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# A modified feasible semi-smooth asymptotically Newton method for nonlinear complementarity problems

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## Abstract

In this paper, a modified feasible semi-smooth asymptotically Newton method for nonlinear complementarity problems is proposed. The global convergence of the method and superlinear convergence are proved under some suitable assumptions. Numerical experiments are included to highlight the efficacy of the modified algorithm.

**MSC:** 90C33

**Keywords:** complementarity problems; asymptotically Newton method; global convergence; superlinear convergence

## 1 Introduction

In this paper we are concerned with nonlinear complementarity problems (NCPs)

$$x \geq 0, \quad F(x) \geq 0, \quad x^T F(x) = 0, \quad (1)$$

where the function  $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is continuously differentiable on  $\mathbb{R}^n$ .

As we know, the nonsmoothing Newton method is one of the important methods for solving NCPs. Recently, there have appeared lots of studies having a strong interest in semi-smoothing Newton method and feasible methods [1–3]. Sun, Robert, and Qi have proposed a feasible semi-smooth asymptotically Newton method for mixed complementarity problems [4]. In this paper, we propose a modified method based on the algorithm in [4], which combines semi-smoothness with feasibility. In [4], one takes the projected gradient direction as the well-defined criterion, that is,

$$\Psi(x_k + \bar{d}_k(\rho^m)) \leq \Psi(x_k) + \sigma \nabla \Psi(x_k)^T \bar{d}_G^k(\rho^m).$$

In this paper, we replace the projected gradient direction with the projected Newton direction, then the criterion is changed into

$$\Psi(x_k + \bar{d}_k(\rho^m)) \leq \Psi(x_k) + \sigma \nabla \Psi(x_k)^T \bar{d}_N^k(\rho^m).$$

The algorithm with a new criterion is proved to have the same properties as the old one under some suitable assumptions.

It is easy to verify that  $x$  is a solution of (1) if and only if it is an optimal solution of the following problems with zero objective value:

$$\begin{aligned} \min \quad & \Psi(x) \\ \text{s.t.} \quad & x \in \mathbb{R}_+^n, \end{aligned} \quad (2)$$

where  $\mathbb{R}_+^n := \{x \in \mathbb{R}^n | x \geq 0\}$ ,  $\Psi(x) = \frac{1}{2} \|\Phi(x)\|^2$ , and  $\Phi(x) := (\phi(F_1(x), x_1), \dots, \phi(F_n(x), x_n))^T$ . Here  $\phi$  is an NCP function. A function  $\phi(a, b) : \mathbb{R}^2 \rightarrow \mathbb{R}$  called an NCP function if

$$a \geq 0, b \geq 0, ab = 0 \Leftrightarrow \phi(a, b) = 0.$$

The functions  $\Psi$ ,  $\Phi$  are assumed to have the following properties:

(H1) the function  $\Phi$  is semi-smooth,

(H2) the function  $\Psi$  is continuously differentiable on  $\mathbb{R}^n$ .

In [4], one combines the projected Newton direction  $\bar{d}_N(\lambda)$  and the projected gradient direction  $\bar{d}_G(\lambda)$  into the design of algorithms, where these two directions are defined as follows:

$$\bar{d}_N(\lambda) = \Pi_{\mathbb{R}_+^n}(x + \lambda d_N) - x, \quad \bar{d}_G(\lambda) = \Pi_{\mathbb{R}_+^n}(x + \lambda d_G) - x.$$

The direction  $d_N$  is a solution (if it exists) of the equation  $\Phi(x) + Vd = 0$ , where  $V \in \partial_B \Phi(x)$ , and  $d_G = -\gamma \nabla \Psi(x)$ . The generalized Jacobian  $\partial_B \Phi(x)$  is in the sense of Clarke [5] and we can note that  $\nabla \Psi(x) = V^T \Phi(x)$ . Then Sun, Robert, and Qi [4] defined the new direction

$$\bar{d}(\lambda) = t^*(\lambda) \bar{d}_G(\lambda) + (1 - t^*(\lambda)) \bar{d}_N(\lambda),$$

where, for any fixed  $\lambda \in [0, 1]$ ,  $t^*(\lambda) \in [0, 1]$  is an optimal solution of the convex quadratic programming problems

$$\min_{t \in [0, 1]} \frac{1}{2} \|\Phi(x) + V[t \bar{d}_G(\lambda) + (1 - t) \bar{d}_N(\lambda)]\|^2. \quad (3)$$

The remainder of the paper is organized as follows: In the next section we state the modified algorithm and some useful results which will be used in subsequent analysis. In Section 3, we analysis the convergence of the algorithm described in the Section 2. Some numerical experiments are report in Section 4.

## 2 The algorithm and preliminaries

In order to analyze the convergence of algorithm, we describe some lemmas that are used in our subsequent analysis. A function  $F$  is said to be BD-regular at  $x$  if the generalized Jacobian  $V \in \partial_B F(x)$  is nonsingular. The concept of semi-smoothness was first introduction by Mifflin [6], and it then was extended by Qi and Sun [7]. We can obtain the properties of semi-smooth function from [8].

