_\varphi^-/q$ with $q(\varphi)$ being as in (2.6) and

Lemma 3.7 with $q(\ddot{\varphi}) = q(\varphi) < p_\varphi^-/q$, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi\big(x, \big(\mathcal{M}^q_{\varphi(\cdot,t)} f\big)(x)\big)\,dx &= \int_{\mathbb{R}^n} \ddot{\varphi}\big(x, \big[\mathcal{M}^q_{\varphi(\cdot,t)} f(x)\big]^q\big)\,dx \\
&= \int_{\mathbb{R}^n} \ddot{\varphi}\big(x, \mathcal{M}_{\varphi(\cdot,t)}\big(|f|^q\big)(x)\big)\,dx \\
&\lesssim \int_{\mathbb{R}^n} \ddot{\varphi}\big(x, \big|f(x)\big|^q\big)\,dx \\
&\lesssim \int_{\mathbb{R}^n} \varphi\big(x, \big|f(x)\big|\big)\,dx. \qquad (3.14)
\end{aligned}$$

By (2.9), we obtain

$$\begin{aligned}
\mathrm{II} &\lesssim \int_0^\infty \frac{1}{\alpha^{1+p_2}} \int_{\mathbb{R}^n} \big|g(x)\big|^{p_2} \varphi(x,\alpha)\,dx\,d\alpha \\
&\sim \int_0^\infty \frac{1}{\alpha^{1+p_2}} \int_{\Omega_\alpha} \big|g(x)\big|^{p_2} \varphi(x,\alpha)\,dx\,d\alpha + \int_0^\infty \frac{1}{\alpha^{1+p_2}} \int_{\Omega_\alpha^\complement} \big|g(x)\big|^{p_2} \varphi(x,\alpha)\,dx\,d\alpha \\
&=: \mathrm{II}_1 + \mathrm{II}_2.
\end{aligned}$$

By (3.13), Lemma 3.2, and (3.14), we conclude that

$$\begin{aligned}
\mathrm{II}_1 &\lesssim \int_0^\infty \frac{1}{\alpha} \int_{\{\mathcal{M}^q_{\varphi(\cdot,t)} f > \alpha\}} \varphi(x,\alpha)\,dx\,d\alpha \\
&\sim \int_{\mathbb{R}^n} \varphi\big(x, \mathcal{M}^q_{\varphi(\cdot,t)} f(x)\big)\,dx \lesssim \int_{\mathbb{R}^n} \varphi\big(x, \big|f(x)\big|\big)\,dx.
\end{aligned}$$

Moreover, by Fubini's theorem, $|g(x)| = |f(x)| \le \mathcal{M}^q_{\varphi(\cdot,t)} f(x)$ for $x \in \Omega_\alpha^\complement$, and the uniformly upper type p_φ^+ property of φ with $p_\varphi^+ < p_2$, we have

$$\begin{aligned}
\mathrm{II}_2 &\sim \int_0^\infty \frac{1}{\alpha^{1+p_2}} \int_{\{\mathcal{M}^q_{\varphi(\cdot,t)} f \le \alpha\}} \big|f(x)\big|^{p_2} \varphi(x,\alpha)\,dx\,d\alpha \\
&\sim \int_{\mathbb{R}^n} \big|f(x)\big|^{p_2} \int_{\mathcal{M}^q_{\varphi(\cdot,t)} f(x)}^\infty \frac{1}{\alpha^{1+p_2}} \varphi(x,\alpha)\,d\alpha\,dx \\
&\lesssim \int_{\mathbb{R}^n} \big|f(x)\big|^{p_2} \int_{|f(x)|}^\infty \frac{1}{\alpha^{1+p_2}} \varphi(x,\alpha)\,d\alpha\,dx \\
&\lesssim \int_{\mathbb{R}^n} \big|f(x)\big|^{p_2} \int_{|f(x)|}^\infty \frac{1}{\alpha^{1+p_2}} \bigg(\frac{\alpha}{|f(x)|}\bigg)^{p_\varphi^+} \varphi\big(x, \big|f(x)\big|\big)\,d\alpha\,dx \\
&\lesssim \int_{\mathbb{R}^n} \big|f(x)\big|^{p_2 - p_\varphi^+} \varphi\big(x, \big|f(x)\big|\big) \int_{|f(x)|}^\infty \alpha^{p_\varphi^+ - p_2 - 1}\,d\alpha\,dx \\
&\sim \int_{\mathbb{R}^n} \varphi\big(x, \big|f(x)\big|\big)\,dx,
\end{aligned}$$

as desired.

