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Hybrid simultaneous algorithms for the split equality problem with applications

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Abstract

The split equality problem has board applications in many areas of applied mathematics. Many researchers studied this problem and proposed various algorithms to solve it. From the literature we know that most algorithms for the split equality problems came from the idea of the projected Landweber algorithm proposed by Byrne and Moudafi (Working paper UAG, 2013), and few algorithms came from the idea of the alternating CQ-algorithm given by Moudafi (Nonlinear Anal. 79:117-121, 2013). Hence, it is important and necessary to give new algorithms from the idea of the alternating CQ-algorithm. In this paper, we first present a hybrid projected Landweber algorithm to study the split equality problem. Next, we propose a hybrid alternating CQ-algorithm to study the split equality problem. As applications, we consider the split feasibility problem and linear inverse problem. Finally, we give numerical results for the split feasibility problem to demonstrate the efficiency of the proposed algorithms.

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Keywords: split equality problem; split feasibility problem; split equality problem; linear inverse problem; simultaneous algorithm

1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We denote the strong convergence and weak convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. The symbols $\mathbb{N}$ and $\mathbb{R}$ are used to denote the sets of positive integers and real numbers, respectively. For each $x \in H$, there is a unique element $\bar{x} \in C$ such that $\|x-\bar{x}\| = \min_{y\in C}\|x-y\|$. In this study, we set $P_C x = \bar{x}$, and P_C is called the metric projection from H onto C.

Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ and $A^* : H_2 \to H_1$ be two linear and bounded operators. Then A^* is called the adjoint of A if $\langle Az, w\rangle = \langle z, A^*w\rangle$ for all $z \in H_1$ and $w \in H_2$. It is known that the adjoint operator of a linear and bounded operator on a Hilbert space always exists and is linear, bounded, and unique. Further, we know that $\|A\| = \|A^*\|$.

Let H_1, H_2, and H_3 be real Hilbert spaces. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be linear and bounded operators with adjoint operators A^* and B^*, respectively. The following problem is the split equality problem, which was studied by Moudafi [2, 3]:

$$\text{(SEP)}\quad \text{Find } \bar{x} \in C \text{ and } \bar{y} \in Q \text{ such that } A\bar{x} = B\bar{y}.$$

Let $\Omega := \{(x,y) \in C \times Q : Ax = By\}$ be the solution set of problem (SEP). Further, we observed that (x,y) is a solution of the split equality problem if and only if

$$\begin{cases} x = P_C(x - \rho_1 A^*(Ax - By)), \\ y = P_Q(y + \rho_2 B^*(Ax - By)), \end{cases}$$

for all $\rho_1 > 0$ and $\rho_2 > 0$ (for details, see [4]).

As mentioned by Moudafi [2], the interest of the split equality problem covers many situations, for instance, in decomposition methods for PDEs, game theory, and intensity modulated radiation therapy (IMRT). For details, see [2, 5, 6]. To solve problem (SEP), Moudafi [3] proposed the alternating CQ-algorithm:

$$\text{(ACQA)} \quad \begin{cases} x_{n+1} := P_C(x_n - \rho_n A^*(Ax_n - By_n)), \\ y_{n+1} := P_Q(y_n + \rho_n B^*(Ax_{n+1} - By_n)), \quad n \in \mathbb{N}, \end{cases}$$

where $H_1 = \mathbb{R}^N$, $H_2 = \mathbb{R}^M$, P_C is the metric projection mapping from H_1 onto C, and P_Q is the metric projection mapping from H_2 onto Q, $\varepsilon > 0$, A is a $J \times N$ matrix, B is a $J \times M$ matrix, λ_A and λ_B are the spectral radii of A^*A and B^*B, respectively, and $\{\rho_n\}$ is a sequence in $(\varepsilon, \min\{\frac{1}{\lambda_A}, \frac{1}{\lambda_B}\} - \varepsilon)$.

In 2013, Byrne and Moudafi [1] presented a simultaneous algorithm, which was called the projected Landweber algorithm, to study the split equality problem

$$\text{(PLA)} \quad \begin{cases} x_{n+1} := P_C(x_n - \rho_n A^*(Ax_n - By_n)), \\ y_{n+1} := P_Q(y_n + \rho_n B^*(Ax_n - By_n)), \quad n \in \mathbb{N}, \end{cases}$$

where $H_1 = \mathbb{R}^N$, $H_2 = \mathbb{R}^M$, P_C is the metric projection mapping from H_1 onto C, and P_Q is the metric projection mapping from H_2 onto Q, $\varepsilon > 0$, A is a $J \times N$ matrix, B is a $J \times M$ matrix, λ_A and λ_B are the spectral radii of A^*A and B^*B, respectively, and $\{\rho_n\}$ is a sequence in $(\varepsilon, \frac{2}{\lambda_A + \lambda_B})$.

Besides, we also observed that Chen *et al.* [7] gave the following modification of (ACQA) by using the Tikhonov regularization method and proved a convergence theorem under suitable conditions:

$$\text{(TRA)} \quad \begin{cases} x_{n+1} := P_C((1 - \varepsilon_n \rho_n)x_n - \rho_n A^*(Ax_n - By_n)), \\ y_{n+1} := P_Q((1 - \varepsilon_n \rho_n)y_n + \rho_n B^*(Ax_{n+1} - By_n)), \quad n \in \mathbb{N}, \end{cases}$$

where $\{\varepsilon_n\}_{n\in\mathbb{N}}$ is a sequence in $(0,\infty)$. Besides, many researchers studied problem (SEP) and gave various algorithms. For more details about the algorithms for the split equality problem, we refer to [8, 9] and related references.

Besides, from the literature we know that most algorithms in the literature come from the idea of the projected Landweber algorithm, and few algorithms come from the idea of the alternating CQ-algorithm. Hence, it is important and necessary to give new algorithms from the idea of the alternating CQ-algorithm. In this paper, motivated by the works mentioned on the split equality problem, we present a hybrid projected Landweber algorithm and a hybrid alternating CQ-algorithm to study the split equality problem and give convergence theorems for the proposed algorithms. As applications, we consider the split feasibility problem and linear inverse problem in real Hilbert spaces. Finally, we give numerical results for the split feasibility problem to demonstrate the efficiency of the proposed algorithms.

