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Boundedness of fractional integrals on weighted Herz spaces with variable exponent

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Abstract

Our aim is to prove the boundedness of fractional integral operators on weighted Herz spaces with variable exponent. Our method is based on the theory on Banach function spaces and the Muckenhoupt theory with variable exponent.

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Keywords: Herz spaces; Muckenhoupt weight; variable exponent

1 Introduction

The boundedness of fractional integrals on function spaces is one of the important problems not in harmonic analysis but also in potential theory and in partial differential equations. Among the development of variable exponent analysis we can list up boundedness of fractional integrals on function spaces with variable exponent. Capone, Cruz-Uribe and Fiorenza [1] have proved the boundedness on Lebesgue spaces with variable exponent, provided that the exponents satisfy the log-Hölder continuous conditions. The conditions on variable exponents have been established by the study of the boundedness of the Hardy-Littlewood maximal operator on spaces with variable exponent [2–8]. Cruz-Uribe, Fiorenza, Martell and Pérez [9] have also proved the boundedness of fractional integrals by virtue of the extrapolation method.

Every Herz space has an interesting norm involving both local and global information and has been studied in harmonic analysis. Lu and Yang [10] have proved the boundedness of fractional integrals on Herz spaces. Inspired by the study of variable exponent analysis and on Herz spaces the first author has defined Herz spaces with variable exponent [11, 12]. Later he has proved the boundedness of fractional integrals on Herz spaces with variable exponent [13]. Almeida and Drihem [14] have also independently proved the boundedness.

Based on the Muckenhoupt theory [15] the modern harmonic analysis has been greatly developed. Recently the generalized Muckenhoupt weights with variable exponent have been considered [16–20]. In particular, Cruz-Uribe, Fiorenza and Neugebauer [17] and Diening and Hästö [18] have independently proved the equivalence between the Muckenhoupt condition and the boundedness of the Hardy-Littlewood maximal operator on weighted Lebesgue spaces in the variable exponent setting. We also note that Cruz-Uribe and Wang [21] have obtained the boundedness of fractional integrals on weighted Lebesgue spaces with variable exponent applying the extrapolation theorem.

In this paper we define weighted Herz spaces with variable exponent and prove the boundedness of fractional integrals on those spaces under proper assumptions on weights and exponents. Our argument is based on the theory on Banach function spaces and on the Muckenhoupt theory with variable exponent. The authors have also considered other problems on boundedness of some operators on weighted Herz spaces with variable exponent in the recent preprints [22, 23].

In this paper we use the following symbols and notation:

1. For any measurable set E , $|E|$ denotes the Lebesgue measure and χ_E means the characteristic function.
2. A locally integrable and positive function defined on \mathbb{R}^n is said to be a weight. We write $w(E) := \int_E w(x) dx$ for a weight w and a measurable set E .
3. If there exists a positive constant C independent of the main parameters such that $A \leq CB$, then we write $A \lesssim B$.

2 Preliminaries

2.1 Variable Lebesgue spaces

Based on the fundamental papers and books [2, 7, 8, 24] we introduce Lebesgue spaces with variable exponent. Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ be a measurable function. The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of all complex-valued measurable functions f defined on \mathbb{R}^n satisfying

$$\rho_p(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty.$$

It is well known that the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ becomes a Banach space equipped with a norm given by

$$\|f\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0 : \rho_p \left(\frac{f}{\lambda} \right) \leq 1 \right\}.$$

Denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot) : \mathbb{R}^n \rightarrow (1, \infty)$ such that

$$1 < p^- := \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x), \quad p^+ := \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x) < \infty. \tag{1}$$

A measurable function $p(\cdot)$ defined on \mathbb{R}^n is said to be globally log-Hölder continuous if it satisfies

$$|p(x) - p(y)| \lesssim \frac{1}{-\log(|x - y|)} \quad (x, y \in \mathbb{R}^n, |x - y| \leq 1/2), \tag{2}$$

$$|p(x) - p_\infty| \lesssim \frac{1}{\log(e + |x|)} \quad (x \in \mathbb{R}^n) \tag{3}$$

for some real number p_∞ . The set of $p(\cdot)$ satisfying (2) and (3) is denoted by $LH(\mathbb{R}^n)$. It is also well known that the Hardy-Littlewood maximal operator M , defined by

$$Mf(x) := \sup_{B: \text{ball}, x \in B} \frac{1}{|B|} \int_B |f(y)| dy$$

is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ whenever $p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$ [3–6].

2.2 The Muckenhoupt weights with variable exponent

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and w be a weight. The weighted variable exponent Lebesgue space $L^{p(\cdot)}(w)$ is the set of all complex-valued measurable functions f such that $fw^{1/p(\cdot)} \in L^{p(\cdot)}(\mathbb{R}^n)$. The space $L^{p(\cdot)}(w)$ is a Banach space equipped with the norm

$$\|f\|_{L^{p(\cdot)}(w)} := \|fw^{1/p(\cdot)}\|_{L^{p(\cdot)}}.$$

Below $p'(\cdot)$ is the conjugate exponent of $p(\cdot)$ given by $1/p(\cdot) + 1/p'(\cdot) = 1$. Now we define the Muckenhoupt classes. We begin with the classical Muckenhoupt A_1 weight.

Definition 1 A weight is said to be a Muckenhoupt A_1 weight if $Mw(x) \lesssim w(x)$ holds for almost every $x \in \mathbb{R}^n$. The set A_1 consists of all Muckenhoupt A_1 weights.

The original Muckenhoupt A_p class with constant exponent $p \in (1, \infty)$ established by Muckenhoupt [15] can be generalized in terms of a variable exponent as follows.

