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# An upper bound for solutions of the Lebesgue-Nagell equation $x^2 + a^2 = y^n$

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**Abstract**

Let $a$ be a positive integer with $a > 1$, and let $(x, y, n)$ be a positive integer solution of the equation $x^2 + a^2 = y^n$, $\gcd(x, y) = 1$, $n > 2$. Using Baker's method, we prove that, for any positive number $\epsilon$, if $n$ is an odd integer with $n > C(\epsilon)$, where $C(\epsilon)$ is an effectively computable constant depending only on $\epsilon$, then $n < (2 + \epsilon)(\log a)/\log y$. Owing to the obvious fact that every solution $(x, y, n)$ of the equation satisfies $n > 2(\log a)/\log y$, the above upper bound is optimal.

**MSC:** 11D61

**Keywords:** exponential diophantine equation; Lebesgue-Nagell equation; upper bound for solutions; Baker's method

## 1 Introduction

Let $\mathbb{Z}$, $\mathbb{N}$ be the sets of all integers and positive integers, respectively. Let $D$ be a positive integer. In 1850, Lebesgue [1] proved that if $D = 1$, then the equation

$$x^2 + D = y^n, \quad x, y, n \in \mathbb{N}, \gcd(x, y) = 1, n > 2 \tag{1.1}$$

has no solutions $(x, y, n)$, which solved a type important case of the famous Catalan's conjecture. From then on, Nagell [2–4] dealt with the solution of (1.1) more systematically for the case of $D > 1$. Therefore, equation (1.1) is called the Lebesgue-Nagell equation (see [5]).

In this paper, we shall discuss an upper bound for solutions of (1.1) when $D > 1$, that is, $D = a^2$, where $a$ is a positive integer with $a > 1$. So equation (1.1) can be expressed as

$$x^2 + a^2 = y^n, \quad x, y, n \in \mathbb{N}, \gcd(x, y) = 1, n > 2. \tag{1.2}$$

This is a type of Lebesgue-Nagell equation leading to more discussions (see [6]). Let $(x, y, n)$ be a solution of (1.2). In 2004, Tengely [7] proved that if $y > 50{,}000$ and $n$ is an odd prime with $n > 9{,}511$, then

$$n < \frac{4 \log a}{\log 50{,}000}. \tag{1.3}$$

Using Baker's method, the following result is proved.

**Theorem** *For any positive number* $\epsilon$, *if* $n$ *is an odd number with* $n > C(\epsilon)$, *then*

$$n < \frac{(2+\epsilon)\log a}{\log y}, \tag{1.4}$$

*where* $C(\epsilon)$ *is an effectively computable constant depending only on* $\epsilon$.

Owing to (1.2) every solution $(x, y, n)$ of the equation satisfies $a^2 < y^n$, then we have

$$n > \frac{2(\log a)}{\log y}. \tag{1.5}$$

Hence comparing (1.4) and (1.5), we see that the upper bound we get in this paper is optimal.

## 2 Preliminaries

**Lemma 2.1** *For a positive odd integer* $n$, *every solution* $(X, Y, Z)$ *of the equation*

$$X^2 + Y^2 = Z^n, \quad X, Y, Z \in \mathbb{N}, \gcd(X, Y) = 1 \tag{2.1}$$

*can be expressed as*

$$\begin{aligned} &Z = f^2 + g^2, \qquad X + Y\sqrt{-1} = \lambda_1(f + \lambda_2 g\sqrt{-1})^n, \quad f, g \in \mathbb{N}, \\ &\gcd(f, g) = 1, \quad \lambda_1, \lambda_2 \in \{1, -1\}. \end{aligned} \tag{2.2}$$

*Proof* See Section 15.2 of [8]. □

Let $\alpha$ be an algebraic number of degree $d$, $c$ be a leading coefficient of the defined polynomial of $\alpha$, $\alpha^{(j)}$ $(j = 1, \dots, d)$ be the whole conjugate numbers of $\alpha$. Then

$$h(\alpha) = \frac{1}{d}\left(\log c + \sum_{j=1}^{d} \log\max\{1, |\alpha^{(j)}|\}\right) \tag{2.3}$$

is called the Weil height of $\alpha$.