2 Main results

In the sequel, we need the following lemma, which is a crucial tool for our results.

Lemma 2.1 [10] *Let C be a nonempty closed convex subset of a real Hilbert space H, and let P_C be the metric projection from H onto C. Then*:

(i) $\langle x - P_C x, P_C x - y\rangle \geq 0$ *for all* $x \in H$ *and* $y \in C$;

(ii) $\|x - P_C x\|^2 + \|P_C x - y\|^2 \leq \|x - y\|^2$ *for all* $x \in H$ *and* $y \in C$;

(iii) $\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle$ *for all* $x, y \in H$.

2.1 Hybrid projected Landweber algorithm

Let H_1, H_2, and H_3 be real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle_{H_i}$ and norm $\|\cdot\|_{H_i}$, $i = 1, 2, 3$. For simplicity, we write $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be linear and bounded operators with adjoint operators A^* and B^*, respectively. Choose $\delta \in (0, 1)$. Let Ω be the solution set of the split equality problem and suppose that $\Omega \neq \emptyset$. Let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence in $(0, \infty)$.

Now we present a hybrid projected Landweber algorithm to study the split equality problem.

Algorithm 2.1 For given $x_n \in H_1$ and $y_n \in H_2$, find the approximate solution by the following iterative process.

Step 1. Compute the next iterate (u_n, v_n) as follows:

$$\begin{cases} u_n = P_C[x_n - \rho_n A^*(Ax_n - By_n)], \\ v_n = P_Q[y_n + \rho_n B^*(Ax_n - By_n)], \end{cases}$$

where $\rho_n > 0$ satisfies

$$\begin{aligned} &\rho_n^2 \left(\left\| A^*(Ax_n - By_n) - A^*(Au_n - Bv_n) \right\|^2 + \left\| B^*(Ax_n - By_n) - B^*(Au_n - Bv_n) \right\|^2 \right) \\ &\quad \leq \delta \|x_n - u_n\|^2 + \delta \|y_n - v_n\|^2, \quad 0 < \delta < 1. \end{aligned} \tag{2.1}$$

Step 2. If $x_n = u_n$ and $y_n = v_n$, then (x_n, y_n) is a solution of problem (SEP) and stop. Otherwise, go to Step 3.

Step 3. Compute the next iterate (x_{n+1}, y_{n+1}) as follows:

$$\begin{cases} D_{(n,1)} := x_n - u_n + \rho_n[A^*(Au_n - Bv_n) - A^*(Ax_n - By_n)], \\ D_{(n,2)} := y_n - v_n - \rho_n[B^*(Au_n - Bv_n) - B^*(Ax_n - By_n)], \\ \alpha_n := \frac{\langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle}{\|D_{(n,1)}\|^2 + \|D_{(n,2)}\|^2}, \\ x_{n+1} = P_C[x_n - \alpha_n D_{(n,1)}], \\ y_{n+1} = P_Q[y_n - \alpha_n D_{(n,2)}]. \end{cases}$$

Next, update $n := n + 1$ and go to Step 1.

Remark 2.1 If $0 < \rho_n \leq \frac{\sqrt{\delta}}{\sqrt{2}(\|A\|^2+\|B\|^2)}$, then (2.1) holds.

Proof Without loss of generality, we may assume that $x_n \neq u_n$ and $y_n \neq v_n$. We know that

$$\begin{aligned}
&\rho_n^2 \cdot \left(\left\|A^*(Ax_n - By_n) - A^*(Au_n - Bv_n)\right\|^2 + \left\|B^*(Ax_n - By_n) - B^*(Au_n - Bv_n)\right\|^2\right) \\
&\quad \leq \rho_n^2 \cdot \left(\left\|A^*\right\|^2 + \left\|B^*\right\|^2\right) \cdot \left\|(Ax_n - By_n) - (Au_n - Bv_n)\right\|^2 \\
&\quad \leq \rho_n^2 \cdot \left(\left\|A^*\right\|^2 + \left\|B^*\right\|^2\right) \cdot \left(\|A\| \cdot \|x_n - u_n\| + \|B\| \cdot \|y_n - v_n\|\right)^2 \\
&\quad \leq 2\rho_n^2 \cdot \left(\|A\|^2 + \|B\|^2\right) \cdot \left(\|A\|^2 \cdot \|x_n - u_n\|^2 + \|B\|^2 \cdot \|y_n - v_n\|^2\right) \\
&\quad \leq 2\rho_n^2 \cdot \left(\|A\|^2 + \|B\|^2\right)^2 \cdot \left(\|x_n - u_n\|^2 + \|y_n - v_n\|^2\right) \\
&\quad \leq 2 \cdot \frac{\delta}{2(\|A\|^2 + \|B\|^2)^2} \cdot \left(\|A\|^2 + \|B\|^2\right)^2 \cdot \left(\|x_n - u_n\|^2 + \|y_n - v_n\|^2\right) \\
&\quad = \delta \cdot \left(\|x_n - u_n\|^2 + \|y_n - v_n\|^2\right).
\end{aligned}$$

Therefore, the proof is completed. □

Theorem 2.1 *Let* $\{\rho_n\}_{n\in\mathbb{N}}$ *be a sequence in* $(0, 2/(\|A\|^2 + \|B\|^2))$ *such that* (2.1) *holds and assume that* $\liminf_{n\to\infty} \rho_n(2 - \rho_n(\|A\|^2 + \|B\|^2)) > 0$. *Then, for the sequence* $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ *in Algorithm* 2.1, *there exists* $(\bar{x}, \bar{y}) \in \Omega$ *such that* $x_n \rightharpoonup \bar{x}$ *and* $y_n \rightharpoonup \bar{y}$ *as* $n \to \infty$.