**Lemma 1** Let  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a locally Lipschitz function, if  $\Phi$  is semi-smooth at  $x$ , for any  $h \rightarrow 0$  and  $V \in \partial_B \Phi(x + h)$ ,

$$\Phi(x + h) - \Phi(x) - Vh = o(\|h\|).$$

**Lemma 2** Let  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a locally Lipschitz function. If  $\Phi$  is BD-regular at a solution  $x^*$  of  $\Phi(x^*) = 0$ , and  $\Phi$  is semi-smooth at  $x^*$ , then there exist a neighborhood  $\mathcal{N}(x^*)$  of  $x^*$  and a constant  $\kappa$  such that for any  $x \in \mathcal{N}(x^*)$

$$\|\Phi(x)\| \geq \kappa \|x - x^*\|.$$

**Lemma 3** Let  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a locally Lipschitz function. Suppose that  $\Phi$  is BD-regular at  $x \in \mathbb{R}^n$ , then there exist a neighborhood  $\mathcal{N}(x)$  of  $x$  and a positive constant  $M$  such that for any  $y \in \mathcal{N}(x)$  and  $V \in \partial_B \Phi(y)$ ,  $V$  is nonsingular and  $\|V^{-1}\| \leq M$ .

The next lemma summarizes the properties of the projection operator, which are very important in our subsequence analysis.

**Lemma 4** The projection operator  $\Pi_X(\cdot)$  satisfies

- (a)  $\|\Pi_X(y) - \Pi_X(z)\| \leq \|y - z\|$  for any  $y, z \in \mathbb{R}^n$ .
- (b) For each  $y \in \mathbb{R}^n$ ,  $(\Pi_X(y) - y)^T (\Pi_X(y) - x) \leq 0$  for any  $x \in X$ .
- (c) For each  $y, z \in \mathbb{R}^n$ ,  $\|\Pi_X(y) - \Pi_X(z)\|^2 \leq (y - z)^T (\Pi_X(y) - \Pi_X(z))$ .

Here  $X$  is a nonempty closed convex subset of  $\mathbb{R}_+^n$ .

It is shown in Lemma 4 that the projection operator  $\Pi_X(\cdot)$  is nonexpansive, that is, we have the property (a), thus the projection operator  $\Pi_X(\cdot)$  is globally Lipschitz continuous on  $X$ . The proof for detail and the more properties about the projection operator can be found in [9]. Here a direction  $d$  is said to be a descent direction of the function  $f(x)$  at  $x$  if and only if  $\nabla f(x)^T d < 0$ . The following two lemmas are very important in the proof of convergence and superlinear convergence, the proof of these two lemmas can be found in [4].

**Lemma 5** Suppose that  $\Phi$  is BD-regular at a solution  $x^*$  of  $\Phi(x^*) = 0$ .  $\Phi$  is semi-smooth at  $x^*$ , then for any  $\rho \in (0, 2)$ , there exist a neighborhood  $\mathcal{N}$  of  $x^*$  such that for any  $\lambda \in (0, 1]$  and  $x \in \mathcal{N} \cap \mathbb{R}_+^n$ ,  $\bar{d}_N(\lambda)$  is a descent direction of  $\Psi$  at  $x$  with

$$\nabla \Psi(x)^T \bar{d}_N(\lambda) \leq -\rho \lambda \Psi(x),$$

$$\bar{d}_N(\lambda) = -\lambda(x - x^*) + \lambda o(\Psi(x)^{\frac{1}{2}}).$$

**Lemma 6** Suppose that  $\Phi$  is BD-regular at a solution  $x^*$  of  $\Phi(x^*) = 0$ ,  $\eta \in (0, 1)$ .  $\Phi$  is semi-smooth at  $x^*$ . Then for any  $\lambda \in (0, 1]$ ,

$$0 < \gamma = \min\{1, \eta \Psi(x) / \|\nabla \Psi(x)\|^2\},$$

and  $x \rightarrow x^*$ . We have

$$\sup_{\lambda \in (0, 1]} \frac{\|\bar{d}(\lambda) - \lambda d_N\|}{\lambda \|d_N\|} = o(1).$$

Lemma 5 shows that the projected Newton direction  $\bar{d}_N$  is a descent direction of  $\Psi(x)$  at  $x$ . In the next proposition we show that the descent of  $\Psi(x)$  at  $x$  along the projected Newton direction  $\bar{d}_N$  is bounded below.

**Proposition 1** *Suppose that  $\Phi$  is BD-regular at a solution  $x^*$  of  $\Phi(x^*) = 0$ .  $\Phi$  is semi-smooth at  $x^*$ , then for any  $h > 2$ , there exist a neighborhood  $\mathcal{N}$  of  $x^*$  such that for any  $\lambda \in (0, 1]$  and  $x \in \mathcal{N} \cap \mathfrak{N}_+^h$ ,*

$$\nabla \Psi(x)^T \bar{d}_N(\lambda) \geq -\lambda h \Psi(x).$$

*Proof* Let  $H(x) := \Phi(x) - \Phi(x^*) - V(x - x^*)$ , then for  $x \in \mathcal{N} \cap \mathfrak{N}_+^h$ ,

$$\begin{aligned} x + \lambda d_N &= x - \lambda V^{-1} \Phi(x) \\ &= x - \lambda V^{-1} (H(x) + \Phi(x^*) + V(x - x^*)) \\ &= x - \lambda V^{-1} (H(x) + V(x - x^*)) \\ &= x - \lambda V^{-1} H(x) - \lambda (x - x^*) \\ &= (1 - \lambda)x + \lambda x^* - \lambda V^{-1} H(x). \end{aligned}$$