Definition 2 Suppose $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. A weight w is said to be an $A_{p(\cdot)}$ weight if

$$\sup_{B:\text{ball}} \frac{1}{|B|} \|w^{1/p(\cdot)} \chi_B\|_{L^{p(\cdot)}} \|w^{-1/p(\cdot)} \chi_B\|_{L^{p'(\cdot)}} < \infty.$$

Our symbol $A_{p(\cdot)}$ slightly differs from that in [17, 18, 21] where the space $L^{p(\cdot)}(w)$ is defined as the set consisting of all f such that $fw \in L^{p(\cdot)}(\mathbb{R}^n)$. If $p(\cdot)$ equals a constant $p \in (1, \infty)$ in Definition 2, then we see immediately that the definition is equivalent to the classical Muckenhoupt class A_p [15].

Diening and Hästö [18] have pointed out that Definition 2 does not directly imply the monotone property of the class $A_{p(\cdot)}$. In order to obtain the property they have generalized the Muckenhoupt class as follows.

Definition 3 Suppose $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. A weight w is said to be an $\tilde{A}_{p(\cdot)}$ weight if

$$\sup_{B:\text{ball}} |B|^{-p_B} \|w \chi_B\|_{L^1} \|w^{-1} \chi_B\|_{L^{p'(\cdot)/p(\cdot)}} < \infty,$$

where $p_B := (\frac{1}{|B|} \int_B \frac{1}{p(x)} dx)^{-1}$ is the harmonic average of $p(\cdot)$ over B . The set $\tilde{A}_{p(\cdot)}$ consists of all $\tilde{A}_{p(\cdot)}$ weights.

Based on the definition $\tilde{A}_{p(\cdot)}$ Diening and Hästö [18], Lemma 3.1, have proved the next monotone property.

Theorem 1 Suppose $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$ and $p(\cdot) \leq q(\cdot)$. Then we have $A_1 \subset \tilde{A}_{p(\cdot)} \subset \tilde{A}_{q(\cdot)}$.

We next state the relation between the generalized Muckenhoupt conditions and the boundedness of the Hardy-Littlewood maximal operator on weighted Lebesgue spaces in the variable exponent setting.

Theorem 2 Suppose $p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$. Then the following three conditions are equivalent:

- (A) $w \in A_{p(\cdot)}$.
- (B) $w \in \tilde{A}_{p(\cdot)}$.
- (C) *The Hardy-Littlewood maximal operator is bounded on the weighted variable Lebesgue space $L^{p(\cdot)}(w)$.*

Cruz-Uribe, Fiorenza and Neugebauer [17] have proved (A) \Leftrightarrow (C). On the other hand, Diening and Hästö [18] have proved (B) \Leftrightarrow (C). By Theorem 2 we can identify $A_{p(\cdot)}$ and $\tilde{A}_{p(\cdot)}$, provided that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$. Combining Theorems 1 and 2 we get the monotone property for the class $A_{p(\cdot)}$, that is, the next corollary is true.

Corollary 1 *If $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$ and $p(\cdot) \leq q(\cdot)$, then we have*

$$A_1 \subset A_{p(\cdot)} \subset A_{q(\cdot)}.$$

In order to state the boundedness of fractional integrals on weighted function spaces we shall define the class $A(p_1(\cdot), p_2(\cdot))$ as follows.

Definition 4 Let $0 < \beta < n$ and $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1/p_2(x) \equiv 1/p_1(x) - \beta/n$. A weight w is said to be an $A(p_1(\cdot), p_2(\cdot))$ weight if

$$\|w\chi_B\|_{L^{p_2(\cdot)}} \|w^{-1}\chi_B\|_{L^{p_1'(\cdot)}} \lesssim |B|^{1-\frac{\beta}{n}}$$

holds for all balls $B \subset \mathbb{R}^n$.

Lemma 1 *Let $0 < \beta < n$ and $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1/p_2(x) \equiv 1/p_1(x) - \beta/n$. Then $w \in A(p_1(\cdot), p_2(\cdot))$ if and only if $w^{p_2(\cdot)} \in A_{1+p_2(\cdot)/p_1'(\cdot)}$.*

Proof Cruz-Uribe and Wang [21], Proposition 5.4, have proved a result similar to the lemma. Komori and Matsuoka [25] have also considered the case with constant exponent. We will prove the lemma by referring to [21, 25].

Note that

$$\frac{p_2(\cdot)}{1 + \frac{p_2(\cdot)}{p_1'(\cdot)}} = \frac{1}{\frac{1}{p_2(\cdot)} + \frac{1}{p_1'(\cdot)}} = \frac{1}{1 - \frac{\beta}{n}}$$

by the assumption on $p_1(\cdot)$ and $p_2(\cdot)$. Below we fix a ball $B \subset \mathbb{R}^n$ arbitrarily. It is easy to see that

$$\begin{aligned} & \frac{1}{|B|} \left\| \left(w^{p_2(\cdot)} \right)^{\left(1 + \frac{p_2(\cdot)}{p_1'(\cdot)}\right)^{-1}} \chi_B \right\|_{L^{1 + \frac{p_2(\cdot)}{p_1'(\cdot)}}} \left\| \left(w^{p_2(\cdot)} \right)^{-\left(1 + \frac{p_2(\cdot)}{p_1'(\cdot)}\right)^{-1}} \chi_B \right\|_{L^{\left(1 + \frac{p_2(\cdot)}{p_1'(\cdot)}\right)'}} \\ &= \frac{1}{|B|} \left\| w^{p_2(\cdot)\left(1 + \frac{p_2(\cdot)}{p_1'(\cdot)}\right)^{-1}} \chi_B \right\|_{L^{1 + \frac{p_2(\cdot)}{p_1'(\cdot)}}} \left\| w^{-p_2(\cdot)\left(1 + \frac{p_2(\cdot)}{p_1'(\cdot)}\right)^{-1}} \chi_B \right\|_{L^{\left(1 + \frac{p_2(\cdot)}{p_1'(\cdot)}\right)'}} \\ &= \frac{1}{|B|} \|w\chi_B\|_{L^{p_2(\cdot)}}^{\frac{1}{1-\beta/n}} \|w^{-1}\chi_B\|_{L^{p_1'(\cdot)}}^{\frac{1}{1-\beta/n}} \end{aligned} \tag{4}$$

holds.