Proof Take any $n \in \mathbb{N}$ and let n be fixed. Take any $(\bar{u}, \bar{v}) \in \Omega$ and let $(\bar{u}, \bar{v})$ be fixed. Then $\bar{u} \in C$, $\bar{v} \in Q$, and $A\bar{u} = B\bar{v}$. First, we set

$$\begin{cases}
\varepsilon_{n,1} := \rho_n[A^*(Au_n - Bv_n) - A^*(Ax_n - By_n)], \\
\varepsilon_{n,2} := \rho_n[B^*(Ax_n - By_n) - B^*(Au_n - Bv_n)].
\end{cases}$$

Then

$$\begin{aligned}
&\langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle \\
&\quad = \langle x_n - u_n, x_n - u_n + \varepsilon_{n,1}\rangle + \langle y_n - v_n, y_n - v_n + \varepsilon_{n,2}\rangle \\
&\quad = \|x_n - u_n\|^2 + \langle x_n - u_n, \varepsilon_{n,1}\rangle + \|y_n - v_n\|^2 + \langle y_n - v_n, \varepsilon_{n,2}\rangle \\
&\quad = \frac{1}{2}\|x_n - u_n\|^2 + \langle x_n - u_n, \varepsilon_{n,1}\rangle + \frac{1}{2}\|x_n - u_n\|^2 \\
&\qquad + \frac{1}{2}\|y_n - v_n\|^2 + \langle y_n - v_n, \varepsilon_{n,2}\rangle + \frac{1}{2}\|y_n - v_n\|^2 \\
&\quad \geq \frac{1}{2}\|x_n - u_n\|^2 + \langle x_n - u_n, \varepsilon_{n,1}\rangle + \frac{1}{2}\|\varepsilon_{n,1}\|^2 \\
&\qquad + \frac{1}{2}\|y_n - v_n\|^2 + \langle y_n - v_n, \varepsilon_{n,2}\rangle + \frac{1}{2}\|\varepsilon_{n,2}\|^2 \\
&\quad = \frac{1}{2}\|x_n - u_n + \varepsilon_{n,1}\|^2 + \frac{1}{2}\|y_n - v_n + \varepsilon_{n,2}\|^2 \\
&\quad = \frac{1}{2}\|D_{(n,1)}\|^2 + \frac{1}{2}\|D_{(n,2)}\|^2. \qquad (2.2)
\end{aligned}$$

By (2.2) we know that

$$\alpha_n := \frac{\langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle}{\|D_{(n,1)}\|^2 + \|D_{(n,2)}\|^2} \geq \frac{1}{2}. \tag{2.3}$$

Next, by Lemma 2.1 we know that

$$\|x_n - \alpha_n D_{(n,1)} - x_{n+1}\|^2 + \|x_{n+1} - \bar{u}\|^2 \leq \|x_n - \alpha_n D_{(n,1)} - \bar{u}\|^2 \tag{2.4}$$

and

$$\|y_n - \alpha_n D_{(n,2)} - y_{n+1}\|^2 + \|y_{n+1} - \bar{v}\|^2 \leq \|y_n - \alpha_n D_{(n,2)} - \bar{v}\|^2. \tag{2.5}$$

Hence, by (2.4),

$$\begin{aligned}
&\|x_n - \bar{u}\|^2 - \|x_{n+1} - \bar{u}\|^2 \\
&\quad \geq \|x_n - \bar{u}\|^2 - \|x_n - \alpha_n D_{(n,1)} - \bar{u}\|^2 + \|x_{n+1} - x_n + \alpha_n D_{(n,1)}\|^2 \\
&\quad \geq \|x_n - \bar{u}\|^2 - \|x_n - \alpha_n D_{(n,1)} - \bar{u}\|^2 \\
&\quad = \|x_n - \bar{u}\|^2 - \|x_n - \bar{u}\|^2 - \alpha_n^2 \|D_{(n,1)}\|^2 + 2\alpha_n \langle x_n - \bar{u}, D_{(n,1)}\rangle \\
&\quad = 2\alpha_n \langle x_n - \bar{u}, D_{(n,1)}\rangle - \alpha_n^2 \|D_{(n,1)}\|^2.
\end{aligned} \tag{2.6}$$

Similarly, we have

$$\|y_n - \bar{v}\|^2 - \|y_{n+1} - \bar{v}\|^2 \geq 2\alpha_n \langle y_n - \bar{v}, D_{(n,2)}\rangle - \alpha_n^2 \|D_{(n,2)}\|^2. \tag{2.7}$$

By (2.6) and (2.7) we get

$$\begin{aligned}
&\|x_{n+1} - \bar{u}\|^2 + \|y_{n+1} - \bar{v}\|^2 \\
&\quad \leq \|x_n - \bar{u}\|^2 + \|y_n - \bar{v}\|^2 - 2\alpha_n \langle x_n - \bar{u}, D_{(n,1)}\rangle - 2\alpha_n \langle y_n - \bar{v}, D_{(n,2)}\rangle \\
&\qquad + \alpha_n^2 \bigl(\|D_{(n,1)}\|^2 + \|D_{(n,2)}\|^2\bigr).
\end{aligned} \tag{2.8}$$

Next, we know that

$$\begin{aligned}
&\langle u_n - \bar{u}, D_{(n,1)}\rangle + \langle v_n - \bar{v}, D_{(n,2)}\rangle \\
&\quad = \bigl\langle u_n - \bar{u}, x_n - u_n + \rho_n \bigl[A^*(Au_n - Bv_n) - A^*(Ax_n - By_n)\bigr]\bigr\rangle \\
&\qquad + \bigl\langle v_n - \bar{v}, y_n - v_n - \rho_n \bigl[B^*(Au_n - Bv_n) - B^*(Ax_n - By_n)\bigr]\bigr\rangle \\
&\quad = \bigl\langle u_n - \bar{u}, x_n - u_n - \rho_n A^*(Ax_n - By_n)\bigr\rangle + \bigl\langle u_n - \bar{u}, \rho_n A^*(Au_n - Bv_n)\bigr\rangle \\
&\qquad + \bigl\langle v_n - \bar{v}, y_n - v_n + \rho_n B^*(Ax_n - By_n)\bigr\rangle - \bigl\langle v_n - \bar{v}, \rho_n B^*(Au_n - Bv_n)\bigr\rangle.
\end{aligned} \tag{2.9}$$

By Lemma 2.1,

$$\bigl\langle u_n - \bar{u}, x_n - \rho_n A^*(Ax_n - By_n) - u_n\bigr\rangle \geq 0 \tag{2.10}$$

and

$$\big\langle v_n - \bar{v}, y_n + \rho_n B^*(Ax_n - By_n) - v_n\big\rangle \geq 0. \tag{2.11}$$