Case (ii): $p_2 = \infty$. In this case, obviously we have $\|Tg\|_\infty \le C\|g\|_\infty$ (see (2.9) with the usual modification). From this and (3.13), it follows that

$$\|Tg\|_\infty \le C\|g\|_\infty \le C\alpha. \qquad (3.15)$$

By Lemma 3.2, Fubini's theorem, Assumption (φ)(ii), and the fact that T is a sublinear operator, we further have

$$\begin{aligned}\int_{\mathbb{R}^n}\varphi(x,|Tf(x)|)\,dx &\sim \int_0^\infty \frac{1}{\alpha}\int_{\{|Tf|>\alpha\}}\varphi(x,\alpha)\,dx\,d\alpha \\ &\lesssim \int_0^\infty \frac{1}{\alpha}\int_{\{|Tf|>2C\alpha\}}\varphi(x,\alpha)\,dx\,d\alpha \\ &\lesssim \int_0^\infty \frac{1}{\alpha}\int_{\{|Th|>C\alpha\}}\varphi(x,\alpha)\,dx\,d\alpha \\ &\quad + \int_0^\infty \frac{1}{\alpha}\int_{\{|Tg|>C\alpha\}}\varphi(x,\alpha)\,dx\,d\alpha \\ &=: \mathrm{I}+\mathrm{II}.\end{aligned}$$

For I, similar to the estimate of I of Case (i), we obtain $\mathrm{I}\lesssim \int_{\mathbb{R}^n}\varphi(x,|f(x)|)\,dx$.

For II, by (3.15), we see that

$$\mathrm{II}\lesssim \int_0^\infty \frac{1}{\alpha}\int_{\{\|Tg\|_\infty>C\alpha\}}\varphi(x,\alpha)\,dx\,d\alpha = 0 \le \int_{\mathbb{R}^n}\varphi(x,|f(x)|)\,dx,$$

as desired.

Using the above estimates of Case (i) and Case (ii), we have, for any $f\in L^\varphi$,

$$\int_{\mathbb{R}^n}\varphi(x,|Tf(x)|)\,dx \lesssim \int_{\mathbb{R}^n}\varphi(x,|f(x)|)\,dx,$$

which, together with Lemma 3.5, implies that T is bounded on L^φ. This finishes the proof of Theorem 2.5. □

In order to prove Theorem 2.9, we need some lemmas.

Let

$$L_0^\infty := \left\{f\in L_c^\infty : \int_{\mathbb{R}^n} f(x)\,dx = 0\right\},$$

where L_c^∞ is the space of bounded measurable functions with compact supports.

Lemma 3.10 *Let φ be a Musielak-Orlicz function satisfying Assumption* (φ) *with $q(\varphi) < p_\varphi^- \le p_\varphi^+ < \infty$, where $q(\varphi)$ is as in* (2.6). *Then L_0^∞ is dense in L^φ.*

Proof Let φ be a Musielak-Orlicz function satisfying Assumption (φ) with $q(\varphi) < p_\varphi^- \le p_\varphi^+ < \infty$, where $q(\varphi)$ is as in (2.6). First, we prove that L_c^∞ is dense in L^φ. For any $f\in L^\varphi$, $j\in\mathbb{Z}_+$ and $x\in\mathbb{R}^n$, let $f_j(x) := f(x)\chi_{B_j}(x)\chi_{\{|f|<j\}}(x)$. Obviously, we have $|f_j(x)-f(x)|\to 0$ ($j\to\infty$) for a.e. $x\in\mathbb{R}^n$. From this, $\varphi(x,\cdot)$ is continuous and strictly increasing for all $x\in\mathbb{R}^n$ (see Remark 3.3), $\varphi(x,|f_j(x)-f(x)|)$ decreasingly converges to 0 as $j\to\infty$ for a.e. $x\in\mathbb{R}^n$ and Levi's theorem, it follows that

$$\int_{\mathbb{R}^n}\varphi(x,|f_j(x)-f(x)|)\,dx\to 0 \quad (j\to\infty),$$

which implies that, L_c^∞ is dense in L^φ.

It remains to prove that L_0^∞ is dense in L_c^∞ with respect to the norm $\|\cdot\|_{L^\varphi}$. For any $f \in L_c^\infty$ with $\operatorname{supp} f \subset B_N$, where N is some integer, integer $j \geq N$ and $x \in \mathbb{R}^n$, let

$$f_j(x) := \left(f(x) - \frac{1}{|B_j|}\int_{B_j} f(y)\,dy\right)\chi_{B_j}(x).$$

Obviously, $\{f_j\}_{j=N}^\infty \subset L_0^\infty$. Thus, for any $x \in \mathbb{R}^n$, we only need to prove

$$\int_{\mathbb{R}^n} \varphi\big(x, \big|f_j(x) - f(x)\big|\big)\,dx \to 0 \quad (j \to \infty).$$