If $w \in A(p_1(\cdot), p_2(\cdot))$, then we have

$$\begin{aligned} & \frac{1}{|B|} \left\| \left(w^{p_2(\cdot)} \right)^{\left(1 + \frac{p_2(\cdot)}{p_1(\cdot)} \right)^{-1}} \chi_B \right\|_{L^{p_1(\cdot)}_{1 + \frac{p_2(\cdot)}{p_1(\cdot)}}} \left\| \left(w^{p_2(\cdot)} \right)^{-\left(1 + \frac{p_2(\cdot)}{p_1(\cdot)} \right)^{-1}} \chi_B \right\|_{L^{\left(1 + \frac{p_2(\cdot)}{p_1(\cdot)} \right)'}} \\ &= \frac{1}{|B|} \|w \chi_B\|_{L^{p_2(\cdot)}}^{\frac{1}{1-\beta/n}} \|w^{-1} \chi_B\|_{L^{p_1(\cdot)}}^{\frac{1}{1-\beta/n}} \lesssim 1. \end{aligned}$$

This implies that $w^{p_2(\cdot)} \in A_{1 + \frac{p_2(\cdot)}{p_1(\cdot)}}$.

If $w^{p_2(\cdot)} \in A_{1 + \frac{p_2(\cdot)}{p_1(\cdot)}}$, then, by (4), we see that

$$\begin{aligned} & \|w \chi_B\|_{L^{p_2(\cdot)}} \|w^{-1} \chi_B\|_{L^{p_1(\cdot)}} \\ &= \left\| \left(w^{p_2(\cdot)} \right)^{\left(1 + \frac{p_2(\cdot)}{p_1(\cdot)} \right)^{-1}} \chi_B \right\|_{L^{p_1(\cdot)}_{1 + \frac{p_2(\cdot)}{p_1(\cdot)}}}^{1-\beta/n} \left\| \left(w^{p_2(\cdot)} \right)^{-\left(1 + \frac{p_2(\cdot)}{p_1(\cdot)} \right)^{-1}} \chi_B \right\|_{L^{\left(1 + \frac{p_2(\cdot)}{p_1(\cdot)} \right)'}}^{1-\beta/n} \\ &\lesssim |B|^{1-\beta/n} \end{aligned}$$

holds. Hence we have $w \in A(p_1(\cdot), p_2(\cdot))$. This completes the proof. □

2.3 Herz spaces with variable exponent

Let $\Omega \subset \mathbb{R}^n$ be a measurable set and w a positive and locally integrable function on Ω . The set $L_{loc}^{p(\cdot)}(\Omega, w)$ consists of all functions f satisfying the following condition: for all compact sets $E \subset \Omega$ there exists a constant $\lambda > 0$ such that

$$\int_E \left| \frac{f(x)}{\lambda} \right|^{p(x)} w(x) \, dx < \infty.$$

Let $l \in \mathbb{Z}$. We use the following notations in order to define Herz spaces:

$$B_l := \{x \in \mathbb{R}^n : |x| \leq 2^l\}, \quad R_l := B_l \setminus B_{l-1}, \quad \chi_l := \chi_{R_l}.$$

Definition 5 Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < q < \infty$, and $\alpha \in \mathbb{R}$. The homogeneous weighted Herz space $\dot{K}_{p(\cdot)}^{\alpha, q}(w)$ is the collection of $f \in L_{loc}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}, w)$ such that

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha, q}(w)} := \left(\sum_{k=-\infty}^{\infty} 2^{\alpha k q} \|f \chi_k\|_{L^{p(\cdot)}(w)} \right)^{1/q} < \infty. \tag{5}$$

Herz spaces with variable exponent were initially defined by the first author [11, 12]. The weighted case has been recently studied by the authors [22, 23].

2.4 Weighted Banach function spaces

We introduce Banach function space and state fundamental properties of it based on the book [26] by Bennett and Sharpley. We additionally show some properties of Banach function spaces in terms of the boundedness of the Hardy-Littlewood maximal operator. We will also consider the weighted case based on [27] by Karlovich and Spitkovsky.

Definition 6 Let \mathcal{M} be the set of all complex-valued measurable functions defined on \mathbb{R}^n , and X a linear subspace of \mathcal{M} .

1. The space X is said to be a Banach function space if there exists a functional $\|\cdot\|_X : \mathcal{M} \rightarrow [0, \infty]$ satisfying the following properties: Let $f, g, f_j \in \mathcal{M}$ ($j = 1, 2, \dots$), then
 - (a) $f \in X$ holds if and only if $\|f\|_X < \infty$.
 - (b) Norm property:
 - i. Positivity: $\|f\|_X \geq 0$.
 - ii. Strict positivity: $\|f\|_X = 0$ holds if and only if $f(x) = 0$ for almost every $x \in \mathbb{R}^n$.
 - iii. Homogeneity: $\|\lambda f\|_X = |\lambda| \cdot \|f\|_X$ holds for all $\lambda \in \mathbb{C}$.
 - iv. Triangle inequality: $\|f + g\|_X \leq \|f\|_X + \|g\|_X$.
 - (c) Symmetry: $\|f\|_X = \| |f| \|_X$.
 - (d) Lattice property: If $0 \leq g(x) \leq f(x)$ for almost every $x \in \mathbb{R}^n$, then $\|g\|_X \leq \|f\|_X$.
 - (e) Fatou property: If $0 \leq f_j(x) \leq f_{j+1}(x)$ for all j and $f_j(x) \rightarrow f(x)$ as $j \rightarrow \infty$ for almost every $x \in \mathbb{R}^n$, then $\lim_{j \rightarrow \infty} \|f_j\|_X = \|f\|_X$.
 - (f) For every measurable set $F \subset \mathbb{R}^n$ such that $|F| < \infty$, $\|\chi_F\|_X$ is finite. Additionally there exists a constant $C_F > 0$ depending only on F so that $\int_F |h(x)| \, dx \leq C_F \|h\|_X$ holds for all $h \in X$.
2. Suppose that X is a Banach function space equipped with a norm $\|\cdot\|_X$. The associated space X' is defined by

$$X' := \{f \in \mathcal{M} : \|f\|_{X'} < \infty\},$$

where

$$\|f\|_{X'} := \sup_g \left\{ \left| \int_{\mathbb{R}^n} f(x)g(x) \, dx \right| : \|g\|_X \leq 1 \right\}.$$

The proof of the following fundamental lemma is found in [26].