Besides, we also have

$$\begin{aligned}
&\big\langle u_n - \bar{u}, A^*(Au_n - Bv_n)\big\rangle - \big\langle v_n - \bar{v}, B^*(Au_n - Bv_n)\big\rangle \\
&\quad = \langle Au_n - A\bar{u}, Au_n - Bv_n\rangle - \langle Bv_n - B\bar{v}, Au_n - Bv_n\rangle \\
&\quad = \langle Au_n - Bv_n - A\bar{u} + B\bar{v}, Au_n - Bv_n\rangle \\
&\quad = \langle Au_n - Bv_n, Au_n - Bv_n\rangle \\
&\quad = \|Au_n - Bv_n\|^2 \geq 0.
\end{aligned} \tag{2.12}$$

So, by (2.9), (2.10), (2.11), and (2.12) we determine that

$$\langle u_n - \bar{u}, D_{(n,1)}\rangle + \langle v_n - \bar{v}, D_{(n,2)}\rangle \geq 0, \tag{2.13}$$

which implies that

$$\langle x_n - \bar{u}, D_{(n,1)}\rangle + \langle y_n - \bar{v}, D_{(n,2)}\rangle \geq \langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle. \tag{2.14}$$

By (2.2), (2.8), and (2.14),

$$\begin{aligned}
&\|x_{n+1} - \bar{u}\|^2 + \|y_{n+1} - \bar{v}\|^2 \\
&\quad \leq \|x_n - \bar{u}\|^2 + \|y_n - \bar{v}\|^2 - 2\alpha_n\langle x_n - \bar{u}, D_{(n,1)}\rangle - 2\alpha_n\langle y_n - \bar{v}, D_{(n,2)}\rangle \\
&\qquad + \alpha_n^2\big(\|D_{(n,1)}\|^2 + \|D_{(n,2)}\|^2\big) \\
&\quad \leq \|x_n - \bar{u}\|^2 + \|y_n - \bar{v}\|^2 - 2\alpha_n\big(\langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle\big) \\
&\qquad + \alpha_n^2\big(\|D_{(n,1)}\|^2 + \|D_{(n,2)}\|^2\big) \\
&\quad = \|x_n - \bar{u}\|^2 + \|y_n - \bar{v}\|^2 - \alpha_n\big(\langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle\big) \\
&\quad \leq \|x_n - \bar{u}\|^2 + \|y_n - \bar{v}\|^2.
\end{aligned} \tag{2.15}$$

So, $\{\|x_n - \bar{u}\|^2 + \|y_n - \bar{v}\|^2\}$ is a decreasing sequence, and $\lim_{n\to\infty}\|x_n - \bar{u}\|^2 + \|y_n - \bar{v}\|^2$ exists. Further, $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are bounded sequences, and

$$\lim_{n\to\infty}\langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle = 0. \tag{2.16}$$

Besides, we know that

$$\begin{aligned}
&\langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle \\
&\quad = \langle x_n - u_n, x_n - u_n + \varepsilon_{n,1}\rangle + \langle y_n - v_n, y_n - v_n + \varepsilon_{n,2}\rangle \\
&\quad = \|x_n - u_n\|^2 + \langle x_n - u_n, \varepsilon_{n,1}\rangle + \|y_n - v_n\|^2 + \langle y_n - v_n, \varepsilon_{n,2}\rangle,
\end{aligned} \tag{2.17}$$

which implies that

$$\begin{aligned}
&\|x_n - u_n\|^2 + \|y_n - v_n\|^2 \\
&\quad = \langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle - \langle x_n - u_n, \varepsilon_{n,1}\rangle - \langle y_n - v_n, \varepsilon_{n,2}\rangle \\
&\quad \le \langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle + \|x_n - u_n\| \cdot \|\varepsilon_{n,1}\| + \|y_n - v_n\| \cdot \|\varepsilon_{n,2}\| \\
&\quad \le \langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle + \frac{1}{2}\bigl(\|x_n - u_n\|^2 + \|\varepsilon_{n,1}\|^2 + \|y_n - v_n\|^2 + \|\varepsilon_{n,2}\|^2\bigr) \\
&\quad \le \langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle + \frac{1+\delta}{2} \cdot \bigl(\|x_n - u_n\|^2 + \|y_n - v_n\|^2\bigr). \qquad (2.18)
\end{aligned}$$

Hence, by (2.18) we derive that

$$(1-\delta)\bigl(\|x_n - u_n\|^2 + \|y_n - v_n\|^2\bigr) \le 2\langle x_n - u_n, D_{(n,1)}\rangle + 2\langle y_n - v_n, D_{(n,2)}\rangle. \qquad (2.19)$$

By (2.16) and (2.19) we know that

$$\lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|y_n - v_n\| = 0. \qquad (2.20)$$

By Lemma 2.1 again,

$$\begin{aligned}
\|u_n - \bar{u}\|^2 &= \bigl\|P_C\bigl[x_n - \rho_n A^*(Ax_n - By_n)\bigr] - P_C[\bar{u}]\bigr\|^2 \\
&\le \bigl\|x_n - \rho_n A^*(Ax_n - By_n) - \bar{u}\bigr\|^2 \\
&\le \|x_n - \bar{u}\|^2 + \rho_n^2\|A\|^2 \cdot \|Ax_n - By_n\|^2 - 2\rho_n\langle Ax_n - By_n, Ax_n - A\bar{u}\rangle. \qquad (2.21)
\end{aligned}$$

Similarly,

$$\|v_n - \bar{v}\|^2 \le \|y_n - \bar{v}\|^2 + \rho_n^2\|B\|^2 \cdot \|Ax_n - By_n\|^2 + 2\rho_n\langle Ax_n - By_n, By_n - B\bar{v}\rangle. \qquad (2.22)$$

By (2.21) and (2.22),

$$\begin{aligned}
&\|u_n - \bar{u}\|^2 + \|v_n - \bar{v}\|^2 \\
&\quad \le \|x_n - \bar{u}\|^2 + \|y_n - \bar{v}\|^2 + \rho_n^2\bigl(\|A\|^2 + \|B\|^2\bigr) \cdot \|Ax_n - By_n\|^2 \\
&\qquad - 2\rho_n\langle Ax_n - By_n, Ax_n - A\bar{u}\rangle + 2\rho_n\langle Ax_n - By_n, By_n - B\bar{v}\rangle \\
&\quad = \|x_n - \bar{u}\|^2 + \|y_n - \bar{v}\|^2 - \rho_n\bigl(2 - \rho_n\bigl(\|A\|^2 + \|B\|^2\bigr)\bigr) \cdot \|Ax_n - By_n\|^2. \qquad (2.23)
\end{aligned}$$