By the convexity of  $\mathfrak{N}_+^h$ , then we have

$$(1 - \lambda)x + \lambda x^* \in \mathfrak{N}_+^h,$$

that is  $\Pi_{\mathfrak{N}_+^h}((1 - \lambda)x + \lambda x^*) = (1 - \lambda)x + \lambda x^*$ . So we have

$$\begin{aligned} \bar{d}_N(\lambda) &= \Pi_{\mathfrak{N}_+^h}(x + \lambda d_N) - x \\ &= \Pi_{\mathfrak{N}_+^h}((1 - \lambda)x + \lambda x^* - \lambda V^{-1} H(x)) - x \\ &= \Pi_{\mathfrak{N}_+^h}((1 - \lambda)x + \lambda x^* - \lambda V^{-1} H(x)) \\ &\quad - \Pi_{\mathfrak{N}_+^h}((1 - \lambda)x + \lambda x^*) + \Pi_{\mathfrak{N}_+^h}((1 - \lambda)x + \lambda x^*) - x \\ &= \Pi_{\mathfrak{N}_+^h}((1 - \lambda)x + \lambda x^* - \lambda V^{-1} H(x)) \\ &\quad - \Pi_{\mathfrak{N}_+^h}((1 - \lambda)x + \lambda x^*) + (-\lambda)(x - x^*) \\ &= -\lambda(x - x^*) + \lambda b_\lambda(x). \end{aligned}$$

Here  $\lambda b_\lambda(x) := \Pi_{\mathfrak{N}_+^h}((1 - \lambda)x + \lambda x^* - \lambda V^{-1} H(x)) - \Pi_{\mathfrak{N}_+^h}((1 - \lambda)x + \lambda x^*)$ . Then by the equality  $\nabla \Psi(x)^T = (V^T \Phi(x))^T$ , Lemma 3, and Lemma 4, we have

$$\begin{aligned} -\nabla \Psi(x)^T \bar{d}_N(\lambda) &= \lambda \nabla \Psi(x)^T (x - x^*) - \lambda \nabla \Psi(x)^T b_\lambda(x) \\ &= \lambda \Phi(x)^T V(x - x^*) - \lambda \Phi(x)^T V b_\lambda(x) \\ &= \lambda \Phi(x)^T (\Phi(x) - \Phi(x^*) - H(x)) - \lambda \Phi(x)^T V b_\lambda(x) \\ &= 2\lambda \Psi(x) - \lambda \Phi(x)^T H(x) - \lambda \Phi(x)^T V b_\lambda(x) \\ &\leq 2\lambda \Psi(x) + \lambda \|\Phi(x)\| \|H(x)\| + \lambda \|\Phi(x)\| \|V\| \|b_\lambda(x)\| \end{aligned}$$

$$\begin{aligned} &\leq 2\lambda\Psi(x) + \lambda\|\Phi(x)\|\|H(x)\| + \lambda\|\Phi(x)\|\|V\|\|V^{-1}H(x)\| \\ &\leq 2\lambda\Psi(x) + \lambda\|\Phi(x)\|(1 + M\|V\|)\|H(x)\|. \end{aligned}$$

By Lemma 1 and Lemma 2, we know that  $\|H(x)\| = o(\|x - x^*\|)$ , for  $x \in \mathcal{N} \cap \mathfrak{N}_+^n$ , so we have

$$\|H(x)\| \leq (h-2)\frac{\kappa\|x - x^*\|}{2(1 + M\|V\|)} \leq (h-2)\frac{\|\Phi(x)\|}{2(1 + M\|V\|)},$$

which implies that

$$\begin{aligned} -\nabla\Psi(x)^T\bar{d}_N(\lambda) &\leq 2\lambda\Psi(x) + \lambda\|\Phi(x)\|(1 + M\|V\|)(h-2)\frac{\|\Phi(x)\|}{2(1 + M\|V\|)} \\ &= 2\lambda\Psi(x) + (h-2)\lambda\Psi(x) = h\lambda\Psi(x). \end{aligned}$$

That is  $\nabla\Psi(x)^T\bar{d}_N(\lambda) \geq -h\lambda\Psi(x)$ .  $\square$

In this section, we recall some useful results which will be used later on first, now we give our modified feasible semi-smooth asymptotically Newton method for solving NCPs.

**Algorithm 1** (A feasible asymptotically Newton method)

**Step 1.** Choose parameters  $\rho, \eta \in (0, 1)$ ,  $h > 2$ ,  $0 < \sigma < \frac{1}{h}$ ,  $p_2 > 2$ ,  $p_1 > 0$ ,  $\epsilon > 0$ . Let  $x_0 \in \mathfrak{N}_+^n$  be an arbitrary initial point. Set  $k := 0$ .

**Step 2.** Choose  $V_k \in \partial_B\Phi(x_k)$ , and compute  $\nabla\Psi(x_k) = V_k^T\Phi(x_k)$ .

**Step 3.** If  $\Psi(x_k) < \epsilon$ , stop. Else set

$$d_G^k = -\gamma_k\nabla\Psi(x_k),$$

where  $\gamma_k = \min\{1, \eta\Psi(x_k)/\|\nabla\Psi(x_k)\|^2\}$ .

**Step 4.** If the linear system

$$\Phi(x_k) + V_k d = 0 \tag{4}$$

has a solution  $d_N^k$  and

$$-\nabla\Psi(x_k)^T d_N^k \geq p_1\|d_N^k\|^{p_2}, \tag{5}$$

then use the direction  $d_N^k$ . Else, set  $d_N^k = d_G^k$ .