Lemma 2 *Let X be a Banach function space. Then the following hold:*

1. *The associated space X' is also a Banach function space.*
2. *(The Lorentz-Luxemburg theorem.) $(X')' = X$ holds, in particular, the norms $\|\cdot\|_{(X)'}$ and $\|\cdot\|_X$ are equivalent.*
3. *(The generalized Hölder inequality.) If $f \in X$ and $g \in X'$, then we have*

$$\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq \|f\|_X \|g\|_{X'}.$$

An easy application of the generalized Hölder inequality gives us the following lemma.

Lemma 3 *If X is a Banach function space, then we have that for all balls B ,*

$$1 \leq \frac{1}{|B|} \|\chi_B\|_X \|\chi_B\|_{X'}.$$

Kováčik and Rákosník [24] have proved that the generalized Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent $p(\cdot)$ is a Banach function space and the associate space is $L^{p'(\cdot)}(\mathbb{R}^n)$ with norm equivalence.

If we assume some conditions for the boundedness of the Hardy-Littlewood maximal operator M on X , then the norm $\|\cdot\|_X$ has properties similar to the classical Muckenhoupt weights.

Lemma 4 *Let X be a Banach function space. Suppose that the Hardy-Littlewood maximal operator M is weakly bounded on X , that is,*

$$\|\chi_{\{Mf>\lambda\}}\|_X \lesssim \lambda^{-1} \|f\|_X \tag{6}$$

is true for all $f \in X$ and all $\lambda > 0$. Then we have

$$\sup_{B:\text{ball}} \frac{1}{|B|} \|\chi_B\|_X \|\chi_B\|_{X'} < \infty. \tag{7}$$

The proof of Lemma 4 is found in the first author’s paper [19], Lemmas 2.4 and 2.5, and [28], Lemmas G’ and H.

Remark 1 If M is bounded on X , that is, $\|Mf\|_X \lesssim \|f\|_X$ holds for all $f \in X$, then we can easily check that (6) holds. On the other hand, if M is bounded on the associate space X' , then Lemma 2 shows that (7) is true.

Below we define weighted Banach function space and give some of its property. Let X be a Banach function space. The set $X_{\text{loc}}(\mathbb{R}^n)$ consists of all measurable functions f such that $f\chi_E \in X$ for any compact set E with $|E| < \infty$. Given a function W such that $0 < W(x) < \infty$ for almost every $x \in \mathbb{R}^n$, $W \in X_{\text{loc}}(\mathbb{R}^n)$ and $W^{-1} \in (X')_{\text{loc}}(\mathbb{R}^n)$, we define the weighted Banach function space

$$X(\mathbb{R}^n, W) := \{f \in \mathcal{M} : fW \in X\}.$$

Then the following hold.

Lemma 5

1. *The weighted Banach function space $X(\mathbb{R}^n, W)$ is a Banach function space equipped the norm*

$$\|f\|_{X(\mathbb{R}^n, W)} := \|fW\|_X.$$

2. *The associate space of $X(\mathbb{R}^n, W)$ is a Banach function space and equals $X'(\mathbb{R}^n, W^{-1})$.*

The properties above naturally arise from those of the usual Banach function spaces and the proof is found in [27].

Remark 2 Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Comparing the definition of $X(\mathbb{R}^n, W)$ with weighted Lebesgue spaces $L^{p(\cdot)}(w^{p(\cdot)})$ and $L^{p'(\cdot)}(w^{-p'(\cdot)})$ respectively, we obtain the following:

1. If we take $X = L^{p(\cdot)}(\mathbb{R}^n)$ and $W = w$, then we have $L^{p(\cdot)}(\mathbb{R}^n, w) = L^{p(\cdot)}(w^{p(\cdot)})$.

2. If we take $X = L^{p(\cdot)}(\mathbb{R}^n)$ and $W = w^{-1}$, then we have $L^{p'(\cdot)}(\mathbb{R}^n, w^{-1}) = L^{p'(\cdot)}(w^{-p'(\cdot)})$. Therefore Lemma 5 yields

$$(L^{p(\cdot)}(w^{p(\cdot)}))' = (L^{p(\cdot)}(\mathbb{R}^n, w))' = L^{p'(\cdot)}(\mathbb{R}^n, w^{-1}) = L^{p'(\cdot)}(w^{-p'(\cdot)}).$$

The next lemma has been initially proved by the first author [12] in the case that $X = L^{p(\cdot)}(\mathbb{R}^n)$, however, his argument depends on Diening’s work [6]. Cruz-Uribe, Hernández and Martell [29] have recently given its alternative proof based on the Rubio de Francia algorithm [30–32]. As is mentioned in the authors’ preprint [22], the proof due to [29] is self-contained and valid for general Banach function spaces X .

Lemma 6 *Let X be a Banach function space. Suppose that M is bounded on the associate space X' . Then there exists a constant $0 < \delta < 1$ such that for all balls $B \subset \mathbb{R}^n$ and all measurable sets $E \subset B$,*

$$\frac{\|\chi_E\|_X}{\|\chi_B\|_X} \lesssim \left(\frac{|E|}{|B|}\right)^\delta.$$

Proof For the reader’s convenience we shall give the proof based on [29]. Let $A := \|M\|_{X' \rightarrow X'}$ and define a function

$$Rg(x) := \sum_{k=0}^\infty \frac{M^k g(x)}{(2A)^k} \quad (g \in X'), \tag{8}$$

where

$$M^k g := \begin{cases} |g| & (k = 0), \\ Mg & (k = 1), \\ M(M^{k-1}g) & (k \geq 2). \end{cases}$$

For every $g \in X$, the function Rg satisfies the following properties:

1. $|g(x)| \leq Rg(x)$ for almost every $x \in \mathbb{R}^n$.
2. $\|Rg\|_{X'} \leq 2\|g\|_{X'}$, namely the operator R is bounded on X' .
3. $M(Rg)(x) \leq 2ARg(x)$, that is, Rg is a Muckenhoupt A_1 weight. We note that the constant $2A$ appearing in the right-hand side is independent of g and x .

