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Estimation of inequalities for warped product semi-slant submanifolds of Kenmotsu space forms

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Abstract
In this paper, we construct the geometric inequalities for the squared norm of the mean curvature and warping functions of warped product semi-slant submanifolds in Kenmotsu space forms. The equality cases are also discussed.

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Keywords: warped products; semi-slant submanifold; inequalities; Kenmotsu space forms

1 Introduction

The theory of warped product manifolds is an emerging research area in differential geometry. The idea of a warped product manifold was first discovered by Bishop and O'Neil (*cf.* [1]) as a manifold of negative curvature. They defined the manifolds based on M_1 and M_2, which are the two Riemannian manifolds of dimensions n_1 and n_2 endowed with Riemannian matrices g_1 and g_2 such that $f : M_1 \to (0,\infty)$ be a positive differentiable function on M_1. Thus, the warped product $M = M_1 \times_f M_2$ is defined based on the product manifold $M_1 \times M_2$ equipped with a metric $g = g_1 + f^2 \cdot g_2$. Moreover, if we consider that $\gamma_1 : M_1 \times M_2 \to M_1$ and $\gamma_2 : M_1 \times M_2 \to M_2$ are the natural projections on M_1 and M_2, respectively, then the metric g on a warped product is defined as

$$\|X\|^2 = \left\|\gamma_1^*(X)\right\|^2 + (fo\gamma_1)^2\left\|\gamma_2^*(X)\right\|^2, \tag{1.1}$$

for any X tangent to TM. The function f is called the warping function. If $f = 1$, then M is called a simply Riemannian product manifold. In contrast, M is denoted a non-trivial warped product manifold when $f \neq 1$. Let $M = M_1 \times_f M_2$ be a non-trivial warped product manifold of an arbitrary Riemannian manifold $\widetilde{M}$. Then

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z, \tag{1.2}$$

for any vector fields $X \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$. Further, ∇ is a Levi-Citvita connection of the induced Riemannian manifold M.

The approach of such type inequalities for warped products in almost Hermitian and almost contact metric manifolds has been an important field for a few decades. Especially, Chen in [2] obtained the sharp relationship between norm of the squared mean curvature and the warping function f of the warped product $M_1 \times_f M_2$ isometrically immersed in a real space form, *i.e.*, we have the following.

Theorem 1.1 *Let $\phi : M_1 \times_f M_2$ be an isometrically immersion of an n-dimensional warped product into $2m$-dimensional real space form $\widetilde{M}(c)$ with constant sectional curvature c. Then*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1 \cdot c,$$

where $n_i = \dim M_i$, $i = 1, 2$, and ∇ is the Lapalcian operator of M_1. Moreover, the equality holds in the above if and only if ϕ is a mixed totally geodesic and $n_1 H_1 = n_2 H_2$ such that H_1 and H_2 are partial mean curvatures.

However, on the base of the literature, we find that several inequalities have been extended to various structures for warped products by many geometers in [3–9]. Therefore other inequalities also appear, in [4, 5, 10–19] for slant submanifolds and semi-slant submanifolds in different curvature forms, which are called Chen inequalities. In addition, it is well known that Atceken [20] studied the non-existence of the warped product semi-slant submanifolds of a Kenmotsu manifold such that structure vector ξ is tangent to the fiber. Meanwhile, Uddin in [21] and Srivastava in [22] proved that the warped product semi-slant submanifold of a Kenmotsu manifold exists in the forms $M = M_T \times_f M_\theta$ and $M = M_\theta \times_f M_T$, except in the case when the structure vector field ξ is tangent to M_T and M_θ, respectively. Moreover, we have studied some inequalities that Cioroboiu [13] and Aktan *et al.* [12] obtained for semi-slant submanifolds by constructing its orthonormal frame but overlooking the suitable conditions for inequalities of a warped semi-slant product. Therefore, one needs to derive the inequalities for the mean curvature and warping functions with slant angles of a warped semi-slant product in Kemotsu space form. In the current paper, we are extending studies like [13] for warped product semi-slant submanifolds in a Kenmotsu space form. We also generalize some other inequalities for CR-warped product submanifolds in special cases because of the warped product of semi-slant generalized CR-warped products in Kenmotsu manifolds. Moreover, the equality cases and geometric inequalities applications related to Wireless Sensor Network are also discussed.