**Lemma 2.2** *For the positive integers* $b_1$ *and* $b_2$, *assume*

$$\Lambda = b_1 \log\alpha - b_2\pi\sqrt{-1}, \tag{2.4}$$

*where* $\log\alpha$ *is principal value of the logarithm of* $\alpha$. *If* $|\alpha| = 1$ *and* $\alpha$ *is not a unit root, then*

$$\log|\Lambda| \geq -8.87AB^2, \tag{2.5}$$

*where*

$$\begin{aligned} &A = \max\left\{20, 10.98|\log\alpha| + \frac{1}{2}dh(\alpha)\right\}, \\ &B = \max\left\{17, \frac{\sqrt{d}}{40}, 5.03 + 2.35\left(\frac{d}{2}\right) + \frac{d}{2}\left(\frac{b_1}{68.9} + \frac{b_2}{2A}\right)\right\}. \end{aligned}$$

*Proof* See Theorem 3 of [9]. □

## 3 Proof of theorem

Let $(x, y, n)$ be a solution of equation (1.2) with $n$ being odd and satisfying

$$n > \frac{(2+\epsilon)\log a}{\log y}. \tag{3.1}$$

By (1.2), we see that equation (2.1) has the solution $(X, Y, Z) = (x, a, y)$. So from Lemma 2.1, we get

$$y = f^2 + g^2, \quad f, g \in \mathbb{N}, \gcd(f, g) = 1, \tag{3.2}$$

$$x + a\sqrt{-1} = \lambda_1(f + \lambda_2 g\sqrt{-1})^n, \quad \lambda_1, \lambda_2 \in \{1, -1\}. \tag{3.3}$$

Assume

$$\theta = f + g\sqrt{-1}, \qquad \bar{\theta} = f - g\sqrt{-1}. \tag{3.4}$$

From (3.2) and (3.4), we have

$$\theta\bar{\theta} = y, \qquad |\theta| = |\bar{\theta}| = \sqrt{y}. \tag{3.5}$$

Let $\alpha = \theta/\bar{\theta}$. From (3.4) and (3.5), we see that $\alpha$ satisfies $|\alpha| = 1$ and

$$y\alpha^2 - 2(f^2 - g^2)\alpha + y = 0. \tag{3.6}$$

Since $\gcd(x, y) = 1$ by (1.1) and $n > 2$, we have $\gcd(x, a) = 1$ and $y$ is odd. And since $\gcd(f, g) = 1$ from (3.2), we see $f$ is odd, $g$ is even, so $\gcd(f^2 + g^2, f^2 - g^2) = \gcd(f^2 + g^2, 2(f^2 - g^2)) = 1$. Hence $y > 1$ and we see that $\alpha$ is not a unit root. And since the discriminant of the polynomial $yz^2 - 2(f^2 - g^2)z + y \in \mathbb{Z}[z]$ is equal to $-16f^2g^2$, we see that $\alpha$ is a quadratic algebraic number, $\alpha$ and $\alpha^{-1}$ are its whole conjugate numbers. Thus by (2.3), we deduce that the Weil height of $\alpha$ is

$$h(\alpha) = \frac{1}{2}\log y. \tag{3.7}$$

Since by (3.3) we have

$$x - a\sqrt{-1} = \lambda_1(f - \lambda_2 g\sqrt{-1})^n, \tag{3.8}$$

from (3.3), (3.4), (3.5), and (3.8), we obtain

$$a = \left|\frac{\theta^n - \bar{\theta}^n}{2\sqrt{-1}}\right| = \frac{1}{2}\left|\theta^n - \bar{\theta}^n\right| = \frac{1}{2}\left|\bar{\theta}^n\right|\left|\left(\frac{\theta}{\bar{\theta}}\right)^n - 1\right| = \frac{y^{n/2}}{2}\left|\alpha^n - 1\right|. \tag{3.9}$$