We can write $Rg(S) = \int_S Rg(x) dx$ for every measurable set $S \subset \mathbb{R}^n$ because Rg is a weight. Thus by virtue of [33], Chapter 7, we can take positive constants C and $\delta < 1$ depending only on A and n so that, for all balls B and all measurable sets $E \subset B$,

$$\frac{Rg(E)}{Rg(B)} \leq C \left(\frac{|E|}{|B|}\right)^\delta$$

holds. Now we fix $g \in X'$ with $\|g\|_{X'} \leq 1$ arbitrarily. By virtue of generalized Hölder’s inequality we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\chi_E(x)g(x)| dx &\leq Rg(E) \leq C \left(\frac{|E|}{|B|}\right)^\delta \cdot Rg(B) \leq C \left(\frac{|E|}{|B|}\right)^\delta \cdot \|\chi_B\|_X \|Rg\|_{X'} \\ &\lesssim \left(\frac{|E|}{|B|}\right)^\delta \|\chi_B\|_X. \end{aligned}$$

Therefore by the duality we get

$$\|\chi_E\|_X \lesssim \sup_g \left\{ \left| \int_{\mathbb{R}^n} \chi_E(x)g(x) \, dx \right| : g \in X', \|g\|_{X'} \leq 1 \right\} \lesssim \left(\frac{|E|}{|B|} \right)^\delta \|\chi_B\|_X.$$

This completes the proof of the lemma. □

3 The main results

3.1 Remarks on boundedness of fractional integrals on Lebesgue spaces

Definition 7 Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and a weight w , we say $(p(\cdot), w)$ is an M -pair if the maximal operator M is bounded on $L^{p(\cdot)}(w^{p(\cdot)})$ and on $L^{p'(\cdot)}(w^{-p'(\cdot)})$.

Let $0 < \beta < n$. Then the fractional integral operator I^β is defined by

$$I^\beta f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\beta}} \, dy.$$

Cruz-Uribe and Wang [21], Corollary 3.7, have obtained the following boundedness of I^β on weighted variable exponent Lebesgue spaces.

Theorem 3 Let $p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$, $0 < \beta < n/p_1^+$ and $\sigma := (n/\beta)'$. Define $p_2(\cdot)$ by $1/p_2(\cdot) \equiv 1/p_1(\cdot) - \beta/n$. Then for all weights w such that $(p_2(\cdot)/\sigma, w^\sigma)$ is an M -pair, I^β is bounded from $L^{p_1(\cdot)}(w^{p_1(\cdot)})$ to $L^{p_2(\cdot)}(w^{p_2(\cdot)})$.

Let $w \in A(p_1(\cdot), p_2(\cdot))$. Note that $p_2(\cdot)/\sigma = 1 + p_2(\cdot)/p_1'(\cdot)$. Lemma 1 and the equivalence

$$w^{p_2(\cdot)} \in A_{1+p_2(\cdot)/p_1'(\cdot)} = A_{p_2(\cdot)/\sigma} \iff w^{-\sigma(p_2(\cdot)/\sigma)'} \in A_{(p_2(\cdot)/\sigma)'}$$

imply that M is bounded on $L^{p_2(\cdot)/\sigma}(w^{\sigma p_2(\cdot)/\sigma})$ and on $L^{(p_2(\cdot)/\sigma)'}(w^{-\sigma(p_2(\cdot)/\sigma)'})$. Therefore, $(p_2(\cdot)/\sigma, w^\sigma)$ is an M -pair whenever

$$w \in A(p_1(\cdot), p_2(\cdot)), \quad p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n) \quad \text{and} \quad 0 < \beta < n/p_1^+.$$

Hence we have the following corollary.

Corollary 2 Let $p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$ and $0 < \beta < n/p_1^+$. Define $p_2(\cdot)$ by $1/p_2(\cdot) \equiv 1/p_1(\cdot) - \beta/n$. If $w \in A(p_1(\cdot), p_2(\cdot))$, then I^β is bounded from $L^{p_1(\cdot)}(w^{p_1(\cdot)})$ to $L^{p_2(\cdot)}(w^{p_2(\cdot)})$.

3.2 Boundedness of fractional integrals on Herz spaces

Let $p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$ and $w^{p_2(\cdot)} \in A_1$. Then the monotone property yields $w^{p_2(\cdot)} \in A_{p_2(\cdot)}$. Hence the Hardy-Littlewood maximal operator M is bounded on $L^{p_2(\cdot)}(w^{p_2(\cdot)})$. On the other hand, by the definition of $A_{p(\cdot)}$ it is easy to see that $w^{p_2(\cdot)} \in A_{p_2(\cdot)}$ implies $w^{-p_2'(\cdot)} \in A_{p_2'(\cdot)}$. Thus M is bounded on $L^{p_2'(\cdot)}(w^{-p_2'(\cdot)})$. Therefore applying Remark 2, Lemma 6, and the Lorentz-Luxemburg theorem we can take constants $\delta_1, \delta_2 \in (0, 1)$ such that

$$\frac{\|\chi_E\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}}{\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}} = \frac{\|\chi_E\|_{(L^{p_2'(\cdot)}(w^{-p_2'(\cdot)})')'}}{\|\chi_B\|_{(L^{p_2'(\cdot)}(w^{-p_2'(\cdot)})')'}} \lesssim \left(\frac{|E|}{|B|} \right)^{\delta_1}, \tag{9}$$