We also have

$$\begin{aligned}
\|u_n - \bar{u}\|^2 + \|v_n - \bar{v}\|^2 = {}& \|u_n - x_n\|^2 + 2\langle u_n - x_n, x_n - \bar{u}\rangle + \|x_n - \bar{u}\|^2 \\
&+ \|v_n - y_n\|^2 + 2\langle v_n - y_n, y_n - \bar{v}\rangle + \|y_n - \bar{v}\|^2. \qquad (2.24)
\end{aligned}$$

By (2.20), (2.23), and (2.24) we get

$$\lim_{n\to\infty} \|Ax_n - By_n\| = 0. \qquad (2.25)$$

Since $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are bounded sequences, there exist subsequences $\{x_{n_k}\}_{k\in\mathbb{N}}$ and $\{y_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$, respectively, such that $x_{n_k} \rightharpoonup \bar{x}$ and $y_{n_k} \rightharpoonup \bar{y}$ for some $\bar{x} \in H_1$ and $\bar{y} \in H_2$. Since $\{x_n\}_{n=2}^{\infty}$ is a sequence in C, we know that $\bar{x} \in C$. Also, $\bar{y} \in Q$. Since $x_{n_k} \rightharpoonup \bar{x}$ and $y_{n_k} \rightharpoonup \bar{y}$, it is easy to see that $Ax_{n_k} \rightharpoonup A\bar{x}$ and $By_{n_k} \rightharpoonup B\bar{y}$ by using the properties of A and B. Further, $Ax_{n_k} - By_{n_k} \rightharpoonup A\bar{x} - B\bar{y}$, and the lower semicontinuity of the squared norm implies

$$\|A\bar{x} - B\bar{y}\|^2 \le \liminf_{k\to\infty} \|Ax_{n_k} - By_{n_k}\|^2 = \lim_{n\to\infty} \|Ax_n - By_n\|^2 = 0. \tag{2.26}$$

Then $A\bar{x} = B\bar{y}$ and $(\bar{x}, \bar{y}) \in \Omega$.

Next, let $\{x'_{n_k}\}$ and $\{y'_{n_k}\}$ be other subsequences of $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ such that $x'_{n_k} \rightharpoonup \hat{x}$ and $y'_{n_k} \rightharpoonup \hat{y}$, respectively. Following the same argument as before, we get that $(\hat{x}, \hat{y}) \in \Omega$. Besides, we have

$$\begin{aligned}
&\|x_n - \bar{x}\|^2 + \|y_n - \bar{y}\|^2 \\
&\quad = \|x_n - \hat{x}\|^2 + \|\hat{x} - \bar{x}\|^2 + 2\langle x_n - \hat{x}, \hat{x} - \bar{x}\rangle \\
&\qquad + \|y_n - \hat{y}\|^2 + \|\hat{y} - \bar{y}\|^2 + 2\langle y_n - \hat{y}, \hat{y} - \bar{y}\rangle
\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}
&\|x_n - \hat{x}\|^2 + \|y_n - \hat{y}\|^2 \\
&\quad = \|x_n - \bar{x}\|^2 + \|\hat{x} - \bar{x}\|^2 + 2\langle x_n - \bar{x}, \bar{x} - \hat{x}\rangle \\
&\qquad + \|y_n - \bar{y}\|^2 + \|\hat{y} - \bar{y}\|^2 + 2\langle y_n - \bar{y}, \bar{y} - \hat{y}\rangle.
\end{aligned} \tag{2.28}$$

Clearly, $\lim_{n\to\infty} \|x_n - \bar{x}\|^2 + \|y_n - \bar{y}\|^2$ exists, and $\lim_{n\to\infty} \|x_n - \hat{x}\|^2 + \|y_n - \hat{y}\|^2$ exists. Hence, by (2.27) we get

$$\begin{aligned}
&\lim_{n\to\infty} \left(\|x_n - \bar{x}\|^2 + \|y_n - \bar{y}\|^2\right) \\
&\quad = \lim_{k\to\infty} \left(\left\|x'_{n_k} - \bar{x}\right\|^2 + \left\|y'_{n_k} - \bar{y}\right\|^2\right) \\
&\quad = \lim_{k\to\infty} \left(\left\|x'_{n_k} - \hat{x}\right\|^2 + \left\|y'_{n_k} - \hat{y}\right\|^2\right) + \|\hat{x} - \bar{x}\|^2 + \|\hat{y} - \bar{y}\|^2 \\
&\qquad + \lim_{k\to\infty} 2\left\langle x'_{n_k} - \hat{x}, \hat{x} - \bar{x}\right\rangle + 2\lim_{k\to\infty} \left\langle y'_{n_k} - \hat{y}, \hat{y} - \bar{y}\right\rangle \\
&\quad = \lim_{k\to\infty} \left(\left\|x'_{n_k} - \hat{x}\right\|^2 + \left\|y'_{n_k} - \hat{y}\right\|^2\right) + \|\hat{x} - \bar{x}\|^2 + \|\hat{y} - \bar{y}\|^2 \\
&\quad = \lim_{n\to\infty} \left(\|x_n - \hat{x}\|^2 + \|y_n - \hat{y}\|^2\right) + \|\hat{x} - \bar{x}\|^2 + \|\hat{y} - \bar{y}\|^2.
\end{aligned} \tag{2.29}$$

Similarly, by (2.28) we have

$$\lim_{n\to\infty} \left(\|x_n - \hat{x}\|^2 + \|y_n - \hat{y}\|^2\right) = \lim_{n\to\infty} \left(\|x_n - \bar{x}\|^2 + \|y_n - \bar{y}\|^2\right) + \|\hat{x} - \bar{x}\|^2 + \|\hat{y} - \bar{y}\|^2. \tag{2.30}$$

By (2.29) and (2.30) we know that $\bar{x} = \hat{x}$ and $\bar{y} = \hat{y}$. Therefore, $x_n \rightharpoonup \bar{x}$ and $y_n \rightharpoonup \bar{y}$, and the proof is completed. □

Remark 2.2 In Theorem 2.1, if we choose $\{\rho_n\}_{n\in\mathbb{N}}$ from $(0, \frac{\delta}{\sqrt{2}(\|A\|^2+\|B\|^2)}]$, then we only need to assume that $\liminf_{n\to\infty} \rho_n > 0$.