By $\operatorname{supp} f \subset B_N \subset B_j$, $1/|B_j| = 1/b^j < 1$ for sufficient large j (since $1/b^j \to 0$ as $j \to \infty$), the uniformly lower type p_φ^- property of φ with $q(\varphi) < p_\varphi^-$, Lemma 3.1, and $\varphi(\cdot, t) \in \mathbb{A}_q(A)$ with some $q \in (q(\varphi), p_\varphi^-)$ for any $t \in (0, \infty)$, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varphi\big(x, \big|f(x) - f_j(x)\big|\big)\,dx \\
&\quad = \int_{\mathbb{R}^n} \varphi\left(x, \left|f(x)\chi_{B_j}(x) - \left(f(x)\chi_{B_j}(x) - \frac{1}{|B_j|}\int_{B_j} f(y)\,dy\,\chi_{B_j}(x)\right)\right|\right)dx \\
&\quad \leq \int_{B_j} \varphi\left(x, \frac{1}{|B_j|}\int_{B_N} \big|f(y)\big|\,dy\right)dx \lesssim \frac{1}{|B_j|^{p_\varphi^-}}\varphi\left(B_j, \int_{B_N} \big|f(y)\big|\,dy\right) \\
&\quad \lesssim b^{-j(p_\varphi^- - q)}\varphi\big(B_0, \|f\|_\infty |B_N|\big) \to 0 \quad (j \to \infty).
\end{aligned}$$

This finishes the proof of Lemma 3.10. □

Since $(\mathbb{R}^n, \rho, dx)$ is a space of homogeneous type in the sense of Coifman and Weiss [33, 34]. On such homogeneous spaces, the following lemma provides an analog of the grid of Euclidean dyadic cubes, which comes from [38], Lemma 2.3; see also [32].

Lemma 3.11 *Let A be a dilation. There exists a collection $\mathcal{Q} := \{Q_\alpha^k \subset \mathbb{R}^n : k \in \mathbb{Z}, \alpha \in I_k\}$ of open subsets, where I_k is certain index set, such that*

(i) $|\mathbb{R}^n \setminus \bigcup_\alpha Q_\alpha^k| = 0$ *for each fixed k and $Q_\alpha^k \cap Q_\beta^k = \emptyset$ if $\alpha \neq \beta$;*
(ii) *for any α, β, k, ℓ with $\ell \geq k$, either $Q_\alpha^k \cap Q_\beta^\ell = \emptyset$ or $Q_\alpha^\ell \subset Q_\beta^k$;*
(iii) *for each (ℓ, β) and each $k < \ell$ there exists a unique α such that $Q_\beta^\ell \subset Q_\alpha^k$;*
(iv) *there exist a certain negative integer v and a positive integer u such that, for all Q_α^k with $k \in \mathbb{Z}$ and $\alpha \in I_k$, there exists $x_{Q_\alpha^k} \in Q_\alpha^k$ satisfying that, for any $x \in Q_\alpha^k$, $x_{Q_\alpha^k} + B_{vk-u} \subset Q_\alpha^k \subset x + B_{vk+u}$.*

In the following, for convenience, we call $\{Q_\alpha^k\}_{k\in\mathbb{Z},\alpha\in I_k}$ in Lemma 3.11 dyadic cubes, k the *level* of the dyadic cube Q_α^k with $k \in \mathbb{Z}$ and $\alpha \in I_k$ and we denote it by $\ell(Q_\alpha^k)$.

Now we recall the definition of the dyadic maximal function. For any given measurable function $f \in L_{\mathrm{loc}}^1$ and $x \in \mathbb{R}^n$, we define the *dyadic maximal function* by $\mathcal{M}_d f(x) := \sup_{k\in\mathbb{Z}} E_k f(x)$, where

$$E_k f(x) := \sum_{Q\in\mathcal{Q}_k}\left(\frac{1}{|Q|}\int_Q \big|f(y)\big|\,dy\right)\chi_Q(x)$$

and $\mathcal{Q}_k := \{Q_\alpha^k : \alpha \in I_k\}$ denotes the set of dyadic cubes as in Lemma 3.11. The following lemma provides the Calderón-Zygmund decomposition in our setting with a non-typical assumption on f instead of the usual $f \in L^1$, which is from [38], Proposition A.5.

Lemma 3.12 *Given a measurable function $f \in L_w^p$ for certain $p \in [1,\infty)$ and $w \in \mathbb{A}_p(A)$, and a positive number λ, then exists a sequence $\{Q_j\}_j \subset \mathcal{Q}$ of disjoint dyadic cubes such that*

(i) $\bigcup_j Q_j = \{\mathcal{M}_d f > \lambda\}$;
(ii) $|f(x)| < \lambda$ *for almost every* $x \notin \bigcup_j Q_j$;
(iii) $\lambda < \frac{1}{|Q_j|}\int_{Q_j} |f(x)|\,dx \leq C\lambda$, *where $C > 1$ is a constant independent of f and λ.*

Let $\mu_f(\lambda) := \sum_j w(Q_j^\lambda)$, where $Q_j^\lambda := Q_j$ is as in Lemma 3.12. We also need the following two lemmas.

Lemma 3.13 *If $f \in L_w^1$ with $w \in \mathbb{A}_1(A)$. Then, for any $\lambda \in (0,\infty)$, there exists a positive constant C such that*

$$w\bigl(\bigl\{\mathcal{M}f > \bigl(b^{3\sigma}C_1 + 1\bigr)\lambda\bigr\}\bigr) \leq C\mu_f(\lambda),$$

where C_1 is a positive constant as in Lemma 3.12(iii).