**Step 5.** Let  $m_k$  be the smallest nonnegative integer  $m$  satisfying

$$\Psi(x_k + \bar{d}_k(\rho^m)) \leq \Psi(x_k) + \sigma\nabla\Psi(x_k)^T\bar{d}_N^k(\rho^m), \tag{6}$$

where, for any  $\lambda \in [0, 1]$ ,

$$\begin{aligned} \bar{d}_k(\lambda) &= t^*(\lambda)\bar{d}_G^k(\lambda) + (1 - t^*(\lambda))\bar{d}_N^k(\lambda), \\ \bar{d}_N^k(\lambda) &= \Pi_{\mathfrak{N}_+^n}(x_k + \lambda d_N^k) - x_k, \quad \bar{d}_G^k(\lambda) = \Pi_{\mathfrak{N}_+^n}(x_k + \lambda d_G^k) - x_k, \end{aligned}$$

and  $t^*(\lambda) \in [0, 1]$  is an optimal solution of the convex quadratic programming problems

$$\min_{t \in [0, 1]} Q_\lambda^k(t) := \frac{1}{2} \left\| \Phi(x_k) + V_k [t \bar{d}_G^k(\lambda) + (1-t) \bar{d}_N^k(\lambda)] \right\|^2, \quad (7)$$

set  $\lambda_k = \rho^{m_k}$ ,  $x_{k+1} = x_k + \bar{d}_k(\lambda_k)$ .

**Step 6.** Set  $k := k + 1$ , and go to Step 3.

The analysis for the global convergence and the convergence rate of the algorithm is reported in the next section.

**Remark 1** The optimal solution  $t^*(\lambda)$  of the convex programming problem state in Step 5 is given as follows. For  $\lambda \in [0, 1]$ , set

$$t(\lambda) = \begin{cases} 0 & \text{if } V(\bar{d}_G(\lambda) - \bar{d}_N(\lambda)) = 0, \\ -\frac{(\Phi(x_k) + V \bar{d}_N(\lambda))^T V(\bar{d}_G(\lambda) - \bar{d}_N(\lambda))}{\|V(\bar{d}_G(\lambda) - \bar{d}_N(\lambda))\|^2} & \text{otherwise.} \end{cases}$$

Then  $t^*(\lambda) = \max\{0, \min\{t(\lambda), 1\}\}$ . The details of the proof can be found in [4].

### 3 The convergence analysis of the algorithm

In this section, we consider the convergence of the algorithm which describe in Section 2. First of all we consider the global convergence of Algorithm 1, then analysis the convergent rate and we see that Algorithm 1 is superlinear. The following theorem shows that Algorithm 1 is well defined.

**Theorem 1** Suppose that  $\{x_k\}$  is a sequence generated by Algorithm 1. Then any accumulation point of  $\{x_k\}$  is a solution of the problem (2).

*Proof* Suppose  $\tilde{x}$  is an accumulation point of  $\{x_k\}$  generated by Algorithm 1. Assume that  $\tilde{x}$  is not a solution of  $\Psi(x) = 0$ , then there exists  $\epsilon_0 > 0$  such that  $\Psi(x_k) > \epsilon_0$ . From  $t^*(\lambda) \in [0, 1]$  being an optimal solution of (7), we have  $Q_\lambda^k(t^*(\lambda)) \leq Q_\lambda^k(0)$ , that is,

$$\begin{aligned} Q_\lambda^k(t^*(\lambda)) &= \frac{1}{2} \left\| \Phi(x_k) + V_k [t^*(\lambda) \bar{d}_G^k(\lambda) + (1-t^*(\lambda)) \bar{d}_N^k(\lambda)] \right\|^2 \\ &= \frac{1}{2} \left\| \Phi(x_k) \right\|^2 + \Phi(x_k)^T V \bar{d}_k(\lambda) + \frac{1}{2} \left\| V_k \bar{d}_k(\lambda) \right\|^2 \\ &= 2\Psi(x_k) + \nabla \Psi(x_k)^T \bar{d}_k(\lambda) + \frac{1}{2} \left\| V_k \bar{d}_k(\lambda) \right\|^2 \\ &\leq Q_\lambda^k(0) = \frac{1}{2} \left\| \Phi(x_k) + V_k \bar{d}_N^k(\lambda) \right\|^2 \\ &= 2\Psi(x_k) + \nabla \Psi(x_k)^T \bar{d}_N^k(\lambda) + \frac{1}{2} \left\| V_k \bar{d}_N^k(\lambda) \right\|^2, \end{aligned}$$

then we have

$$\begin{aligned} \nabla \Psi(x_k)^T \bar{d}_k(\lambda) &\leq \nabla \Psi(x_k)^T \bar{d}_N^k(\lambda) + \frac{1}{2} \left\| V_k \bar{d}_N^k(\lambda) \right\|^2 - \frac{1}{2} \left\| V_k \bar{d}_k(\lambda) \right\|^2 \\ &\leq \nabla \Psi(x_k)^T \bar{d}_N^k(\lambda) + \frac{1}{2} \left\| V_k \bar{d}_N^k(\lambda) \right\|^2. \end{aligned} \quad (8)$$