2 Preliminaries

An odd $(2m + 1)$-dimensional smooth manifold $\widetilde{M}$ is called a *Kenmotsu* manifold, if it is consisting in an endomorphism φ of its tangent bundle $T\widetilde{M}$, a structure vector field ξ, and a 1-form η satisfying the following:

$$\varphi^2 = -I + \eta \oplus \xi, \qquad \eta(\xi) = 1, \qquad \eta o \varphi = 0, \tag{2.1}$$

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \qquad \eta(U) = g(U, \xi), \tag{2.2}$$

and the structure equation is given by

$$(\widetilde{\nabla}_U \varphi)V = g(\varphi U, V) - \eta(V)\varphi U, \tag{2.3}$$

$$\widetilde{\nabla}_U \xi = U - \eta(U)\xi, \tag{2.4}$$

for any U, V tangent to $\widetilde{M}$ (see [23]). The curvature tensor $\widetilde{R}$ for Kenmotsu space forms is defined as

$$\begin{aligned} \widetilde{R}(X,Y,Z,W) = {} & \frac{c-3}{4}\bigl\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\bigr\} \\ & + \frac{c+1}{4}\bigl\{g(X,\varphi W)g(Y,\varphi Z) - g(X,\varphi Z)g(Y,\varphi W) \\ & - 2g(X,\varphi Y)g(Z,\varphi W) - g(X,W)\eta(Y)\eta(Z) \\ & + g(X,Z)\eta(X)\eta(W) - g(Y,Z)\eta(X)\eta(W) \\ & + g(Y,W)\eta(X)\eta(Z)\bigr\}, \end{aligned} \tag{2.5}$$

where c is a function of the constant φ-sectional curvature of $\widetilde{M}$ (see [3].

Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$ with induced metric g; if ∇ and $\nabla^{\perp}$ are the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively, then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_U V = \nabla_U V + h(U,V), \tag{2.6}$$

$$\widetilde{\nabla}_U N = -A_N U + \nabla^{\perp}_U N, \tag{2.7}$$

for each $U, V \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively, for the immersion of M into $\widetilde{M}$. They are related as

$$g\bigl(h(U,V),N\bigr) = g(A_N U, V), \tag{2.8}$$

where g denotes the Riemannian metric on $\widetilde{M}$ as well as the metric induced on M. Moreover, for a submanifold M, the Gauss equation is defined as

$$\begin{aligned} \widetilde{R}(U,V,Z,W) = {} & R(U,V,Z,W) + g\bigl(h(U,Z),h(V,W)\bigr) \\ & - g\bigl(h(U,W),h(V,Z)\bigr), \end{aligned} \tag{2.9}$$

for any $U, V, Z, W \in \Gamma(TM)$, where $\widetilde{R}$ and R are the curvature tensors on $\widetilde{M}$ and M, respectively. The mean curvature vector H for an orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ of the tangent space TM on M is defined by

$$H = \frac{1}{n}\operatorname{trace}(h) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{2.10}$$

where $n = \dim M$. In addition, we set

$$h^r_{ij} = g\bigl(h(e_i,e_j),e_r\bigr) \quad \text{and} \quad \|P\|^2 = \sum_{i,j=1}^{n} g^2(\varphi e_i, e_j). \tag{2.11}$$

Furthermore, the scalar curvature ρ for a submanifold M of an almost contact manifold $\widetilde{M}$ is given by

$$\rho = \sum_{1\leq i\neq j\leq n} K(e_i \wedge e_j), \tag{2.12}$$

where $K(e_i \wedge e_j)$ is the sectional curvature of plane section spanned by e_i and e_j. Let G_r be a r-plane section on TM and $\{e_1, e_2, \dots, e_r\}$ any orthonormal basis of G_r. Then the scalar curvature $\rho(G_r)$ of G_r is given by