Proof Let us first prove the inequality

$$\mathcal{M}f(x) \leq \bigl(b^{3\sigma}C_1 + 1\bigr)\lambda \quad \text{if } x \in \Bigl(\bigcup_j x_{Q_j} + B_{\ell(Q_j)\nu+u+2\sigma}\Bigr)^{\complement}, \tag{3.16}$$

where $x_{Q_j} + B_{\ell(Q_j)\nu+u+2\sigma}$ is the dilated ball containing Q_j as in Lemma 3.11(iv). Pick any dilated ball $y + B_l$ such that $x \in y + B_l$. When $y + B_l \subseteq (\bigcup_j Q_j)^{\complement}$, by Lemma 3.12(ii), we have

$$\frac{1}{|y + B_l|}\int_{y+B_l} \bigl|f(z)\bigr|\,dz \leq \lambda. \tag{3.17}$$

In the following, we assume that $y + B_l \nsubseteq (\bigcup_j Q_j)^{\complement}$, and therefore, $(y + B_l) \cap (\bigcup_j Q_j) \neq \emptyset$. Thus, there exists at least one j such that $(y + B_l) \cap (x_{Q_j} + B_{\ell(Q_j)\nu+u}) \neq \emptyset$. So there exists some $\tilde{x} \in (y + B_l) \cap (x_{Q_j} + B_{\ell(Q_j)\nu+u})$. By $\tilde{x} \in y + B_l$, we have $y \in \tilde{x} + B_l$. From this, $\tilde{x} \in x_{Q_j} + B_{\ell(Q_j)\nu+u}$, and (2.1), it follows that

$$x \in y + B_l \subset \tilde{x} + B_l + B_l \subset x_{Q_j} + B_{\ell(Q_j)\nu+u} + B_{l+\sigma}. \tag{3.18}$$

If $l + \sigma \leq \ell(Q_j)\nu + u$, we obtain $x \in x_{Q_j} + B_{\ell(Q_j)\nu+u+\sigma}$, which contradicts with $x \notin \bigcup_j x_{Q_j} + B_{\ell(Q_j)\nu+u+2\sigma}$. Therefore, we have $l + \sigma > \ell(Q_j)\nu + u$. By this and (3.18), we obtain

$$x_{Q_j} + B_{\ell(Q_j)\nu+u} \subset x_{Q_j} + B_{l+\sigma} \tag{3.19}$$

and $x \in x_{Q_j} + B_{l+2\sigma}$. From this and (2.1), we conclude that $x_{Q_j} \in x + B_{l+2\sigma}$ and

$$x_{Q_j} + B_{l+\sigma} \subset x + B_{l+2\sigma} + B_{l+\sigma} \subset x + B_{l+3\sigma}. \tag{3.20}$$

By $Q_j \subset x_{Q_j} + B_{\ell(Q_j)v+u}$ (see Lemma 3.11(iv)), (3.19), and (3.20), we have

$$\bigcup_{j:Q_j\cap(y+B_l)\neq\emptyset} Q_j \subset x + B_{l+3\sigma}.$$

Using this, (ii) and (iii) of Lemma 3.12, and the disjoint property of $\{Q_j\}_j$, we find that

$$\begin{aligned}\int_{y+B_l} |f(z)|\,dz &= \int_{(y+B_l)\cap(\bigcup_j Q_j)^\complement} |f(z)|\,dz + \int_{(y+B_l)\cap(\bigcup_j Q_j)} |f(z)|\,dz \\ &\leq \lambda b^l + \sum_{j:Q_j\cap(y+B_l)\neq\emptyset} C_1\lambda|Q_j| \\ &= \lambda b^l + C_1\lambda\Bigg|\bigcup_{j:Q_j\cap(y+B_l)\neq\emptyset} Q_j\Bigg| \\ &\leq \lambda b^l + C_1\lambda b^{l+3\sigma} = \big(b^{3\sigma}C_1 + 1\big)\lambda b^l,\end{aligned}$$

where C_1 is a positive constant as in Lemma 3.12(iii). That is,

$$\frac{1}{|y+B_l|}\int_{y+B_l} f(z)\,dz \leq \big(b^{3\sigma}C_1 + 1\big)\lambda. \tag{3.21}$$

Combining the estimates (3.17) and (3.21), we conclude that (3.16) holds true. Then, by (3.16), Lemma 3.11(iv), and Lemma 3.1 with $w \in \mathbb{A}_1(A)$, we obtain