From Step 4 of Algorithm 1, first we note that  $d_N^k = -V_k^{-1}\Phi(x_k)$  if the linear equation (4) has a solution and (5) is satisfied. Then we have

$$\begin{aligned}\|\nabla\Psi(x_k)\|\|d_N^k\| &\geq -\nabla\Psi(x_k)^T d_N^k \geq p_1\|d_N^k\|^{p_2}, \\ \|d_N^k\| &\leq (p_1^{-1}\|\nabla\Psi(x_k)\|)^{\frac{1}{p_2-1}}.\end{aligned}$$

Else  $d_N^k = d_G^k = -\gamma_k\nabla\Psi(x_k)$ . Then from the continuity of  $\nabla\Psi(x_k)$ , we have

$$\|d_N^k\| \leq \max\{(p_1^{-1}\|\nabla\Psi(x_k)\|)^{\frac{1}{p_2-1}}, \gamma_k\|\nabla\Psi(x_k)\|\} \leq \kappa_1,$$

where  $\kappa_1 > 0$ . From  $d_G^k = -\gamma_k\nabla\Psi(x_k)$ , the formula above is equal to  $\max\{\|d_N^k\|, \|d_G^k\|\} \leq \kappa_1$ . By Lemma 4, we have  $\|\bar{d}_N^k(\lambda)\| = \|\Pi_{\mathbb{R}_+^n}(x_k + \lambda d_N) - x_k\| = \|\Pi_{\mathbb{R}_+^n}(x_k + \lambda d_N) - \Pi_{\mathbb{R}_+^n}(x_k)\| \leq \lambda\|d_N\|$ , similarly  $\|\bar{d}_G^k(\lambda)\| \leq \lambda\|d_G\|$ , and for  $t^*(\lambda) \in [0, 1]$ ,

$$\begin{aligned}\|\bar{d}_k(\lambda)\| &= \|t^*(\lambda)\bar{d}_G^k(\lambda) + (1-t^*(\lambda))\bar{d}_N^k(\lambda)\| \leq t^*(\lambda)\|\bar{d}_G^k(\lambda)\| + (1-t^*(\lambda))\|\bar{d}_N^k(\lambda)\| \\ &\leq t^*(\lambda)\lambda\|d_G\| + (1-t^*(\lambda))\lambda\|d_N\| \leq t^*(\lambda)\lambda\kappa_1 + (1-t^*(\lambda))\lambda\kappa_1 = \lambda\kappa_1.\end{aligned}\quad (9)$$

By the upper semi-continuity of the generalized Jacobian [9],  $\|V_k\| \leq \kappa_2$ , where  $\kappa_2 > 0$ . Then combine (8) with the inequality  $\|\bar{d}_N^k(\lambda)\| \leq \lambda\kappa_1$ , and we have

$$\begin{aligned}\nabla\Psi(x_k)^T \bar{d}_k(\lambda) &\leq \nabla\Psi(x_k)^T \bar{d}_N^k(\lambda) + \frac{1}{2}\|V_k\bar{d}_N^k(\lambda)\|^2 \\ &\leq \nabla\Psi(x_k)^T \bar{d}_N^k(\lambda) + \frac{1}{2}\|V_k\|^2\|\bar{d}_N^k(\lambda)\|^2 \\ &\leq \nabla\Psi(x_k)^T \bar{d}_N^k(\lambda) + \frac{1}{2}\kappa_2^2(\lambda\kappa_1)^2 \\ &= \nabla\Psi(x_k)^T \bar{d}_N^k(\lambda) + \frac{1}{2}\tau\lambda^2,\end{aligned}\quad (10)$$

where  $\tau = (\kappa_1\kappa_2)^2$ . Then by the uniformly continuity of  $\nabla\Psi(x_k)$ , for any  $\varepsilon > 0$  and the inequality (9), there exists a number  $\bar{\lambda} > 0$ , for all  $\lambda \in [0, \bar{\lambda}]$ , and

$$\begin{aligned}\Psi(x_k + \bar{d}_k(\lambda)) &= \Psi(x_k) + \nabla\Psi(x_k)^T \bar{d}_k(\lambda) \\ &\quad + \int_0^1 [\nabla\Psi(x_k + t\bar{d}_k(\lambda)) - \nabla\Psi(x_k)]^T \bar{d}_k(\lambda) dt \\ &\leq \Psi(x_k) + \nabla\Psi(x_k)^T \bar{d}_N^k(\lambda) + \frac{1}{2}\tau\lambda^2 \\ &\quad + \int_0^1 \|\nabla\Psi(x_k + t\bar{d}_k(\lambda)) - \nabla\Psi(x_k)\| dt \|\bar{d}_k(\lambda)\| \\ &\leq \Psi(x_k) + \nabla\Psi(x_k)^T \bar{d}_N^k(\lambda) + \frac{1}{2}\tau\lambda^2 \\ &\quad + \int_0^1 \|\nabla\Psi(x_k + t\bar{d}_k(\lambda)) - \nabla\Psi(x_k)\| dt \lambda\kappa_1 \\ &\leq \Psi(x_k) + \nabla\Psi(x_k)^T \bar{d}_N^k(\lambda) + \frac{1}{2}\tau\lambda^2 + \varepsilon\lambda\kappa_1\end{aligned}$$

$$\begin{aligned}
 &= \Psi(x_k) + \sigma \nabla \Psi(x_k)^T \bar{d}_N^k(\lambda) + (1 - \sigma) \nabla \Psi(x_k)^T \bar{d}_N^k(\lambda) \\
 &\quad + \frac{1}{2} \tau \lambda^2 + \varepsilon \lambda \kappa_1.
 \end{aligned} \tag{11}$$

