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Shafer-type inequalities for inverse trigonometric functions and Gauss lemniscate functions

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Abstract
In this paper, we present Shafer-type inequalities for inverse trigonometric functions and Gauss lemniscate functions.

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1 Introduction

Shafer [1] indicated several elementary quadratic approximations of selected functions without proof. Subsequently, Shafer [2] established these results as analytic inequalities. For example, Shafer [2] proved that for $x > 0$,

$$\frac{8x}{3+\sqrt{25+\frac{80}{3}x^2}} < \arctan x. \tag{1.1}$$

The inequality (1.1) can also be found in [3]. Also in [2], Shafer proved that for $0 < x < 1$,

$$\frac{8x}{3\sqrt{1-x^2}+\sqrt{25+\frac{5}{3}x^2}} < \arcsin x. \tag{1.2}$$

Zhu [4] proved that the function

$$F(x) = \frac{(\frac{8x}{\arctan x}-3)^2-25}{x^2}$$

is strictly decreasing for $x > 0$, and

$$\lim_{x\to 0^+} F(x) = \frac{80}{3} \quad \text{and} \quad \lim_{x\to\infty} F(x) = \frac{256}{\pi^2}.$$

From this one derives the following double inequality:

$$\frac{8x}{3+\sqrt{25+\frac{80}{3}x^2}} < \arctan x < \frac{8x}{3+\sqrt{25+\frac{256}{\pi^2}x^2}}, \quad x > 0. \tag{1.3}$$

The constants 80/3 and $256/\pi^2$ are the best possible. In [4], (1.3) is called Shafer-type inequality.

Using the Maple software, we find that

$$\arctan x\left(3+\sqrt{25+\frac{80}{3}x^2}\right)=8x+\frac{32}{4{,}725}x^7-\frac{64}{4{,}725}x^9+\frac{25{,}376}{1{,}299{,}375}x^{11}-\cdots.$$

This fact motivated us to present a new upper bound for $\arctan x$, which is the first aim of the present paper.

Theorem 1.1 *For* $x>0$,

$$\arctan x<\frac{8x+\frac{32}{4{,}725}x^7}{3+\sqrt{25+\frac{80}{3}x^2}}. \tag{1.4}$$

The second aim of the present paper is to develop (1.2) to produce a symmetric double inequality.

Theorem 1.2 *For* $0<x<1$, *we have*

$$\frac{8x}{3\sqrt{1-x^2}+\sqrt{25+ax^2}}<\arcsin x<\frac{8x}{3\sqrt{1-x^2}+\sqrt{25+bx^2}} \tag{1.5}$$

with the best possible constants

$$a=\frac{5}{3}=1.666666\ldots \quad \textit{and} \quad b=\frac{256-25\pi^2}{\pi^2}=0.938223\ldots. \tag{1.6}$$

Recently, some famous inequalities for trigonometric and inverse trigonometric functions have been improved (see, for example, [5–8]).

The lemniscate, also called the lemniscate of Bernoulli, is the locus of points (x,y) in the plane satisfying the equation $(x^2+y^2)^2=x^2+y^2$. In polar coordinates (r,θ), the equation becomes $r^2=\cos(2\theta)$ and its arc length is given by the function

$$\operatorname{arcsl} x=\int_0^x\frac{1}{\sqrt{1-t^4}}\,\mathrm{d}t,\quad |x|\le 1, \tag{1.7}$$

where $\operatorname{arcsl} x$ is called the arc lemniscate sine function studied by Gauss in 1797-1798. Another lemniscate function investigated by Gauss is the hyperbolic arc lemniscate sine function, defined as

$$\operatorname{arcslh} x=\int_0^x\frac{1}{\sqrt{1+t^4}}\,\mathrm{d}t,\quad x\in\mathbb{R}. \tag{1.8}$$

Functions (1.7) and (1.8) can be found (see [9], Chapter 1, [10], p.259 and [11–19]).

Another pair of lemniscate functions, the arc lemniscate tangent arctl and the hyperbolic arc lemniscate tangent arctlh, have been introduced in [12], (3.1)-(3.2). Therein it has been proven that

$$\operatorname{arctl} x=\operatorname{arcsl}\left(\frac{x}{\sqrt[4]{1+x^4}}\right),\quad x\in\mathbb{R} \tag{1.9}$$

and

$$\operatorname{arctlh} x = \operatorname{arcslh}\left(\frac{x}{\sqrt[4]{1-x^4}}\right), \quad |x| < 1 \tag{1.10}$$

(see [12], Proposition 3.1).

In analogy with (1.1), we here establish Shafer-type inequalities for the lemniscate functions, which is the last aim of the present paper.