According to the maximum modulus principle, for any complex number $z$, we are sure that

$$\left|e^z - 1\right| \geq \frac{1}{2} \tag{3.10}$$

or

$$\left|e^{z}-1\right| \geq \frac{2}{\pi}|z-k\pi\sqrt{-1}|, \quad k\in\mathbb{Z}. \tag{3.11}$$

Assume $\alpha = e^{z}$. If (3.10) holds, then from (3.9), we can deduce that

$$a \geq \frac{y^{n/2}}{4}. \tag{3.12}$$

Combining (3.1) and (3.12), we get

$$4 > y^{\epsilon n/2(2+\epsilon)}. \tag{3.13}$$

However, since $y \geq 5$ by (3.2), we see that (3.13) does not hold when $n > 2(2+\epsilon)/\epsilon$. Hence, we only need to discuss the case when (3.11) holds.

Owing to $a^{2+\epsilon} < y^{n}$ by (3.1), if (3.11) holds, then from (3.9) and (3.11) we have

$$y^{n/(2+\epsilon)} > a \geq \frac{y^{n/2}}{\pi}|n\log\alpha - k\pi\sqrt{-1}|, \quad k\in\mathbb{N}, k\leq n. \tag{3.14}$$

Let

$$\Lambda = n\log\alpha - k\pi\sqrt{-1}. \tag{3.15}$$

By (3.14) and (3.15), we see

$$\log\pi - \log|\Lambda| \geq \frac{\epsilon n}{2(2+\epsilon)}\log y. \tag{3.16}$$

Since we have proved that $\alpha$ is not only a quadratic algebraic number but also a non-unit root with $|\alpha| = 1$, and the degree of $\alpha$ is 2, from Lemma 2.2, by (3.7), we see that $\Lambda$ satisfies (2.5), where

$$A = \max\left\{20, 10.98|\log\alpha| + \frac{1}{2}\log y\right\}, \tag{3.17}$$

$$B = \max\left\{17, 7.38 + \log\left(\frac{n}{2A} + \frac{k}{68.9}\right)\right\}. \tag{3.18}$$

Since $y \geq 5$ and the principal value of the logarithm of $\alpha$ satisfies $|\log\alpha| \leq \pi$, we deduce by (3.17) that

$$A \leq 10.98\pi + \frac{1}{2}\log y. \tag{3.19}$$

By (3.14) and (3.17), we have $k \leq n$ and $1/(2A) \leq 0.025$, respectively, therefore if $n > 68.9 \times 10^{8}$, then by (3.18) we get

$$B < 7.38 + \log(0.04n) < 4.17 + \log n. \tag{3.20}$$

Hence from (2.5), (3.16), (3.19), and (3.20), we have

$$\log\pi + 8.87\left(10.98\pi + \frac{1}{2}\log y\right)(4.17 + \log n)^2 > \frac{\epsilon n}{2(2+\epsilon)}\log y. \tag{3.21}$$

Since $y \geq 5$, we see by (3.21) that

$$\frac{2(2+\epsilon)}{\epsilon}\bigl(1 + 194.56(4.17 + \log n)^2\bigr) > n. \tag{3.22}$$

From (3.22), we get $n < C'(\epsilon)$, where $C'(\epsilon)$ is an effectively computable constant depending only on $\epsilon$. Let

$$C(\epsilon) = \max\left\{68.9 \times 10^8, \frac{2(2+\epsilon)}{\epsilon}, C'(\epsilon)\right\}. \tag{3.23}$$

We see by (3.23) that $C(\epsilon)$ is also an effectively computable constant depending only on $\epsilon$, and to sum up, we can deduce when $n > C(\epsilon)$, the solution $(x, y, n)$ of equation (1.2) does not satisfy (3.1), so (1.4) holds definitely. Therefore, we completed the proof of the theorem.

**Competing interests**

The author declares that they have no competing interests.

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