Proof Since $\rho_n \in (0, \frac{\delta}{\sqrt{2}(\|A\|^2+\|B\|^2)}]$, we have

$$\rho_n\bigl(\|A\|^2 + \|B\|^2\bigr) \leq \sqrt{2} \cdot \rho_n \cdot \bigl(\|A\|^2 + \|B\|^2\bigr) \leq \delta, \quad \forall n \in \mathbb{N}, \tag{2.31}$$

which implies that

$$\bigl(2 - \rho_n\bigl(\|A\|^2 + \|B\|^2\bigr)\bigr) \geq 2 - \delta > 1, \quad \forall n \in \mathbb{N}. \tag{2.32}$$

Since $\liminf_{n\to\infty} \rho_n > 0$, we may assume that there is κ such that $\rho_n \geq \kappa > 0$ for all $n \in \mathbb{N}$. Hence, we determine

$$\rho_n\bigl(2 - \rho_n\bigl(\|A\|^2 + \|B\|^2\bigr)\bigr) \geq \kappa \cdot (2 - \delta) > \kappa, \quad \forall n \in \mathbb{N}. \tag{2.33}$$

By (2.33) we get the conclusion of Remark 2.2. □

2.2 Hybrid alternating CQ-algorithm

In this subsection, we present a hybrid alternating CQ-algorithm to study the split equality problem.

Algorithm 2.2 For given $x_n \in H_1$ and $y_n \in H_2$, find the approximate solution by the following iterative process.

Step 1. Compute the next iterate (u_n, v_n) as follows:

$$\begin{cases} u_n = P_C[x_n - \rho_n A^*(Ax_n - By_n)], \\ v_n = P_Q[y_n + \rho_n B^*(Au_n - By_n)], \end{cases}$$

where $\rho_n > 0$ satisfies

$$\begin{aligned} &\rho_n^2\bigl(\bigl\|A^*(Ax_n - By_n) - A^*(Au_n - Bv_n)\bigr\|^2 + \bigl\|B^*(Au_n - By_n) - B^*(Au_n - Bv_n)\bigr\|^2\bigr) \\ &\quad \leq \delta\|x_n - u_n\|^2 + \delta\|y_n - v_n\|^2, \quad 0 < \delta < 1. \end{aligned} \tag{2.34}$$

Step 2. If $x_n = u_n$ and $y_n = v_n$, then (x_n, y_n) is a solution of problem (SEP) and stop. Otherwise, go to Step 3.

Step 3. Compute the next iterate (x_{n+1}, y_{n+1}) as follows:

$$\begin{cases} D_{(n,1)} := x_n - u_n + \rho_n[A^*(Au_n - Bv_n) - A^*(Ax_n - By_n)], \\ D_{(n,2)} := y_n - v_n - \rho_n[B^*(Au_n - Bv_n) - B^*(Au_n - By_n)], \\ \alpha_n := \frac{\langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle}{\|D_{(n,1)}\|^2 + \|D_{(n,2)}\|^2}, \\ x_{n+1} = P_C[x_n - \alpha_n D_{(n,1)}], \\ y_{n+1} = P_Q[y_n - \alpha_n D_{(n,2)}]. \end{cases}$$

Next, update $n := n + 1$ and go to Step 1.

Remark 2.3 If $0 < \rho_n \leq \frac{\sqrt{\delta}}{\max\{\sqrt{2}\cdot\|A\|^2, \sqrt{2\cdot\|A\|^2\cdot|B\|^2+\|B\|^3}\}}$, then (2.34) holds.

Proof Without loss of generality, we may assume that $x_n \neq u_n$ and $y_n \neq v_n$. We have

$$\begin{aligned}
&\rho_n^2 \cdot \left(\left\|A^*(Ax_n - By_n) - A^*(Au_n - Bv_n)\right\|^2 + \left\|B^*(Au_n - By_n) - B^*(Au_n - Bv_n)\right\|^2\right) \\
&\quad \leq \rho_n^2 \cdot \left(\left\|A^*\right\|^2 \cdot \left\|(Ax_n - By_n) - (Au_n - Bv_n)\right\|^2 + \|B\|^3 \cdot \|y_n - v_n\|^2\right) \\
&\quad \leq \rho_n^2 \cdot \left(\left\|A^*\right\|^2 \cdot \left(\|Ax_n - Au_n\| + \|By_n - Bv_n\|\right)^2 + \|B\|^3 \cdot \|y_n - v_n\|^2\right) \\
&\quad \leq \rho_n^2 \cdot \left(\left\|A^*\right\|^2 \cdot \left(2\|Ax_n - Au_n\|^2 + 2\|By_n - Bv_n\|^2\right) + \|B\|^3 \cdot \|y_n - v_n\|^2\right) \\
&\quad \leq \rho_n^2 \cdot \left(2\left\|A^*\right\|^4 \cdot \|x_n - u_n\|^2 + \left(2\|A\|^2 \cdot \|B\|^2 + \|B\|^3\right) \cdot \|y_n - v_n\|^2\right) \\
&\quad \leq \delta\|x_n - u_n\|^2 + \delta\|y_n - v_n\|^2.
\end{aligned}$$

Therefore, the proof is completed. □

Theorem 2.2 *Let* $\{\rho_n\}_{n\in\mathbb{N}}$ *be a sequence in* $(0, 1/\max\{\|A\|^2, \|B\|^2\})$ *such that* (2.34) *holds and assume that* $\liminf_{n\to\infty}\rho_n(1 - \rho_n\|A\|^2) > 0$ *or* $\liminf_{n\to\infty}\rho_n(1 - \rho_n\|B\|^2) > 0$. *Then, for the sequence* $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ *in Algorithm* 2.2, *there exists* $(\bar{x}, \bar{y}) \in \Omega$ *such that* $x_n \rightharpoonup \bar{x}$ *and* $y_n \rightharpoonup \bar{y}$ *as* $n \to \infty$.