Theorem 1.3 *For* $0 < x < 1$,

$$\frac{10x}{5+\sqrt{25-10x^4}} < \operatorname{arcsl} x \tag{1.11}$$

and

$$\frac{10x}{5+\sqrt{25-15x^4}} < \operatorname{arctlh} x. \tag{1.12}$$

Theorem 1.4 *For* $x > 0$,

$$\frac{95x}{80+\sqrt{225+285x^4}} < \operatorname{arcslh} x. \tag{1.13}$$

We present the following conjecture.

Conjecture 1.1 *For* $x > 0$,

$$\operatorname{arcslh} x < \frac{95x + \frac{931}{2{,}925}x^{13}}{80+\sqrt{225+285x^4}} \tag{1.14}$$

and

$$\frac{1{,}210x}{940+9\sqrt{900+1{,}210x^4}} < \operatorname{arctl} x < \frac{1{,}210x + \frac{2{,}078{,}417}{280{,}800}x^{13}}{940+9\sqrt{900+1{,}210x^4}}. \tag{1.15}$$

2 Lemmas

The following lemmas have been proved in [17].

Lemma 2.1 *For* $|x| < 1$,

$$\operatorname{arcsl} x = \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(4n+1)\cdot n!}x^{4n+1} = x + \frac{1}{10}x^5 + \frac{1}{24}x^9 + \cdots. \tag{2.1}$$

Lemma 2.2 *For* $0 < x < 1$,

$$\operatorname{arctlh} x = \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{3}{4})}{\Gamma(\frac{3}{4})\cdot(4n+1)\cdot n!}x^{4n+1} = x + \frac{3}{20}x^5 + \frac{7}{96}x^9 + \cdots. \tag{2.2}$$

3 Proofs of Theorems 1.1 to 1.4

Proof of Theorem 1.1 The inequality (1.11) is obtained by considering the function $f(x)$ defined by

$$f(x) = \arctan x - \frac{8x + \frac{32}{4{,}725}x^7}{3 + \sqrt{25 + \frac{80}{3}x^2}}, \quad x > 0.$$

Differentiation yields

$$f'(x) = -\frac{2g(x)}{(1+x^2)(9+\sqrt{225+240x^2})^2\sqrt{225+240x^2}},$$

where

$$g(x) = \left(-7{,}875 - 2{,}100x^2 + 112x^6 + 112x^8\right)\sqrt{225+240x^2} + 118{,}125 + 94{,}500x^2 + 2{,}800x^6 + 5{,}360x^8 + 2{,}560x^{10}.$$

We now show that

$$g(x) > 0, \quad x > 0. \tag{3.1}$$

By an elementary change of variable

$$t = \sqrt{225+240x^2}, \quad t > 15,$$

the inequality (3.1) is equivalent to

$$\begin{aligned}
&\left[-7{,}875 - 2{,}100\left(\frac{t^2-225}{240}\right) + 112\left(\frac{t^2-225}{240}\right)^3 + 112\left(\frac{t^2-225}{240}\right)^4\right]t \\
&\quad + 118{,}125 + 94{,}500\left(\frac{t^2-225}{240}\right) + 2{,}800\left(\frac{t^2-225}{240}\right)^3 + 5{,}360\left(\frac{t^2-225}{240}\right)^4 \\
&\quad + 2{,}560\left(\frac{t^2-225}{240}\right)^5 \\
&= \frac{(2t^6 + 141t^5 + 4{,}515t^4 + 93{,}690t^3 + 1{,}562{,}400t^2 + 24{,}053{,}625t + 362{,}626{,}875)(t-15)^4}{622{,}080{,}000} \\
&> 0 \quad \text{for } t > 15,
\end{aligned}$$

which is true. Hence, we have

$$g(x) > 0 \quad \text{and} \quad f'(x) < 0 \quad \text{for } x > 0.$$

So, $f(x)$ is strictly decreasing for $x > 0$, and we have

$$f(x) < f(0) = 0, \quad x > 0.$$

The proof is complete. □

Remark 3.1 Let $x_0 = 1.4243\ldots$. Then we have

$$\frac{8x + \frac{32}{4{,}725}x^7}{3 + \sqrt{25 + \frac{80}{3}x^2}} < \frac{8x}{3 + \sqrt{25 + \frac{256}{\pi^2}x^2}}, \quad 0 < x < x_0.$$

This shows that for $0 < x < x_0$, the upper bound in (1.11) is better than the upper bound in (1.3). In fact, for $x \to 0$, we have

$$\arctan x - \frac{8x}{3 + \sqrt{25 + \frac{256}{\pi^2}x^2}} = O(x^3),$$
$$\arctan x - \frac{8x}{3 + \sqrt{25 + \frac{80}{3}x^2}} = O(x^7),$$

and

$$\arctan x - \frac{8x + \frac{32}{4{,}725}x^7}{3 + \sqrt{25 + \frac{80}{3}x^2}} = O(x^9).$$