$$\rho(G_r) = \sum_{1\leq i\neq j\leq r} K(e_i \wedge e_j). \tag{2.13}$$

Let $\widetilde{M}$ be a Kenmotsu manifold with an almost contact structure (φ, ξ, η) and M be a submanifold tangent to the structure vector field ξ isometrically immersed in $\widetilde{M}$. Then M is called invariant if $\varphi(T_pM) \subseteq T_pM$, and M is called anti-invariant if $\varphi(T_pM) \subset T_p^{\perp}M$ for every $p \in M$ where T_pM denotes the tangent bundle of M at the point p. Moreover, M is called a slant submanifold if all non-zero vectors U tangent to M at a point p, the angle of $\theta(U)$ between φU and T_pM are constant, *i.e.*, they do not depend on the choice of $p \in M$ and $U \in \Gamma(T_pM - \langle\xi(p)\rangle)$ (see [24]). Except invariant, anti-invariant, and slant submanifolds, there are several other classes of submanifolds determined by the behavior of the tangent space of the submanifold under the action of a one-one tensor field φ of an ambient manifold.

Recently, Cabrerizo *et al.* [24] extended of the mentioned definition into a characterization for a slant submanifold in a contact metric manifold. In fact, they have obtained the following theorem.

Theorem 2.1 *Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$ such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2 = \delta(-I + \eta \otimes \xi). \tag{2.14}$$

Furthermore, in such a case, if θ is a slant angle, then it satisfies $\delta = \cos^2\theta$.

Hence, we have the following relations which are consequences of Theorem 2.1, *i.e.*,

$$g(PU, PV) = \cos^2\theta\big\{g(U,V) - \eta(U)\eta(V)\big\}, \tag{2.15}$$

$$g(FU, FV) = \sin^2\theta\big\{g(U,V) - \eta(U)\eta(V)\big\}. \tag{2.16}$$

There is another class, which is called a semi-slant submanifold. The notion of semi-slant submanifolds were defined and studied by Papaghiuc in [25] as a natural generalization of CR-submanifolds of almost Hermitian manifolds in terms of the slant distribution and it was later extended to the setting of contact manifolds by Cabrerizo [26]. One defined these submanifolds as follows.

Definition 2.2 Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$. Then M is said to be a semi-slant submanifold if there exist two orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ such that

(i) $TM = \mathcal{D} \oplus \mathcal{D}^\theta \oplus \langle \xi \rangle$ where $\langle \xi \rangle$ is a 1-dimensional distribution spanned by ξ:
(ii) $\mathcal{D}$ is invariant, *i.e.*, $\varphi(\mathcal{D}) \subseteq \mathcal{D}$;
(iii) $\mathcal{D}^\theta$ is slant distribution with slant angle $\theta \neq 0, \pi/2$.

Assume that $\phi : M = M_1 \times_f M_2 \to \widetilde{M}$ is an isometric immersion of a warped product $M_1 \times_f M_2$ into a Riemannian manifold of $\widetilde{M}$ of constant section curvature c. Suppose that n_1, n_2, and n are the dimensions of M_1, M_2, and $M_1 \times_f M_2$, respectively. Then for unit vector fields X, Z tangent to M_1, M_2, respectively,

$$\begin{aligned} K(X \wedge Z) &= g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) \\ &= \frac{1}{f}\bigl\{(\nabla_X X)f - X^2 f\bigr\}. \end{aligned} \tag{2.17}$$

Let us assume a local orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ such that $e_1, e_2, \ldots, e_{n_1}$ tangent to M_1 and $e_{n_1+1}, \ldots, e_n$ are tangent to M_2. Then

$$\sum_{1 \leq i \leq n_1} \sum_{n_1+1 \leq j \leq n} K(e_i \wedge e_j) = \frac{n_2 \cdot \Delta f}{f}. \tag{2.18}$$

Lemma 2.3 [27] *Let $a_1, a_2, \ldots, a_n, a_{n+1}$ be $n+1$ $(n \geq 2)$ be real number such that*

$$\left(\sum_{i=1}^n a_i\right)^2 = (n-1)\left(\sum_{i=1}^n a_i^2 + a_{n+1}\right).$$

Then $2a_1 \cdot a_2 \geq a_3$ with the equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_k$.