Proof Take any $n \in \mathbb{N}$ and let n be fixed. Take any $(\bar{u}, \bar{v}) \in \Omega$ and let $(\bar{u}, \bar{v})$ be fixed. Then $\bar{u} \in C$, $\bar{v} \in Q$, and $A\bar{u} = B\bar{v}$. First, we set

$$\begin{cases}
\varepsilon_{n,1} := \rho_n[A^*(Au_n - Bv_n) - A^*(Ax_n - By_n)], \\
\varepsilon_{n,2} := \rho_n[B^*(Au_n - By_n) - B^*(Au_n - Bv_n)].
\end{cases}$$

Then

$$\langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle \geq \frac{1}{2}\|D_{(n,1)}\|^2 + \frac{1}{2}\|D_{(n,2)}\|^2. \tag{2.35}$$

By (2.35) we have that

$$\alpha_n := \frac{\langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle}{\|D_{(n,1)}\|^2 + \|D_{(n,2)}\|^2} \geq \frac{1}{2}. \tag{2.36}$$

Next, by Lemma 2.1 we have

$$\|x_n - \alpha_n D_{(n,1)} - x_{n+1}\|^2 + \|x_{n+1} - \bar{u}\|^2 \leq \|x_n - \alpha_n D_{(n,1)} - \bar{u}\|^2 \tag{2.37}$$

and

$$\|y_n - \alpha_n D_{(n,2)} - y_{n+1}\|^2 + \|y_{n+1} - \bar{v}\|^2 \leq \|y_n - \alpha_n D_{(n,2)} - \bar{v}\|^2. \tag{2.38}$$

Hence, by (2.37),

$$\|x_n - \bar{u}\|^2 - \|x_{n+1} - \bar{u}\|^2 \geq 2\alpha_n\langle x_n - \bar{u}, D_{(n,1)}\rangle - \alpha_n^2\|D_{(n,1)}\|^2. \tag{2.39}$$

Also, by (2.38),

$$\|y_n - \bar{v}\|^2 - \|y_{n+1} - \bar{v}\|^2 \geq 2\alpha_n \langle y_n - \bar{v}, D_{(n,2)} \rangle - \alpha_n^2 \|D_{(n,2)}\|^2. \tag{2.40}$$

By (2.39) and (2.40) we get

$$\begin{aligned} &\|x_{n+1} - \bar{u}\|^2 + \|y_{n+1} - \bar{v}\|^2 \\ &\quad \leq \|x_n - \bar{u}\|^2 + \|y_n - \bar{v}\|^2 - 2\alpha_n \langle x_n - \bar{u}, D_{(n,1)} \rangle - 2\alpha_n \langle y_n - \bar{v}, D_{(n,2)} \rangle \\ &\qquad + \alpha_n^2 \bigl(\|D_{(n,1)}\|^2 + \|D_{(n,2)}\|^2\bigr). \end{aligned} \tag{2.41}$$

Next, we have

$$\begin{aligned} &\langle u_n - \bar{u}, D_{(n,1)} \rangle + \langle v_n - \bar{v}, D_{(n,2)} \rangle \\ &\quad = \bigl\langle u_n - \bar{u}, x_n - u_n + \rho_n \bigl[A^*(Au_n - Bv_n) - A^*(Ax_n - By_n)\bigr]\bigr\rangle \\ &\qquad + \bigl\langle v_n - \bar{v}, y_n - v_n - \rho_n \bigl[B^*(Au_n - Bv_n) - B^*(Au_n - By_n)\bigr]\bigr\rangle \\ &\quad = \bigl\langle u_n - \bar{u}, x_n - u_n - \rho_n A^*(Ax_n - By_n)\bigr\rangle + \bigl\langle u_n - \bar{u}, \rho_n A^*(Au_n - Bv_n)\bigr\rangle \\ &\qquad + \bigl\langle v_n - \bar{v}, y_n - v_n + \rho_n B^*(Au_n - By_n)\bigr\rangle - \bigl\langle v_n - \bar{v}, \rho_n B^*(Au_n - Bv_n)\bigr\rangle. \end{aligned} \tag{2.42}$$

By Lemma 2.1,

$$\bigl\langle u_n - \bar{u}, x_n - \rho_n A^*(Ax_n - By_n) - u_n \bigr\rangle \geq 0 \tag{2.43}$$

and

$$\bigl\langle v_n - \bar{v}, y_n + \rho_n B^*(Au_n - By_n) - v_n \bigr\rangle \geq 0. \tag{2.44}$$

Besides, we also have

$$\bigl\langle u_n - \bar{u}, A^*(Au_n - Bv_n)\bigr\rangle - \bigl\langle v_n - \bar{v}, B^*(Au_n - Bv_n)\bigr\rangle = \|Au_n - Bv_n\|^2 \geq 0. \tag{2.45}$$

So, by (2.42), (2.43), (2.44), and (2.45) we determine that

$$\langle u_n - \bar{u}, D_{(n,1)} \rangle + \langle v_n - \bar{v}, D_{(n,2)} \rangle \geq 0, \tag{2.46}$$

which implies that

$$\langle x_n - \bar{u}, D_{(n,1)} \rangle + \langle y_n - \bar{v}, D_{(n,2)} \rangle \geq \langle x_n - u_n, D_{(n,1)} \rangle + \langle y_n - v_n, D_{(n,2)} \rangle. \tag{2.47}$$

By (2.35), (2.41), and (2.47),

$$\begin{aligned} &\|x_{n+1} - \bar{u}\|^2 + \|y_{n+1} - \bar{v}\|^2 \\ &\quad \leq \|x_n - \bar{u}\|^2 + \|y_n - \bar{v}\|^2 - 2\alpha_n \langle x_n - \bar{u}, D_{(n,1)} \rangle - 2\alpha_n \langle y_n - \bar{v}, D_{(n,2)} \rangle \\ &\qquad + \alpha_n^2 \bigl(\|D_{(n,1)}\|^2 + \|D_{(n,2)}\|^2\bigr) \end{aligned}$$

$$\begin{aligned}
&\leq \|x_n - \bar{u}\|^2 + \|y_n - \bar{v}\|^2 - 2\alpha_n\big(\langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle\big) \\
&\quad + \alpha_n^2\big(\|D_{(n,1)}\|^2 + \|D_{(n,2)}\|^2\big) \\
&= \|x_n - \bar{u}\|^2 + \|y_n - \bar{v}\|^2 - \alpha_n\big(\langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle\big) \\
&\leq \|x_n - \bar{u}\|^2 + \|y_n - \bar{v}\|^2. \qquad (2.48)
\end{aligned}$$