$$\frac{\|\chi_E\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}{\|\chi_B\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}} \lesssim \left(\frac{|E|}{|B|}\right)^{\delta_2} \tag{10}$$

for all balls B and all measurable sets $E \subset B$. Now we take a positive number β so that $0 < \beta < n(\delta_1 + \delta_2)$. Then we can take a real number α such that $-n\delta_1 < \alpha < n\delta_2 - \beta$ because the choice of β shows $-n\delta_1 < n\delta_2 - \beta$. Using $p_2(\cdot)$ and β , we additionally define $p_1(\cdot)$ so that $1/p_2(\cdot) \equiv 1/p_1(\cdot) - \beta/n$ holds. We see that $p_1(\cdot)$ satisfies $p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$ and $0 < \beta < n/p_1^+$. Applying the monotone property again we have $w^{p_2(\cdot)} \in A_1 \subset A_{1+p_2(\cdot)/p_1^+(\cdot)}$. Thus by Lemma 1 we get $w \in A(p_1(\cdot), p_2(\cdot))$.

Therefore in the setting as above we can apply Corollary 2 to obtain the boundedness of the fractional integral operator I^β on Herz spaces.

Theorem 4 *Let $0 < q_1 \leq q_2 < \infty$, $p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$, $w^{p_2(\cdot)} \in A_1$, $\delta_1, \delta_2 \in (0, 1)$ be the constants appearing in (9) and (10) respectively, $0 < \beta < n(\delta_1 + \delta_2)$ and $-n\delta_1 < \alpha < n\delta_2 - \beta$. Define $p_1(\cdot)$ by $1/p_2(\cdot) \equiv 1/p_1(\cdot) - \beta/n$. Then the fractional integral operator I^β is a bounded operator from $\dot{K}_{p_2(\cdot)}^{\alpha, q_2}(w^{p_2(\cdot)})$ to $\dot{K}_{p_1(\cdot)}^{\alpha, q_1}(w^{p_1(\cdot)})$.*

Proof Let $f \in \dot{K}_{p_1(\cdot)}^{\alpha, q_1}(w^{p_1(\cdot)})$. Then, by the Jensen inequality, we have

$$\begin{aligned} \|I^\beta f\|_{\dot{K}_{p_2(\cdot)}^{\alpha, q_2}(w^{p_2(\cdot)})}^{q_1} &= \left(\sum_{k=-\infty}^{\infty} 2^{\alpha q_2 k} \|(I^\beta f)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}^{q_2} \right)^{\frac{q_1}{q_2}} \\ &\leq \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \|(I^\beta f)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}^{q_1}. \end{aligned} \tag{11}$$

Let $f_j := f\chi_j$ for any $j \in \mathbb{Z}$. Then $f = \sum_{j=-\infty}^{\infty} f_j$. So we have

$$\begin{aligned} \|I^\beta f\|_{\dot{K}_{p_2(\cdot)}^{\alpha, q_2}(w^{p_2(\cdot)})}^{q_1} &\leq \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \|(I^\beta f)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}^{q_1} \\ &\leq \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left(\sum_{j \leq k-2} \|(I^\beta f_j)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \right)^{q_1} \\ &\quad + \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left(\sum_{j=k-1}^{k+1} \|(I^\beta f_j)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \right)^{q_1} \\ &\quad + \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left(\sum_{j \geq k+2} \|(I^\beta f_j)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \right)^{q_1} \\ &=: U_1 + U_2 + U_3. \end{aligned} \tag{12}$$

Step 1. We estimate U_1 . By the definition of I^β and the generalized Hölder inequality, we obtain

$$\begin{aligned} |(I^\beta f_j)(x)|\chi_k(x) &\lesssim \chi_k(x) \int_{\mathbb{R}^n} |x-y|^{\beta-n} |f_j(y)| \, dy \\ &\lesssim 2^{k(\beta-n)} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \chi_k(x). \end{aligned}$$

By taking the $L^{p_2(\cdot)}(w^{p_2(\cdot)})$ -norm and using Lemma 4, we have

$$\begin{aligned} & \| (I^\beta f_j) \chi_k \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\ & \lesssim 2^{k(\beta-n)} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \| \chi_j \|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \| \chi_{B_k} \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\ & = 2^{k\beta} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \| \chi_j \|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} 2^{-kn} \| \chi_{B_k} \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\ & \lesssim 2^{k\beta} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \| \chi_j \|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \| \chi_{B_k} \|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}^{-1}. \end{aligned}$$

By (10), we see that

$$\begin{aligned} & \| (I^\beta f_j) \chi_k \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\ & \lesssim 2^{k\beta} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \| \chi_j \|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \| \chi_{B_k} \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}^{-1} \\ & = 2^{k\beta} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \| \chi_j \|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \| \chi_{B_j} \|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}^{-1} \frac{\| \chi_{B_j} \|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}{\| \chi_{B_k} \|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}} \\ & \lesssim 2^{k\beta} 2^{n\delta_2(j-k)} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \| \chi_j \|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \| \chi_{B_j} \|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}^{-1}. \end{aligned} \tag{13}$$

By the obvious inequality $2^{j\beta} \chi_{B_j}(x) \lesssim (I^\beta f_{B_j})(x)$ and the boundedness of $I^\beta : L^{p_1(\cdot)}(w^{p_1(\cdot)}) \rightarrow L^{p_2(\cdot)}(w^{p_2(\cdot)})$, we have

$$\| \chi_{B_j} \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \lesssim 2^{-j\beta} \| I^\beta \chi_{B_j} \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \lesssim 2^{-j\beta} \| \chi_{B_j} \|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}.$$

By using Lemma 4 again, we obtain

$$\begin{aligned} \| \chi_{B_j} \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} & \lesssim 2^{-j\beta} \| \chi_{B_j} \|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \lesssim 2^{j(n-\beta)} \| \chi_{B_j} \|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}^{-1} \\ & \lesssim 2^{j(n-\beta)} \| \chi_j \|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}^{-1}. \end{aligned} \tag{14}$$