3 Main inequalities

In this section, as applications of very famous studied of Nolker in [28], we obtain the following inequality for warped product semi-slant submanifolds of Kenmotsu space form such that ξ is tangent to the first factor of the warped product, *i.e.*, we have the following.

Theorem 3.1 *Assume that $\phi : M = M_T \times_f M_\theta \to \widetilde{M}(c)$ is an isometric immersion from a warped product semi-slant $M_T \times_f M_\theta$ into a Kenmotsu space form $\widetilde{M}(c)$ such that c is a φ-sectional constant curvature and ξ is tangent to M_T. Then*:

(i) *The relation between warping function and the squared norm of mean curvature is obtained*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + \frac{c-3}{4}n_1 - \frac{c+1}{4n_2}\bigl[3d_1 + d_2\bigl(2 + 3\cos^2\theta\bigr)\bigr], \tag{3.1}$$

where $n_i = \dim M_i, i = T, \theta$, and Δ is the Laplacian operator on M_T.

(ii) *The equality case holds in* (3.1) *if and only if $n_1 \cdot H_T = n_2 \cdot H_\theta$, where H_T and H_θ are partially mean curvature vectors on M_T and M_θ, respectively. Moreover, ϕ is a mixed totally geodesic immersion.*

Proof Let $M_T \times_f M_\theta$ be a warped product semi-slant submanifold in Kenmotsu space form $\widetilde{M}(c)$. Based on the Gauss formula (2.9) and (2.5), we derive

$$\begin{aligned} 2\rho &= \frac{c-3}{4}n(n-1) + \frac{(c+1)}{4}\left[3\sum_{1\leq i\neq k\leq n} g^2(Pe_i, e_k) - 2(n-1)\right] \\ &\quad + n^2\|H\|^2 - \|h\|^2. \end{aligned} \tag{3.2}$$

As well we are concerned that M is a proper semi-slant submanifold of Kenmotsu space form $\widetilde{M}(c)$. Thus, we define the following frame according to Cioroboiu in [13], *i.e.*,

$$\begin{aligned} &e_1, e_2 = \varphi e_1, \ldots, e_{2d_1-1}, e_{2d_1} = \varphi e_{2d_1-1}, \\ &e_{2d_1+1}, e_{2d_1+2} = \sec\theta Pe_{2d_1+1}, \ldots \quad e_{2d_1+2d_2-1}, e_{2d_1+2d_2} = \sec\theta Pe_{2d_1+2d_2-1}, \\ &e_{2d_1+2d_2+1} = \xi, \end{aligned}$$

Clearly, we derive

$$g^2(\varphi e_i, e_{i+1}) = \begin{cases} 1, & \text{for } i \in \{1,2,\ldots,2d_1-1\}, \\ \cos^2\theta, & \text{for } i \in \{2d_1+1,\ldots,2d_1+2d_2-1\}. \end{cases} \tag{3.3}$$

From (3.2) and (3.3), it follows that

$$\begin{aligned} 2\rho &= \frac{c-3}{4}n(n-1) + \frac{(c+1)}{4}\left[d_1 + d_2 \cdot \left(3\cos^2\theta - 2\right)\right] \\ &\quad + n^2\|H\|^2 - \|h\|^2. \end{aligned} \tag{3.4}$$

Now we consider

$$\delta = 2\rho - \frac{c-3}{4}n(n-1) - \frac{(c+1)}{2}\left[d_1 + d_2 \cdot \left(3\cos^2\theta - 2\right)\right] - \frac{n}{2}\|H\|^2. \tag{3.5}$$