$$\begin{aligned}&w\big(\big\{\mathcal{M}f > \big(b^{3\sigma}C_1 + 1\big)\lambda\big\}\big) \\ &\quad= w\left(\left\{x \in \left(\bigcup_j x_{Q_j} + B_{\ell(Q_j)v+u+2\sigma}\right)^\complement : \mathcal{M}f(x) > \big(b^{3\sigma}C_1 + 1\big)\lambda\right\}\right) \\ &\qquad+ w\left(\left\{x \in \left(\bigcup_j x_{Q_j} + B_{\ell(Q_j)v+u+2\sigma}\right) : \mathcal{M}f(x) > \big(b^{3\sigma}C_1 + 1\big)\lambda\right\}\right) \\ &\quad= w\left(\left\{x \in \left(\bigcup_j x_{Q_j} + B_{\ell(Q_j)v+u+2\sigma}\right) : \mathcal{M}f(x) > \big(b^{3\sigma}C_1 + 1\big)\lambda\right\}\right) \\ &\quad\leq w\left(\bigcup_j x_{Q_j} + B_{\ell(Q_j)v+u+2\sigma}\right) \leq \sum_j w(x_{Q_j} + B_{\ell(Q_j)v+u+2\sigma}) \\ &\quad\lesssim b^{2\sigma+2u}\sum_j w(x_{Q_j} + B_{\ell(Q_j)v-u}) \lesssim \sum_j w(Q_j) = \mu_f(\lambda).\end{aligned}$$

This finishes the proof of Lemma 3.13. □

Lemma 3.14 *Suppose that $f \in L^1_w$ with $w \in \mathbb{A}_1(A)$. Then, for any $\alpha \in (0,\infty)$ and $\beta \in (0,\infty)$ satisfying $\beta b^{2u} < 1/2$, there exists a positive constant C such that*

$$\mu_f(\alpha) \leq w\big(\big\{\mathcal{M}^\sharp f > \alpha\beta/2\big\}\big) + \frac{\beta b^{2u}}{1-\beta b^{2u}} C\mu_f\big(2^{-1}C_1^{-1}\alpha\big),$$

where C_1 is a positive constant as in Lemma 3.12(iii).

Proof Let $\nu := 2^{-1}C_1^{-1}\alpha$, and make the Calderón-Zygmund decompositions of f with the heights ν and α, respectively (see Lemma 3.12). Since $\nu < \alpha$, from the proof of Lemma 3.12, we see that each cube Q_k^α in Calderón-Zygmund decomposition with the height α must be included in some cube Q_j^ν in Calderón-Zygmund decomposition with the height ν. Thus, by letting

$$\mathrm{I} := \{Q_j^\nu : j \in \mathrm{J}\}, \quad \text{where } \mathrm{J} := \{j : Q_j^\nu \subset \{\mathcal{M}^\sharp f > \alpha\beta/2\}\}$$

and

$$\mathrm{II} := \{Q_j^\nu : j \notin \mathrm{J}\},$$

we have

$$\mu_f(\alpha) = \sum_k w(Q_k^\alpha) = \sum_{Q\in\mathrm{I}} \sum_{Q_k^\alpha \subset Q} w(Q_k^\alpha) + \sum_{Q\in\mathrm{II}} \sum_{Q_k^\alpha \subset Q} w(Q_k^\alpha).$$

For simplicity, for any dyadic cube Q, let $B^Q := x_Q + B_{\ell(Q)\nu+u}$ and $B_Q := x_Q + B_{\ell(Q)\nu-u}$.

On the one hand, when $Q \in \mathrm{II}$, there exists some $x \in Q$ such that

$$\mathcal{M}^\sharp f(x) \le \alpha\beta/2.$$

From this and $Q \subset B^Q$ (see Lemma 3.11(iv)), it follows that

$$\frac{1}{|B^Q|}\int_{B^Q} |f(y) - f_{B^Q}|\,dy \le \alpha\beta/2. \tag{3.22}$$

Moreover, by Lemma 3.12(iii), we conclude that

$$|f_Q| \le C_1\nu = \alpha/2. \tag{3.23}$$

Thus, by Lemma 3.12(iii), the disjoint property of $\{Q_k^\alpha\}_k$, (3.23), Lemma 3.11(iv), and (3.22), we have

$$\begin{aligned}
\sum_{Q_k^\alpha\subset Q} |Q_k^\alpha| &< \frac{1}{\alpha}\sum_{Q_k^\alpha\subset Q}\int_{Q_k^\alpha} |f(y)|\,dy \\
&\le \frac{1}{\alpha}\sum_{Q_k^\alpha\subset Q}\int_{Q_k^\alpha} |f(y) - f_{B^Q}|\,dy \\
&\quad + \frac{1}{\alpha}\sum_{Q_k^\alpha\subset Q} |Q_k^\alpha||f_{B^Q} - f_Q| + \frac{1}{\alpha}|f_Q|\sum_{Q_k^\alpha\subset Q}|Q_k^\alpha| \\
&\le \frac{1}{\alpha}\int_{\bigcup_{Q_k^\alpha\subset Q} Q_k^\alpha} |f(y) - f_{B^Q}|\,dy \\
&\quad + \frac{1}{\alpha}\sum_{Q_k^\alpha\subset Q}|Q_k^\alpha|\frac{1}{|Q|}\int_Q |f(y) - f_{B^Q}|\,dy + \frac{1}{2}\sum_{Q_k^\alpha\subset Q}|Q_k^\alpha| \\
&\le \frac{1}{\alpha}\int_{B^Q} |f(y) - f_{B^Q}|\,dy
\end{aligned}$$