Combining (13) and (14) and using Lemma 3, we get

$$\begin{aligned} & \| (I^\beta f_j) \chi_k \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\ & \lesssim 2^{k\beta} 2^{n\delta_2(j-k)} 2^{j(n-\beta)} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \| \chi_{B_j} \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}^{-1} \| \chi_{B_j} \|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}^{-1} \\ & = 2^{(\beta-n\delta_2)(k-j)} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \left(2^{-jn} \| \chi_{B_j} \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \| \chi_{B_j} \|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'} \right)^{-1} \\ & \lesssim 2^{(\beta-n\delta_2)(k-j)} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}. \end{aligned}$$

Thus we get $U_1 \lesssim \sum_{k=-\infty}^\infty \left(\sum_{j \leq k-2} 2^{\alpha j} 2^{(\beta-n\delta_2+\alpha)(k-j)} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1}$. Note that $\beta - n\delta_2 + \alpha < 0$. We consider the two cases ‘ $1 < q_1 < \infty$ ’ and ‘ $0 < q_1 \leq 1$ ’.

If $1 < q_1 < \infty$, then, by using the Hölder inequality, we obtain

$$\begin{aligned} U_1 & \lesssim \sum_{k=-\infty}^\infty \left(\sum_{j \leq k-2} 2^{\alpha j} 2^{(\beta-n\delta_2+\alpha)(k-j)} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\ & \lesssim \sum_{k=-\infty}^\infty \left(\sum_{j \leq k-2} 2^{\alpha j q_1} 2^{(\beta-n\delta_2+\alpha)(k-j)q_1/2} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}^{q_1} \right) \left(\sum_{j \leq k-2} 2^{(\beta-n\delta_2+\alpha)(k-j)q_1'/2} \right)^{q_1/q_1'} \end{aligned}$$

$$\begin{aligned} &\lesssim \sum_{k=-\infty}^{\infty} \sum_{j \leq k-2} 2^{\alpha j q_1} 2^{(\beta-n\delta_2+\alpha)(k-j)q_1/2} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}^{q_1} \\ &= \sum_{j=-\infty}^{\infty} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}^{q_1} \sum_{k \leq j-2} 2^{(\beta-n\delta_2+\alpha)(k-j)q_1/2} \\ &\lesssim \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1}(w^{p_1(\cdot)})}^{q_1}. \end{aligned}$$

If $0 < q_1 \leq 1$, then by using the Jensen inequality we obtain

$$\begin{aligned} U_1 &\lesssim \sum_{k=-\infty}^{\infty} \left(\sum_{j \leq k-2} 2^{\alpha j} 2^{(\beta-n\delta_2+\alpha)(k-j)} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\ &\lesssim \sum_{k=-\infty}^{\infty} \sum_{j \leq k-2} 2^{\alpha j q_1} 2^{(\beta-n\delta_2+\alpha)(k-j)q_1} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}^{q_1} \\ &= \sum_{j=-\infty}^{\infty} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}^{q_1} \sum_{k \leq j-2} 2^{(\beta-n\delta_2+\alpha)(k-j)q_1} \\ &\lesssim \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1}(w^{p_1(\cdot)})}^{q_1}. \end{aligned}$$

Step 2. We estimate U_2 . Using Corollary 2 and $-1 \leq k - j \leq 1$, it is easy to see that

$$\begin{aligned} U_2 &= \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left(\sum_{j=k-1}^{k+1} \|(I^\beta f_j) \chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \right)^{q_1} \\ &= \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{k+1} 2^{\alpha(k-j)} 2^{\alpha j} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\ &\lesssim \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1}(w^{p_1(\cdot)})}^{q_1}. \end{aligned}$$

Step 3. We estimate U_3 . Using the generalized Hölder inequality, we have, for every $j, k \in \mathbb{Z}$ with $j \geq k + 2$,

$$\begin{aligned} |I^\beta(f_j) \chi_k(x)| &\lesssim 2^{j(-n+\beta)} \int_{B_j} |f(y)| \, dy \cdot \chi_k(x) \\ &\lesssim 2^{j(-n+\beta)} \|f_j w\|_{L^{p_1(\cdot)}} \| \chi_j \|_{L^{p'_1(\cdot)}(w^{-p'_1(\cdot)})} \chi_k(x). \end{aligned}$$

By taking the $L^{p_2(\cdot)}(w^{p_2(\cdot)})$ -norm and (9), we have

$$\|(I^\beta f_j) \chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \tag{15}$$

$$\begin{aligned} &\lesssim 2^{j(-n+\beta)} \|f_j w\|_{L^{p_1(\cdot)}} \| \chi_j \|_{L^{p'_1(\cdot)}(w^{-p'_1(\cdot)})} \| \chi_k \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\ &\lesssim 2^{j(-n+\beta)} \|f_j w\|_{L^{p_1(\cdot)}} \| \chi_j \|_{L^{p'_1(\cdot)}(w^{-p'_1(\cdot)})} \| \chi_j \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \frac{\| \chi_k \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}}{\| \chi_j \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}} \end{aligned}$$

$$\lesssim 2^{j(-n+\beta)} 2^{n\delta_1(k-j)} \|f_j w\|_{L^{p_1(\cdot)}} \| \chi_j \|_{L^{p'_1(\cdot)}(w^{-p'_1(\cdot)})} \| \chi_j \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}. \tag{16}$$

By the definition of $A(p_1(\cdot), p_2(\cdot))$, we obtain

$$\begin{aligned} \|\chi_j\|_{L^{p_1(\cdot)}(w^{-p_1(\cdot)})} \|\chi_j\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\lesssim \|\chi_{B_j}\|_{L^{p_1(\cdot)}(w^{-p_1(\cdot)})} \|\chi_{B_j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\ &\lesssim \|w^{-1}\chi_{B_j}\|_{L^{p_1(\cdot)}} \|w\chi_{B_j}\|_{L^{p_2(\cdot)}} \\ &\lesssim 2^{jn(1-\beta/n)}. \end{aligned}$$