Proof of Theorem 1.2 The double inequality (1.11) can be written for $0 < x < 1$ as

$$b < F(x) < a,$$

where

$$F(x) = \frac{(\frac{8x}{\arcsin x} - 3\sqrt{1-x^2})^2 - 25}{x^2}, \quad 0 < x < 1.$$

By an elementary change of variable,

$$x = \sin t, \quad 0 < t < \frac{\pi}{2},$$

we have

$$G(t) = F(\sin t) = \frac{(\frac{8\sin t}{t} - 3\cos t)^2 - 25}{\sin^2 t}, \quad 0 < t < \frac{\pi}{2}.$$

We now prove that $F(x)$ is strictly decreasing for $0 < x < 1$. It suffices to show that $G(t)$ is strictly decreasing for $0 < t < \pi/2$. Differentiation yields

$$\begin{aligned}
-\frac{t^3 \sin^3 t}{16} G'(t) &= 8\sin^3 t - 3t^2 \sin t - (2t^3 + 3t)\cos t + 3t\cos^3 t \\
&= 8\left(\frac{3\sin t - \sin(3t)}{4}\right) - 3t^2 \sin t - (2t^3 + 3t)\cos t \\
&\quad + 3t\left(\frac{\cos(3t) + 3\cos t}{4}\right) \\
&= (6 - 3t^2)\sin t - 2\sin(3t) + \frac{3}{4}t\cos(3t) - \left(\frac{3}{4}t + 2t^3\right)\cos t \\
&= \frac{16}{945}t^9 - \frac{16}{4{,}725}t^{11} + \sum_{n=6}^{\infty}(-1)^n u_n(t),
\end{aligned} \tag{3.2}$$

where

$$u_n(t) = \frac{21 + 2n + 48n^2 + 64n^3 + (6n - 21) \cdot 9^n}{4 \cdot (2n+1)!} t^{2n+1}.$$

Elementary calculations reveal that for $0 < t < \pi/2$ and $n \geq 6$,

$$\begin{aligned}
\frac{u_{n+1}(t)}{u_n(t)} &= \frac{t^2(135 + 290n + 240n^2 + 64n^3 + (54n - 135) \cdot 9^n)}{2(n+1)(2n+3)(21 + 2n + 48n^2 + 64n^3 + (6n - 21) \cdot 9^n)} \\
&< \frac{(\pi/2)^2}{n+1} \frac{135 + 290n + 240n^2 + 64n^3 + (54n - 135) \cdot 9^n}{2(2n+3)(21 + 2n + 48n^2 + 64n^3 + (6n - 21) \cdot 9^n)} \\
&< \frac{135 + 290n + 240n^2 + 64n^3 + (54n - 135) \cdot 9^n}{2(2n+3)(21 + 2n + 48n^2 + 64n^3 + (6n - 21) \cdot 9^n)}
\end{aligned}$$

and

$$\begin{aligned}
&2(2n+3)\bigl(21 + 2n + 48n^2 + 64n^3 + (6n - 21) \cdot 9^n\bigr) \\
&\quad - \bigl(135 + 290n + 240n^2 + 64n^3 + (54n - 135) \cdot 9^n\bigr) \\
&\quad = \bigl(24n^2 - 102n + 9\bigr)9^n + 256n^4 + 512n^3 + 56n^2 - 194n - 9 > 0.
\end{aligned}$$

We then obtain, for $0 < t < \pi/2$ and $n \geq 6$,

$$\frac{u_{n+1}(t)}{u_n(t)} < 1.$$

Hence, for every $t \in (0, \pi/2)$, the sequence $n \longmapsto u_n(t)$ is strictly decreasing for $n \geq 6$. We then obtain from (3.2)

$$-\frac{t^3 \sin^3 t}{16} G'(t) > t^9 \left(\frac{16}{945} - \frac{16}{4{,}725} t^2 \right) > 0, \quad 0 < t < \frac{\pi}{2},$$

which implies $G'(t) < 0$ for $0 < t < \pi/2$. Hence, $G(t)$ is strictly decreasing for $0 < t < \pi/2$, and $F(x)$ is strictly decreasing for $0 < x < 1$. So, we have

$$\frac{256 - 25\pi^2}{\pi^2} = \lim_{t \to 1} F(t) < F(x) = \frac{(\frac{8x}{\arcsin x} - 3\sqrt{1 - x^2})^2 - 25}{x^2} < \lim_{t \to 0} F(t) = \frac{5}{3}$$

for all $x \in (0, 1)$, with the constants $5/3$ and $(256 - 25\pi^2)/\pi^2$ being best possible. The proof is complete. □