By Lemma 5, for all  $\lambda \in (0, \lambda']$ , we have

$$\begin{aligned}
 \Psi(x_k + \bar{d}_k(\lambda)) &\leq \Psi(x_k) + \sigma \nabla \Psi(x_k)^T \bar{d}_N^k(\lambda) - (1 - \sigma) \rho \lambda \Psi(x_k) + \frac{1}{2} \tau \lambda^2 + \varepsilon \lambda \kappa_1 \\
 &\leq \Psi(x_k) + \sigma \nabla \Psi(x_k)^T \bar{d}_N^k(\lambda) - (1 - \sigma) \rho \lambda \epsilon_0 + \frac{1}{2} \tau \lambda^2 + \varepsilon \lambda \kappa_1 \\
 &\leq \Psi(x_k) + \sigma \nabla \Psi(x_k)^T \bar{d}_N^k(\lambda),
 \end{aligned} \tag{12}$$

where  $\tilde{\lambda} = \frac{1-\sigma}{\tau} \rho \epsilon_0$ ,  $\varepsilon = \frac{1-\sigma}{2\kappa_1} \rho \epsilon_0$ ,  $\lambda' = \min\{\tilde{\lambda}, \tilde{\lambda}\}$ , and the second inequality is a result from the assumption that  $\Psi(x_k) > \epsilon_0$ . The last inequality holds because

$$-(1 - \sigma) \rho \epsilon_0 + \frac{1}{2} \tau \lambda + \varepsilon \kappa_1 \leq -(1 - \sigma) \rho \epsilon_0 + \frac{1}{2} \tau \frac{1 - \sigma}{\tau} \rho \epsilon_0 + \frac{1 - \sigma}{2\kappa_1} \rho \epsilon_0 \kappa_1 = 0.$$

From Step 5 and (12), we note that for  $\lambda_k \geq \frac{\lambda'}{\rho}$ ,  $\Psi(x_k + \bar{d}_k(\lambda_k)) > \Psi(x_k) + \sigma \nabla \Psi(x_k)^T \bar{d}_N^k(\lambda_k)$ . By the continuity of  $\Psi(x)$ , we have

$$\infty > \sum [\Psi(x_k + \bar{d}_k(\lambda_k)) - \Psi(x_k)] > \sum [\sigma \nabla \Psi(x_k)^T \bar{d}_N^k(\lambda_k)].$$

From Lemma 5, we obtain

$$\rho \lambda_k \Psi(x_k) \leq -\nabla \Psi(x_k)^T \bar{d}_N^k(\lambda_k) \rightarrow 0. \tag{13}$$

But

$$\rho \lambda_k \Psi(x_k) \geq \rho \frac{\lambda'}{\rho} \epsilon_0 = \lambda' \epsilon_0 > 0.$$

It is clear that there exists a contradiction. So the assumption that  $\tilde{x}$  is not a solution of (2) is not true. That is, any accumulation point of the sequence  $\{x_k\}$  is a solution of (2).  $\square$

It is easy to see from the proof of Theorem 1 that the Algorithm 1 is always well defined. We begin with a non-solution point, Algorithm 1 always going to the stage that (6) is satisfied. In other words,  $m_k$  always will be a finite number, that is, Algorithm 1 is well defined. In the next theorem we analyze the superlinear convergence of Algorithm 1.

**Theorem 2** Suppose that  $x_k$  is a sequence generator by Algorithm 1, and  $x^*$  is an accumulation point of  $x_k$ , a solution of  $\Phi(x^*) = 0$ . If  $\Phi(x)$  is BD-regular at  $x^*$ , then the sequence  $\{x^k\}$  generalized by Algorithm 1 converges to  $x^*$  superlinearly.

*Proof* By  $\sup_{\lambda \in (0,1]} \frac{\|\bar{d}(\lambda) - \lambda d_N\|}{\lambda \|d_N\|} = o(1)$  from Lemma 6, we have  $\|\bar{d}(1) - d_N\| = o(\|d_N\|)$ , then

$$\begin{aligned}
 \|x_k + \bar{d}_k(1) - x^*\| &= \|x_k + d_N^k + o(\|d_N^k\|) - x^*\| \\
 &= \|x_k - V_k^{-1} \Phi(x_k) + o(\|d_N\|) - x^*\|
 \end{aligned}$$



$$\begin{aligned}
 &\leq \|V_k^{-1}\| \|\Phi(x_k) - V_k(x_k - x^*)\| + o(\|d_N\|) \\
 &\leq \|V_k^{-1}\| \|\Phi(x_k) - \Phi(x^*) - V_k(x_k - x^*)\| + o(\|V_k^{-1}\Phi(x_k)\|) \\
 &= o(\|x_k - x^*\|) = o(\|\Phi(x_k)\|),
 \end{aligned} \tag{14}$$

where the last two equalities are results from Lemma 1, Lemma 2, and Lemma 3.