Hence we have

$$\begin{aligned} &\|(I^\beta f_j)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\ &\lesssim 2^{j(-n+\beta)} 2^{n\delta_1(k-j)} \|f_j w\|_{L^{p_1(\cdot)}} \|\chi_j\|_{L^{p_1(\cdot)}(w^{-p_1(\cdot)})} \|\chi_j\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\ &\lesssim 2^{j(-n+\beta)} 2^{n\delta_1(k-j)} 2^{jn(1-\beta/n)} \|f_j w\|_{L^{p_1(\cdot)}} \\ &= 2^{n\delta_1(k-j)} \|f_j w\|_{L^{p_1(\cdot)}}. \end{aligned}$$

Therefore we see that

$$\begin{aligned} U_3 &= \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left(\sum_{j \geq k+2} \|(I^\beta f_j)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \right)^{q_1} \\ &\lesssim \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left(\sum_{j \geq k+2} 2^{n\delta_1(k-j)} \|f_j w\|_{L^{p_1(\cdot)}} \right)^{q_1} \\ &\lesssim \sum_{k=-\infty}^{\infty} \left(\sum_{j \geq k+2} 2^{(\alpha+n\delta_1)(k-j)} 2^{\alpha j} \|f_j w\|_{L^{p_1(\cdot)}} \right)^{q_1}. \end{aligned}$$

Note that $\alpha + n\delta_1 > 0$. We consider the two cases: ‘ $1 < q_1 < \infty$ ’ and ‘ $0 < q_1 \leq 1$ ’.

If $1 < q_1 < \infty$, then by using the Hölder inequality we obtain

$$\begin{aligned} U_3 &\lesssim \sum_{k=-\infty}^{\infty} \left(\sum_{j \geq k+2} 2^{(\alpha+n\delta_1)(k-j)} 2^{\alpha j} \|f_j w\|_{L^{p_1(\cdot)}} \right)^{q_1} \\ &\lesssim \sum_{k=-\infty}^{\infty} \left(\sum_{j \geq k+2} 2^{(\alpha+n\delta_1)(k-j)q_1/2} 2^{\alpha q_1 j} \|f_j w\|_{L^{p_1(\cdot)}}^{q_1} \right) \left(\sum_{j \geq k+2} 2^{(\alpha+n\delta_1)(k-j)q_1'/2} \right)^{q_1/q_1'} \\ &\lesssim \sum_{j=-\infty}^{\infty} 2^{\alpha q_1 j} \|f_j w\|_{L^{p_1(\cdot)}}^{q_1} \sum_{k \leq j-2} 2^{(\alpha+n\delta_1)(k-j)q_1/2} \\ &\lesssim \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1}(w^{p_1(\cdot)})}^{q_1}. \end{aligned}$$

If $0 < q_1 \leq 1$, then by using the Jensen inequality we obtain

$$\begin{aligned} U_3 &\lesssim \sum_{k=-\infty}^{\infty} \left(\sum_{j \geq k+2} 2^{(\alpha+n\delta_1)(k-j)} 2^{\alpha j} \|f_j w\|_{L^{p_1(\cdot)}} \right)^{q_1} \\ &\lesssim \sum_{k=-\infty}^{\infty} \sum_{j \geq k+2} 2^{(\alpha+n\delta_1)(k-j)q_1} 2^{\alpha q_1 j} \|f_j w\|_{L^{p_1(\cdot)}}^{q_1} \end{aligned}$$

$$\begin{aligned} &\lesssim \sum_{j=-\infty}^{\infty} 2^{\alpha q_1 j} \|f_j w\|_{L^{p_1(\cdot)}}^{q_1} \sum_{k \leq j-2} 2^{(\alpha+n\delta_1)(k-j)q_1} \\ &\lesssim \|f\|_{K_{p_1(\cdot)}^{\alpha, q_1}(w^{p_1(\cdot)})}^{q_1}. \end{aligned}$$

Consequently we have proved the theorem. □

3.3 The non-homogeneous case

In this paper we have defined the weighted Herz space with variable exponent and proved the boundedness of the fractional integrals on the spaces in the homogeneous case. Our argument is also valid for the non-homogeneous case.

Definition 8 Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < q < \infty$, and $\alpha \in \mathbb{R}$. For non-negative integer k , let

$$C_k := \begin{cases} R_k & (k \geq 1), \\ B_0 & (k = 0). \end{cases}$$

The non-homogeneous weighted Herz space $K_{p(\cdot)}^{\alpha, q}(w)$ is the collection of $f \in L_{loc}^{p(\cdot)}(\mathbb{R}^n, w)$ such that

$$\|f\|_{K_{p(\cdot)}^{\alpha, q}(w)} := \left(\sum_{k=0}^{\infty} 2^{\alpha k q} \|f \chi_{C_k}\|_{L^{p(\cdot)}(w)} \right)^{1/q} < \infty.$$

Theorem 5 Let $0 < q_1 \leq q_2 < \infty$, $p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$, $w^{p_2(\cdot)} \in A_1$, $\delta_1, \delta_2 \in (0, 1)$ be the constants appearing in (9) and (10) respectively, $0 < \beta < n(\delta_1 + \delta_2)$, and $-n\delta_1 < \alpha < n\delta_2 - \beta$. Define $p_1(\cdot)$ by $1/p_2(\cdot) \equiv 1/p_1(\cdot) - \beta/n$. Then the fractional integral operator I^β is a bounded operator from $K_{p_2(\cdot)}^{\alpha, q_2}(w^{p_2(\cdot)})$ to $K_{p_1(\cdot)}^{\alpha, q_1}(w^{p_1(\cdot)})$.

Remark 3 In the main theorems we have assumed that $w^{p_2(\cdot)} \in A_1$ to obtain the boundedness of I^β . One may think that the condition $w^{p_2(\cdot)} \in A_1$ is too strong. In this paper we could not prove the main theorems under weaker conditions on w .

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contributed equally to this work. All authors read and approved the final manuscript.

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