$$+\frac{1}{\alpha}\sum_{Q_k^\alpha\subset Q}\left|Q_k^\alpha\right|\frac{|B^Q|}{|Q|}\frac{1}{|B^Q|}\int_{B^Q}\left|f(y)-f_{B^Q}\right|dy+\frac{1}{2}\sum_{Q_k^\alpha\subset Q}\left|Q_k^\alpha\right|$$

$$\le\frac{\beta}{2}\left|B^Q\right|+\frac{\beta}{2}\sum_{Q_k^\alpha\subset Q}\left|Q_k^\alpha\right|\frac{|B^Q|}{|B_Q|}+\frac{1}{2}\sum_{Q_k^\alpha\subset Q}\left|Q_k^\alpha\right|.$$

From this, it follows that

$$\sum_{Q_k^\alpha\subset Q}\left|Q_k^\alpha\right|<\beta\left|B^Q\right|+\beta b^{2u}\sum_{Q_k^\alpha\subset Q}\left|Q_k^\alpha\right|. \tag{3.24}$$

Choose β small enough such that $\beta b^{2u}<1/2$. By this, the disjoint property of $\{Q_k^\alpha\}_k$, (3.24), and Lemma 3.11(iv), we further have

$$\left|\bigcup_{Q_k^\alpha\subset Q}Q_k^\alpha\right|=\sum_{Q_k^\alpha\subset Q}\left|Q_k^\alpha\right|<\frac{\beta}{1-\beta b^{2u}}\left|B^Q\right|\le\frac{\beta b^{2u}}{1-\beta b^{2u}}|Q|. \tag{3.25}$$

Moreover, by $Q\subset B^Q$ (see Lemma 3.11(iv)), we obtain $\bigcup_{Q_k^\alpha\subset Q}Q_k^\alpha\subset B^Q$. Therefore, by Lemma 3.11(iv) and Lemma 3.1 with $w\in\mathbb{A}_1(A)$, we obtain

$$\frac{w(\bigcup_{Q_k^\alpha\subset Q}Q_k^\alpha)}{w(Q)}\lesssim\frac{w(\bigcup_{Q_k^\alpha\subset Q}Q_k^\alpha)}{w(B_Q)}\lesssim\frac{w(\bigcup_{Q_k^\alpha\subset Q}Q_k^\alpha)}{b^{-2u}w(B^Q)}\lesssim\frac{|\bigcup_{Q_k^\alpha\subset Q}Q_k^\alpha|}{|Q|}.$$

From this, the disjoint property of $\{Q_k^\alpha\}_k$ again, and (3.25), we deduce that

$$\sum_{Q_k^\alpha\subset Q}w\left(Q_k^\alpha\right)=w\left(\bigcup_{Q_k^\alpha\subset Q}Q_k^\alpha\right)\lesssim\frac{|\bigcup_{Q_k^\alpha\subset Q}Q_k^\alpha|}{|Q|}w(Q)\lesssim\frac{\beta b^{2u}}{1-\beta b^{2u}}w(Q).$$

Therefore, we have

$$\sum_{Q\in\mathrm{II}}\sum_{Q_k^\alpha\subset Q}w\left(Q_k^\alpha\right)\lesssim\frac{\beta b^{2u}}{1-\beta b^{2u}}\sum_{Q\in\mathrm{II}}w(Q)\lesssim\frac{\beta b^{2u}}{1-\beta b^{2u}}\mu_f(\nu). \tag{3.26}$$

On the other hand,

$$\sum_{Q\in\mathrm{I}}\sum_{Q_k^\alpha\subset Q}w\left(Q_k^\alpha\right)\le\sum_{Q\in\mathrm{I}}w(Q)\le w\left(\left\{\mathcal{M}^\sharp f>\alpha\beta/2\right\}\right),$$

by which, together with (3.26), we finally finish the proof of Lemma 3.14. □

In the following, for any $\lambda\in(0,\infty)$, let $\mu_f(\lambda,\lambda):=\sum_j\varphi(Q_j^\lambda,\lambda)$, where $Q_j^\lambda:=Q_j$ is as in Lemma 3.12.

Proof of Theorem 2.9 Let φ be a Musielak-Orlicz function satisfying $\varphi(\cdot,t)\in\mathbb{A}_1(A)$ for any $t\in(0,\infty)$ and Assumption (φ)(ii) with $p_\varphi^-\in(1,\infty)$ and $q\in(1,\infty)$ close to 1 such that $1<p_\varphi^-/q$. For any $f\in L_0^\infty$, f is a multiple of $(1,\infty,0)_{\varphi(\cdot,t)}$-atom. From the assumption of