From the locally Lipschitz continuity of  $\Phi(x)$ ,

$$\begin{aligned}
 \Psi(x_k + \bar{d}_k(1)) &= \frac{1}{2} \|\Phi(x_k + \bar{d}_k(1))\|^2 = \frac{1}{2} \|\Phi(x_k + \bar{d}_k(1)) - \Phi(x^*)\|^2 \\
 &= O(\|x_k + \bar{d}_k(1) - x^*\|^2) = o(\|\Phi(x_k)\|^2) = o(\|\Psi(x_k)\|).
 \end{aligned}$$

By Proposition 1 and  $\sigma < \frac{1}{h}$  from Algorithm 1, we know that

$$\begin{aligned}
 \Psi(x_k) + \sigma \nabla \Psi(x_k)^T \bar{d}_N(1) &\geq \Psi(x_k) - h\sigma \Psi(x_k) = (1 - h\sigma)\Psi(x_k) \\
 &\geq o(\|\Psi(x_k)\|) = \Psi(x_k + \bar{d}_k(1)).
 \end{aligned}$$

That is, (6) from Algorithm 1 being satisfied, then we have  $x_{k+1} = x_k + \bar{d}_k(1)$ . From (14),

$$\|x_{k+1} - x^*\| = \|x_k + \bar{d}_k(1) - x^*\| = o(\|x_k - x^*\|),$$

thus we obtain the superlinearity of Algorithm 1.  $\square$

#### 4 Numerical experiments

In this section we present some numerical experiments for the algorithm proposed in Section 2. The algorithm was implemented in Matlab and run in a Matlab 7.0.1 workstation.

In the table of numerical results, SP denotes the starting point  $x_0$ ; IN denotes the iterative number; FV denotes the final value of  $\Psi(x_k)$ ; CPU denotes the CPU time in seconds for solving a problems;  $\bar{x}$  denotes the final value of  $x_k$ , which is the numerical solution of the test problem. Throughout our computational experiments, the parameters in Algorithm 1 are as follows:

$$\eta = 0.6, \quad \sigma = 0.1, \quad p_1 = 10^{-6}, \quad p_2 = 2.2, \quad k \max = 100.$$

In the following we give a detailed description of the numerical experiments.

**Example 1** This is a linear complementarity problem, which is the 76th problem in the Hock-Schittkowski collection. The test function  $F$  is given as follows:

$$F(x) = \begin{pmatrix} 2x_1 - x_3 + x_5 + 3x_6 - 1 \\ x_2 + 2x_5 + x_6 - x_7 - 3 \\ -x_1 + 2x_3 + x_4 + x_5 + 2x_6 - 4x_7 + 1 \\ x_3 + x_4 + x_5 - x_6 - 1 \\ -x_1 - 2x_2 - x_3 - x_4 + 5 \\ -3x_1 - x_2 - 2x_3 + x_4 + 4 \\ x_2 + 4x_3 - 1.5 \end{pmatrix}.$$

**Table 1** Numerical results for Example 1

SP	Iter	FV	CPU
(0,...,0)	6	9.5727e-013	0.0620
(0.5,...,0.5)	5	2.9598e-013	0.0150
(-0.5,...,-0.5)	6	9.5727e-013	0.0310
(1,...,1)	6	7.0301e-024	0.0160
(-1,...,-1)	6	9.5727e-013	0.0320
(-100,...,-100)	6	9.5727e-013	0.0320
(10,...,10)	9	2.0099e-015	0.0160
(50,...,50)	10	8.2935e-019	0.0160
(100,...,100)	10	2.4238e-015	0.0160
(1000,...,1000)	11	2.9654e-022	0.0150

**Table 2** Numerical results for Example 2 ( $n = 100$ )

SP	Iter	FV	CPU
(0,...,0)	10	9.7021e-013	0.2500
(0.5,...,0.5)	16	2.2473e-016	0.2810
(-0.5,...,-0.5)	7	1.6175e-016	0.1570
(1,...,1)	17	2.2481e-016	0.2970
(-1,...,-1)	7	1.6173e-016	0.1410
(10,...,10)	20	2.2869e-014	0.3430
(-10,...,-10)	7	1.6170e-016	0.1400
(100,...,100)	23	9.1849e-013	0.4060
(-100,...,-100)	7	1.6170e-016	0.1410

The solution of the LCP is  $x^* \simeq (0.2727, 2.0909, 0, 0.5454, 0.4545, 0, 0)^T$ . There is also a test by Ma [10]. We executed this problem 10 times from different initial points. The numerical results are listed in Table 1.

**Example 2** Fathi problem. This is a linear complementarity problem, which comes from Fathi [11]. There is also a test by Ma [10] and Xu [12]. The test function  $F := Mx + q$ , the matrix  $M$ , and the vector  $q$  are given as follows:

$$\begin{aligned}
 [M]_{ii} &= 4(i-1) + 1, \quad i = 1, \dots, n; \\
 [M]_{ij} &= [M]_{ii} + 1, \quad i = 1, \dots, n-1, j = i+1, \dots, n; \\
 [M]_{ij} &= [M]_{jj} + 1, \quad j = 1, \dots, n-1, i = j+1, \dots, n; \\
 q &= (-1, -1, \dots, -1)^T.
 \end{aligned}$$

It is easy to see that  $M$  is a positive definite matrix. The solution of the LCP is  $x^* = (1, 0, \dots, 0)^T$ . We executed this problem 9 times from different initial points. The numerical results are listed in Table 2.

**Example 3** Muty problem. This problem is the fifth example of Kanzow [13], which is also tested by Ma [10]. It is a linear complementarity problem that the matrix here is a  $P$ -matrix.