Proof of Theorem 1.3 By (2.1), we find that for $0 < x < 1$,

$$\begin{aligned}
\bigl(25 - 10x^4\bigr) - \left(\frac{10x}{\operatorname{arcsl} x} - 5 \right)^2 &> \bigl(25 - 10x^4\bigr) - \left(\frac{10x}{x + \frac{1}{10}x^5 + \frac{1}{24}x^9} - 5 \right)^2 \\
&= \frac{10x^8(3{,}120 - 1{,}344x^4 - 120x^8 - 25x^{12})}{(120 + 12x^4 + 5x^8)^2}.
\end{aligned}$$

Noting that

$$3{,}120 - 1{,}344t - 120t^2 - 25t^3 > 0 \quad \text{for } 0 < t < 1,$$

we obtain, for $0 < x < 1$,

$$\left(25 - 10x^4\right) - \left(\frac{10x}{\operatorname{arcsl} x} - 5\right)^2 > 0,$$

which implies (1.11).

By (2.2), we find that for $0 < x < 1$,

$$\begin{aligned}\left(25 - 15x^4\right) - \left(\frac{10x}{\operatorname{arctlh} x} - 5\right)^2 &> \left(25 - 15x^4\right) - \left(\frac{10x}{x + \frac{1}{10}x^5 + \frac{1}{24}x^9} - 5\right)^2 \\ &= \frac{15x^8 A(x)}{(24{,}960 + 3{,}744x^4 + 1{,}820x^8 + 1{,}155x^{12})^2},\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}A(x) = {}&\left(115{,}947{,}520 - 71{,}285{,}760x^8 - 4{,}204{,}200x^{16}\right) \\ &+ x^4\left(87{,}320{,}064 - 11{,}961{,}040x^8 - 1{,}334{,}025x^{16}\right).\end{aligned}$$

Noting that for $0 < t < 1$,

$$115{,}947{,}520 - 71{,}285{,}760t - 4{,}204{,}200t^2 > 0$$

and

$$87{,}320{,}064 - 11{,}961{,}040t - 1{,}334{,}025t^2 > 0,$$

we obtain $A(x) > 0$ for $0 < x < 1$. From (3.3), we obtain (1.12). The proof is complete. □

Proof of Theorem 1.4 The inequality (1.13) is obtained by considering the function $h(x)$ defined by

$$h(x) = \operatorname{arcslh} x - \frac{95x}{80 + \sqrt{225 + 285x^4}}, \quad x > 0.$$

Differentiation yields

$$h'(x) = \frac{1}{\sqrt{1 + x^4}} - \frac{475(16\sqrt{225 + 285x^4} + 45 - 57x^4)}{(80 + \sqrt{225 + 285x^4})^2\sqrt{225 + 285x^4}}.$$

By an elementary change of variable

$$t = \sqrt{225 + 285x^4}, \quad x > 0 \qquad \left(\text{or } x = \sqrt[4]{\frac{t^2 - 225}{285}}, t > 15\right), \tag{3.4}$$

we have

$$\begin{aligned}
&\frac{1}{\sqrt{1+x^4}} - \frac{475(16\sqrt{225+285x^4}+45-57x^4)}{(80+\sqrt{225+285x^4})^2\sqrt{225+285x^4}} \\
&\quad = \frac{285}{\sqrt{17{,}100+285t^2}} + \frac{95(t^2-80t-450)}{t(80+t)^2} = \frac{95I(t)}{t(80+t)^2},
\end{aligned}$$

where

$$I(t) = \frac{19{,}200t+480t^2+3t^3}{\sqrt{17{,}100+285t^2}} + t^2 - 80t - 450, \quad t > 15.$$

We now prove that

$$h'(x) > 0, \quad x > 0.$$

It suffices to show that

$$I(t) > 0, \quad t > 15.$$

Differentiation yields

$$\begin{aligned}
I'(t) &= \frac{6(192{,}000+9{,}600t+90t^2+80t^3+t^4)}{(60+t^2)\sqrt{17{,}100+285t^2}} + 2t - 80, \\
I''(t) &= \frac{6(576{,}000-565{,}200t-4{,}800t^2+150t^3+t^5)}{(60+t^2)^2\sqrt{17{,}100+285t^2}} + 2,
\end{aligned}$$

and

$$I'''(t) = \frac{10{,}800(-18{,}840-1{,}920t+1{,}271t^2+8t^3)}{(60+t^2)^3\sqrt{17{,}100+285t^2}} > 0 \quad \text{for } t > 15.$$

Thus, we have, for $t > 15$,

$$I''(t) > I''(15) = 0 \quad \Longrightarrow \quad I'(t) > I'(15) = 0 \quad \Longrightarrow \quad I(t) > I(15) = 0.$$

Hence, $h'(x) > 0$ holds for $x > 0$, and we have

$$h(x) > h(0) = 0, \quad x > 0.$$

The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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