Theorem 2.9, it follows that $Tf \in L^1_{\varphi(\cdot,t)}$. Therefore, for any $\lambda \in (0,\infty)$, by $|f(x)| \le \mathcal{M}_A f(x)$ for any a.e. $x \in \mathbb{R}^n$, Lemma 3.2, Assumption (φ)(ii), and Lemma 3.13, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} \varphi(x, |Tf(x)|)\,dx\\ &\quad\lesssim \int_{\mathbb{R}^n} \varphi(x, \mathcal{M}_A(Tf)(x))\,dx \sim \int_0^\infty \frac{1}{\lambda}\varphi(\{\mathcal{M}_A(Tf) > \lambda\}, \lambda)\,d\lambda\\ &\quad\lesssim \int_0^\infty \frac{1}{\lambda}\varphi(\{\mathcal{M}_A(Tf) > (b^{3\sigma}C_1 + 1)\lambda\}, \lambda)\,d\lambda \lesssim \int_0^\infty \frac{1}{\lambda}\mu_{Tf}(\lambda,\lambda)\,d\lambda,\end{aligned}$$

where C_1 is a positive constant as in Lemma 3.12(iii).

Moreover, by Lemma 3.14 and Assumption (φ)(ii) again, we further have

$$\begin{aligned}&\int_0^\infty \frac{1}{\lambda}\mu_{Tf}(\lambda,\lambda)\,d\lambda\\ &\quad\lesssim \int_0^\infty \frac{1}{\lambda}\varphi(\{\mathcal{M}^\sharp(Tf) > \lambda\beta/2\}, \lambda)\,d\lambda + \frac{\beta b^{2u}}{1-\beta b^{2u}}\int_0^\infty \frac{1}{\lambda}\mu_{Tf}(\lambda,\lambda)\,d\lambda\\ &\quad\lesssim \int_0^\infty \frac{1}{\lambda}\varphi(\{\mathcal{M}^\sharp(Tf) > \lambda\}, \lambda)\,d\lambda + \frac{\beta b^{2u}}{1-\beta b^{2u}}\int_0^\infty \frac{1}{\lambda}\mu_{Tf}(\lambda,\lambda)\,d\lambda.\end{aligned}$$

Taking β small enough such that $\beta b^{2u}/(1-\beta b^{2u}) < 1$, by Lemma 3.2, we obtain

$$\begin{aligned}\int_0^\infty \frac{1}{\lambda}\mu_{Tf}(\lambda,\lambda)\,d\lambda &\lesssim \int_0^\infty \frac{1}{\lambda}\varphi(\{\mathcal{M}^\sharp(Tf) > \lambda\}, \lambda)\,d\lambda\\ &\sim \int_{\mathbb{R}^n} \varphi(x, \mathcal{M}^\sharp(Tf)(x))\,dx.\end{aligned}$$

Therefore, we have

$$\int_{\mathbb{R}^n} \varphi(x, |Tf(x)|)\,dx \lesssim \int_{\mathbb{R}^n} \varphi(x, \mathcal{M}^\sharp(Tf)(x))\,dx. \tag{3.27}$$

For any $\lambda \in (0,\infty)$, let

$$\Omega_\lambda := \{\mathcal{M}^q_{\varphi(\cdot,\lambda)} f > \lambda\},$$

where $\mathcal{M}^q_{\varphi(\cdot,\lambda)} f$ is as in (3.2).

By checking the proof of Theorem 2.5 and using the same notation as in the proof of Theorem 2.5 and an anisotropic variant of the Whitney covering lemma (see [30], Lemma 2.3) associated with Ω_λ, then we may define

$$g(x) := \begin{cases} f(x), & \text{for } x \in \Omega_\lambda^\complement,\\ \sum_i (\pi_{x_i+B_{\ell_i}} f\eta_i)(x)\chi_i(x), & \text{for } x \in \Omega_\lambda,\end{cases}$$

and, for any $x \in \mathbb{R}^n$,

$$h(x) := \sum_i h_i(x), \quad \text{where } h_i(x) := f(x)\chi_i(x) - (\pi_{x_i+B_{\ell_i}} f\eta_i)(x)\chi_i(x).$$

Obviously, for any $x \in \mathbb{R}^n$, we have $f(x) = g(x) + h(x)$.

Moreover, for any $x \in \mathbb{R}^n$, set

$$a_i(x) := \frac{h_i(x)}{C\lambda\varphi(x_i + B_{\ell_i}, \lambda)},$$

where C is a constant as in (3.11), then a_i is a $(1, q, 0)_{\varphi(\cdot,\lambda)}$-atom by checking the proof of Theorem 2.5 with $\varphi(\cdot, \lambda) \in \mathbb{A}_1(A) \subset \mathbb{A}_q(A)$. Therefore, we have

$$h(x) = C\lambda \sum_i \varphi(x_i + B_{\ell_i}, \lambda)a_i(x) \in H^{1,q,0}_{\varphi(\cdot,\lambda)}(A)$$

and

$$\|h\|_{H^{1,q,0}_{\varphi(\cdot,\lambda)}(A)} \lesssim C\lambda \sum_i \varphi(x_i + B_{\ell_i - 2\sigma}, \lambda) \lesssim C\lambda\varphi(\Omega_\lambda, \lambda). \tag{3.28}$$