**Table 3** Numerical results for Example 3

SP	$n = 32$			$n = 100$		
	Iter	FV	CPU	Iter	FV	CPU
(0,...,0)	7	1.4147e-016	0.0470	8	4.2376e-014	0.2190
(1,...,1)	9	2.7573e-021	0.0470	10	3.3580e-014	0.1870
(-1,...,-1)	6	3.1293e-020	0.0310	6	4.3639e-020	0.0940
(-10,...,-10)	6	3.8994e-021	0.0310	6	4.1741e-021	0.0940
(10,...,10)	12	5.2178e-017	0.0630	14	5.2324e-022	0.2500
(100,...,100)	15	1.0121e-013	0.0620	17	8.1853e-018	0.3130
(1000,...,1000)	19	1.2999e-021	0.0630	20	1.7651e-014	0.3590

**Table 4** Numerical results for Example 4

SP	Iter	FV	CPU
(0,...,0)	21	1.0292e-015	0.0310
(0.5,...,0.5)	18	1.0960e-016	0.0630
(1,...,1)	18	9.5526e-020	0.0320
(3,...,3)	40	5.4668e-018	0.0470
(5,...,5)	100	3.3390e-009	0.0620
(2,1,0,1,2)	21	2.4842e-018	0.0310
(0,1,0,1,0)	21	4.7668e-018	0.0320
(0.5,1,0.5,2,0)	20	4.3317e-014	0.0310
(1,2,3,4,5)	29	8.5428e-016	0.0470

The test function  $F := Mx + q$ , the matrix  $M$ , and the vector  $q$  are given as follows:

$$M = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad q = (-1, \dots, -1)^T.$$

The solution of the LCP is  $x^* = (0, 0, \dots, 1)^T$ . We executed this problem 7 times from different initial points. The numerical results are listed in Table 3.

**Example 4** This problem is a nonlinear complementarity problem. The function  $F(x) : \mathbb{R}^5 \rightarrow \mathbb{R}^5$  is given as follows:

$$F_j(x) = 2(x_j - j + 2) \exp \left\{ \sum_{i=1}^5 (x_i - i + 2)^2 \right\}, \quad 1 \leq j \leq 5.$$

This nonlinear complementarity problem has one degenerate solution  $(0, 0, 1, 2, 3)$ . We executed this problem 9 times from different initial points. The numerical results are listed in Table 4.

**Table 5 Numerical results for Example 5**

SP	Iter	FV	CPU	$\bar{x}$
(0,...,0)	5	9.5495e-018	0.0160	(1.2247,0,0,0.5000)
(1,...,1)	9	8.9335e-021	0.0320	(1.2247,0,0,0.5000)
(-1,...,-1)	6	1.0947e-018	0.0160	(1.2247,0,0,0.5000)
(10,...,10)	14	1.2045e-025	0.0310	(1.2247,0,0,0.5000)
(-10,...,-10)	11	7.3075e-016	0.0160	(1.2247,0,0,0.5000)
(100,...,100)	17	2.4086e-023	0.0320	(1.2247,0,0,0.5000)
(-100,...,-100)	6	2.4643e-023	0.0310	(1.2247,0,0,0.5000)
(10 <sup>3</sup> ,...,10 <sup>3</sup> )	26	2.1254e-018	0.0310	(1.2247,0,0,0.5000)
(-10 <sup>3</sup> ,..., -10 <sup>3</sup> )	6	1.9403e-021	0.0160	(1.2247,0,0,0.5000)

**Table 6 Numerical results for Example 6**

SP	Iter	FV	CPU	$\bar{x}$
(0,...,0)	1	0	0.0160	(0.0000,0.0000,0.0000,0.0000)
(1,...,1)	4	6.2183e-016	0.0470	(0.9508,0.0000,0.0000,0.0000)
(5,...,5)	6	0	0.0160	(2.9845,0.0000,0.0000,0.0000)
(10,...,10)	4	7.2927e-032	0.0150	(2.9964,0.0000,0.0000,0.0000)
(30,...,30)	7	0	0.0310	(2.8521,0.0000,0.0000,0.0000)
(60,...,60)	5	0	0.0160	(3.0000,0.0000,0.0000,0.0000)

**Example 5** Kojima-Shindo problem. This is a nonlinear complementarity problem, it is the third example of Jiang and Qi [14]. The test function  $F(x)$  is given as follows:

$$F(x) = \begin{pmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{pmatrix}.$$

This nonlinear complementarity problem has one nondegenerate solution (1, 0, 3, 0) and one degenerate solution ( $\sqrt{6}/2, 0, 0, 1/2$ ). We executed this problem 9 times from different initial points. The numerical results are listed in Table 5.

**Example 6** Modified Mathiesen problem. It is a nonlinear complementarity problem, which is the fifth example of Jiang and Qi [14]. There is also a test by Ma [10]. The test function  $F(x)$  is given as follows:

$$F(x) = \begin{pmatrix} -x_2 + x_3 + x_4 \\ x_1 - (4.5x_3 + 2.7x_4)/(x_2 + 1) \\ 5 - x_1 - (0.5x_3 + 0.3x_4)/(x_3 + 1) \\ 3 - x_1 \end{pmatrix}.$$

This example has infinitely many solutions  $(\lambda, 0, 0, 0)$ , where  $\lambda \in [0, 3]$ . We executed this problem 7 times from different initial points. The numerical results of Example 6 are listed in Table 6.

## 5 Conclusions

In this paper, based on the semi-smoothing asymptotically Newton method, we present a modified feasible semi-smooth asymptotically Newton method for nonlinear complementarity problems. We can achieve the global convergence and the local superlinear conver-

gence with several mild assumptions. The numerical experiments reported in Section 4 show that the modified algorithm is effective.

# Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

# Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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