Then, from (3.4) and (3.5),

$$n^2\|H\|^2 = 2\left(\delta - \|h\|^2\right). \tag{3.6}$$

Thus for a locally orthonormal frame $\{e_1, e_2, \ldots, e_n\}$, equation (3.6) takes the form

$$\left(\sum_{r=n+1}^{2m+1}\sum_{i=1}^{n} h_{ii}^r\right)^2 = 2\left(\delta + \sum_{r=n+1}^{2m+1}\sum_{i=1}^{n}\left(h_{ii}^r\right)^2 + \sum_{r=n+1}^{2m+1}\sum_{i<j=1}^{n}\left(h_{ij}^r\right)^2 + \sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{n}\left(h_{ij}^r\right)^2\right), \tag{3.7}$$

which implies that

$$\begin{aligned} \frac{1}{2}\left(h_{11}^{n+1} + \sum_{i=2}^{n_1} h_{ii}^{n+1} + \sum_{t=n_1+1}^{n} h_{tt}^{n+1}\right)^2 &= \delta + \left(h_{11}^{n+1}\right)^2 + \sum_{i=2}^{n_1}\left(h_{ii}^{n+1}\right)^2 \\ &\quad + \sum_{t=n_1+1}^{n}\left(h_{tt}^{n+1}\right)^2 - \sum_{2\leq j\neq l\leq n_1} h_{jj}^{n+1}h_{ll}^{n+1} \end{aligned}$$

$$- \sum_{n_1+1\le t\neq s\le n} h_{tt}^{n+1}h_{ss}^{n+1} + \sum_{i<j=1}^{n}\left(h_{ij}^{n+1}\right)^2 + \sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{n}\left(h_{ij}^{r}\right)^2. \tag{3.8}$$

Now consider that $a_1 = h_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$, and $a_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}$ and applying Lemma 2.3 in (3.8). Then we derive

$$\frac{\delta}{2} + \sum_{i<j=1}^{n}\left(h_{ij}^{n+1}\right)^2 + \frac{1}{2}\sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{n}\left(h_{ij}^{r}\right)^2 \le \sum_{2\le j\neq l\le n_1} h_{jj}^{n+1}h_{ll}^{n+1} + \sum_{n_1+1\le t\neq s\le n} h_{tt}^{n+1}h_{ss}^{n+1}, \tag{3.9}$$

with equality holding in (3.9) if and only if

$$\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}. \tag{3.10}$$

Further, (2.12) and (2.18) imply that

$$\frac{n_2\cdot\Delta f}{f} = \rho - \sum_{1\le j\neq k\le n_1} K(e_i\wedge e_k) - \sum_{n_1+1\le t\neq s\le n} K(e_t\wedge e_s). \tag{3.11}$$

From (2.5) we obtain

$$\begin{aligned}\frac{n_2\cdot\Delta f}{f} &= \rho - \frac{c-3}{8}n_1(n_1-1) + \frac{(c+1)}{4}(n_1-1)\\ &\quad - \frac{3(c+1)}{4}\sum_{1\le i\neq k\le n_1} g^2(Pe_i,e_k) - \sum_{r=1}^{2m+1}\sum_{2\le j\neq k\le n_1}\left(h_{jj}^{r}h_{kk}^{r} - \left(h_{jk}^{r}\right)^2\right)\\ &\quad - \frac{3(c+1)}{4}\sum_{n_1+1\le t\neq s\le n} g^2(Pe_t,e_s) - \frac{c-3}{8}n_2(n_2-1)\\ &\quad - \sum_{r=1}^{2m+1}\sum_{n_1+1\le t\neq s\le n}\left(h_{tt}^{r}h_{ss}^{r} - \left(h_{ts}^{r}\right)^2\right).\end{aligned} \tag{3.12}$$

Thus, from (3.9) and (3.12), it is easily observed that

$$\begin{aligned}\frac{n_2\cdot\Delta f}{f} &\le \rho - \frac{c-3}{8}n(n-1) + \frac{c-3}{4}n_1\cdot n_2 + \frac{c+1}{4}(n_1-1)\\ &\quad - \frac{3(c+1)}{4}(n_1-1) - \frac{3(c+1)}{4}n_2\cdot\cos^2\theta - \frac{\delta}{2}.\end{aligned} \tag{3.13}$$

Hence, using (3.6), then the inequality (3.13) reduces to

$$\frac{n_2\cdot\Delta f}{f} \le \frac{n^2}{4}\|H\|^2 + \frac{c-3}{4}n_1\cdot n_2 + \frac{c+1}{4}\left(-3d_1 - 3d_2\cos^2\theta - 2d_2\right). \tag{3.14}$$