By $|g(x)| \lesssim C\lambda$ for a.e. $x \in \mathbb{R}^n$ (see (3.12)) and (2.11), we obtain

$$\|Tg\|_{\mathrm{BMO}(A)} \lesssim C\lambda \quad \textit{i.e.} \quad \mathcal{M}^\sharp(Tg)(x) \lesssim C\lambda \quad \text{for a.e. } x \in \mathbb{R}^n.$$

From this and T being a sublinear operator, it follows that

$$\begin{aligned}
&\varphi\bigl(\bigl\{\mathcal{M}^\sharp(Tf) > (C+1)\lambda\bigr\}, \lambda\bigr) \\
&\quad \leq \varphi\bigl(\bigl\{\mathcal{M}^\sharp(Tg) > C\lambda\bigr\}, \lambda\bigr) + \varphi\bigl(\bigl\{\mathcal{M}^\sharp(Th) > \lambda\bigr\}, \lambda\bigr) \\
&\quad = \varphi\bigl(\bigl\{\mathcal{M}^\sharp(Th) > \lambda\bigr\}, \lambda\bigr).
\end{aligned} \tag{3.29}$$

Thus, by Lemma 3.2, the uniformly upper type p_φ^+ property of φ, and (3.29), we see that

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi\bigl(x, \mathcal{M}^\sharp(Tf)(x)\bigr)\, dx &\sim \int_0^\infty \frac{1}{\lambda}\varphi\bigl(\bigl\{\mathcal{M}^\sharp(Tf) > (C+1)\lambda\bigr\}, (C+1)\lambda\bigr)\, d\lambda \\
&\lesssim \int_0^\infty \frac{1}{\lambda}\varphi\bigl(\bigl\{\mathcal{M}^\sharp(Th) > \lambda\bigr\}, \lambda\bigr)\, d\lambda.
\end{aligned}$$

From this and (3.27), it follows that

$$\int_{\mathbb{R}^n} \varphi\bigl(x, |Tf(x)|\bigr)\, dx \lesssim \int_0^\infty \frac{1}{\lambda}\varphi\bigl(\bigl\{\mathcal{M}^\sharp(Th) > \lambda\bigr\}, \lambda\bigr)\, d\lambda. \tag{3.30}$$

Since $f \in L_0^\infty$, we have $f \in L^1_{\mathrm{loc}}$. Therefore, for any $x \in \mathbb{R}^n$, by $\mathcal{M}^\sharp f(x) \lesssim 2\mathcal{M}_A f(x)$, Assumption (φ)(ii), the boundedness of $\mathcal{M}_A$ from $L^1_{\varphi(\cdot,\lambda)}$ to weak $L^1_{\varphi(\cdot,\lambda)}$ with $\varphi(\cdot, \lambda) \in \mathbb{A}_1(A)$ (see [30], Proposition 2.6(ii)), (2.10), $H^1_{\varphi(\cdot,\lambda)}(A) = H^{1,q,0}_{\varphi(\cdot,\lambda)}(A)$ with equivalent norms (see [30], Theorem 5.5), (3.28), Lemma 3.2, and (3.14) with $1 = q(\varphi) < p_\varphi^-/q$, we obtain

$$\begin{aligned}
&\int_0^\infty \frac{1}{\lambda}\varphi\bigl(\bigl\{\mathcal{M}^\sharp(Th) > \lambda\bigr\}, \lambda\bigr)\, d\lambda \\
&\quad \lesssim \int_0^\infty \frac{1}{\lambda}\varphi\left(\left\{\mathcal{M}_A(Th) > \frac{\lambda}{2}\right\}, \lambda\right) d\lambda \lesssim \int_0^\infty \frac{1}{\lambda}\left(\frac{\|Th\|_{L^1_{\varphi(\cdot,\lambda)}}}{\lambda}\right) d\lambda
\end{aligned}$$

$$\begin{aligned}
&\lesssim \int_0^\infty \frac{1}{\lambda}\left(\frac{\|h\|_{H^{1,q,0}_{\varphi(\cdot,\lambda)}(A)}}{\lambda}\right) d\lambda \lesssim \int_0^\infty \frac{1}{\lambda}\varphi(\Omega_\lambda,\lambda)\,d\lambda \\
&\sim \int_0^\infty \frac{1}{\lambda}\int_{\{\mathcal{M}^q_{\varphi(\cdot,\lambda)}f>\lambda\}} \varphi(x,\lambda)\,dx\,d\lambda \sim \int_{\mathbb{R}^n} \varphi\bigl(x,\mathcal{M}^q_{\varphi(\cdot,\lambda)}f(x)\bigr)\,dx \\
&\lesssim \int_{\mathbb{R}^n} \varphi\bigl(x,\bigl|f(x)\bigr|\bigr)\,dx,
\end{aligned}$$

which, together with (3.30) and Lemma 3.10, completes the proof of Theorem 2.9. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BL conceived of the study. JL, RS, and BL carried out the two anisotropic interpolation theorems of Musielak-Orlicz type, participated in the sequence alignment, and drafted the manuscript. Moreover, all authors read and approved the final manuscript.

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