This implies the inequality (3.1). The equality sign holds in (3.1) if and only if the terms left in (3.9), and (3.10) imply that

$$\sum_{r=n+2}^{2m+1}\sum_{j=1}^{n_1} h_{jj}^r = \sum_{r=n+2}^{2m+1}\sum_{i=n_1+1}^{n} h_{ii}^r = 0, \tag{3.15}$$

and $n_1 \cdot H_T = n_2 \cdot H_\theta$, where H_T and H_θ are partially mean curvature vectors on M_T and M_θ, respectively. Moreover, from (3.9), we find that

$$h_{ij}^r = 0, \quad \text{for each } 1 \le i \le n_1, n_1 + 1 \le j \le n, n + 1 \le r \le 2m + 1. \tag{3.16}$$

It means that ϕ is a mixed totally geodesic immersion. But the converse of (3.16) may not be true in a warped semi-slant product in Kenmotsu space form. Thus the proof of the theorem is completed. □

Theorem 3.2 *Let $\phi : M = M_\theta \times_f M_T \to \widetilde{M}(c)$ be an isometric immersion of a warped product semi-slant $M_\theta \times_f M_T$ into a Kenmotsu space form $\widetilde{M}(c)$ such that ξ is tangent to M_θ. Then*

(i) *The relation between warping function and the norm of the squared mean curvature is given by*

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2}\|H\|^2 + \frac{c-3}{4}n_1 - \frac{c+1}{4n_2}\left(3d_2 + d_1\left\{2 + 3\cos^2\theta\right\}\right), \tag{3.17}$$

where $n_i = \dim M_i, i = T, \theta$, and Δ is the Laplacian operator on M_θ.

(ii) *The equality case holds in* (3.17) *if and only if $n_1 \cdot H_T = n_2 \cdot H_\theta$, where H_T and H_θ are partially mean curvature vector fields on M_T and M_θ, respectively, and ϕ is a mixed totally geodesic immersion.*

Proof The proof of Theorem 3.2 is similar to Theorem 3.1 by reversing and considering that the structure vector field ξ is normal to the fiber. □

In the sense of Papaghiuc, *i.e.*, the generalization of semi-slant submanifolds, we directly obtain the following corollaries by using Theorem 3.1, Theorem 3.2, and $\theta = \frac{\pi}{2}$.

Corollary 3.3 *Assume that $\phi : M = M_T \times_f M_\perp \to \widetilde{M}(c)$ is an isometric immersion of a CR-warped product $M_T \times_f M_\perp$ into a Kenmotsu space form $\widetilde{M}(c)$ with c a φ-sectional constant curvature such that ξ is tangent to M_T. Then*

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2}\|H\|^2 + \frac{c-3}{4}n_1 - \frac{c+1}{4n_2}(3d_1 + 2d_2), \tag{3.18}$$

where $n_i = \dim M_i, i = T, \perp$, and Δ is the Laplacian operator on M_T.

Corollary 3.4 *Let $\phi : M = M_\perp \times_f M_T \to \widetilde{M}(c)$ be an isometric immersion of a CR-warped product submanifold $M_\perp \times_f M_T$ into a Kenmotsu space form $\widetilde{M}(c)$ such that ξ is tangent* *to $M_\perp$. Then*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + \frac{c-3}{4}n_1 - \frac{c+1}{4n_2}(3d_2 + 2d_1), \tag{3.19}$$

where $n_i = \dim M_i$, $i = T, \perp$, and Δ is the Laplacian operator on $M_\perp$.

4 Applications

Any geometric inequality reflects a free or constrained optimum problem with suitable strategies for improved bandwidth management in wireless communications due to the dynamically changing traffic conditions and network performance. Except, some applications of geometric inequalities can be found in Wireless Sensor Networks related to Power balanced coverage-time optimization and Coverage by randomly deployed sensors [29]. Therefore, some applications of geometric inequalities can be found in computer sciences.

Competing interests
The authors declared that they have no competing interests.

Authors' contributions
All authors have equally contributed in this work. All authors read and approved the final manuscript.

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