So, $\{\|x_n - \bar{u}\|^2 + \|y_n - \bar{v}\|^2\}$ is a decreasing sequence, $\lim_{n\to\infty} \|x_n - \bar{u}\|^2 + \|y_n - \bar{v}\|^2$ exists, $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are bounded sequences, and

$$\lim_{n\to\infty} \langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle = 0. \qquad (2.49)$$

Besides, we have

$$\begin{aligned}
&\langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle \\
&\quad = \|x_n - u_n\|^2 + \langle x_n - u_n, \varepsilon_{n,1}\rangle + \|y_n - v_n\|^2 + \langle y_n - v_n, \varepsilon_{n,2}\rangle, \qquad (2.50)
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|x_n - u_n\|^2 + \|y_n - v_n\|^2 \\
&\quad \leq \langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle + \frac{1+\delta}{2} \cdot \big(\|x_n - u_n\|^2 + \|y_n - v_n\|^2\big). \qquad (2.51)
\end{aligned}$$

Hence, by (2.51) we derive that

$$(1-\delta)\big(\|x_n - u_n\|^2 + \|y_n - v_n\|^2\big) \leq 2\langle x_n - u_n, D_{(n,1)}\rangle + 2\langle y_n - v_n, D_{(n,2)}\rangle. \qquad (2.52)$$

By (2.49) and (2.52) we get that

$$\lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|y_n - v_n\| = 0. \qquad (2.53)$$

By Lemma 2.1 again,

$$\begin{aligned}
\|u_n - \bar{u}\|^2 &= \big\|P_C\big[x_n - \rho_n A^*(Ax_n - By_n)\big] - P_C[\bar{u}]\big\|^2 \\
&\leq \big\|x_n - \rho_n A^*(Ax_n - By_n) - \bar{u}\big\|^2 \\
&\leq \|x_n - \bar{u}\|^2 + \rho_n^2\|A\|^2 \cdot \|Ax_n - By_n\|^2 \\
&\quad - 2\rho_n\langle Ax_n - By_n, Ax_n - A\bar{u}\rangle \\
&= \|x_n - \bar{u}\|^2 - \rho_n \cdot \big(2 - \rho_n\|A\|^2\big) \cdot \|Ax_n - By_n\|^2 \\
&\quad - 2\rho_n\langle Ax_n - By_n, By_n - A\bar{u}\rangle. \qquad (2.54)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|v_n - \bar{v}\|^2 &= \big\|P_Q\big[y_n + \rho_n B^*(Au_n - By_n)\big] - P_Q[\bar{v}]\big\|^2 \\
&\leq \big\|y_n + \rho_n B^*(Au_n - By_n) - \bar{v}\big\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq \|y_n - \bar{v}\|^2 + \rho_n^2 \|B\|^2 \cdot \|Au_n - By_n\|^2 \\
&\quad + 2\rho_n \langle Au_n - By_n, By_n - B\bar{v}\rangle \\
&= \|y_n - \bar{v}\|^2 - \rho_n\bigl(2 - \rho_n \|B\|^2\bigr) \cdot \|Au_n - By_n\|^2 \\
&\quad + 2\rho_n \langle Au_n - By_n, Au_n - B\bar{v}\rangle.
\end{aligned} \tag{2.55}$$

We also have

$$2\langle Ax_n - By_n, By_n - A\bar{u}\rangle = \|Ax_n - A\bar{u}\|^2 - \|Ax_n - By_n\|^2 - \|By_n - A\bar{u}\|^2 \tag{2.56}$$

and

$$2\langle Au_n - By_n, Au_n - B\bar{v}\rangle = \|Au_n - B\bar{v}\|^2 + \|Au_n - By_n\|^2 - \|By_n - B\bar{v}\|^2. \tag{2.57}$$

By (2.54), (2.55), (2.56), and (2.57),

$$\begin{aligned}
&\|u_n - \bar{u}\|^2 + \|v_n - \bar{v}\|^2 \\
&\quad \leq \|x_n - \bar{u}\|^2 + \|y_n - \bar{v}\|^2 - \rho_n\bigl(1 - \rho_n \|A\|^2\bigr) \cdot \|Ax_n - By_n\|^2 \\
&\qquad - \rho_n\bigl(1 - \rho_n \|B\|^2\bigr) \cdot \|Au_n - By_n\|^2 + \rho_n\bigl(\|Au_n - A\bar{u}\|^2 - \|Ax_n - A\bar{u}\|^2\bigr) \\
&\quad \leq \|x_n - \bar{u}\|^2 + \|y_n - \bar{v}\|^2 - \rho_n\bigl(1 - \rho_n \|A\|^2\bigr) \cdot \|Ax_n - By_n\|^2 \\
&\qquad - \rho_n\bigl(1 - \rho_n \|B\|^2\bigr) \cdot \|Au_n - By_n\|^2 \\
&\qquad + \rho_n \cdot \|A\| \cdot \|u_n - x_n\| \cdot \bigl(\|Au_n - A\bar{u}\| + \|Ax_n - A\bar{u}\|\bigr).
\end{aligned} \tag{2.58}$$

We also have

$$\begin{aligned}
\|u_n - \bar{u}\|^2 + \|v_n - \bar{v}\|^2 &= \|u_n - x_n\|^2 + 2\langle u_n - x_n, x_n - \bar{u}\rangle + \|x_n - \bar{u}\|^2 \\
&\quad + \|v_n - y_n\|^2 + 2\langle v_n - y_n, y_n - \bar{v}\rangle + \|y_n - \bar{v}\|^2.
\end{aligned} \tag{2.59}$$

Case 1: $\liminf_{n\to\infty} \rho_n(1 - \rho_n \|A\|^2) > 0$.
By (2.53), (2.58), and (2.59) we get

$$\lim_{n\to\infty} \|Ax_n - By_n\| = 0. \tag{2.60}$$

Case 2: Suppose that $\liminf_{n\to\infty} \rho_n(1 - \rho_n \|B\|^2) > 0$.
By (2.53), (2.58), and (2.59) we get

$$\lim_{n\to\infty} \|Au_n - By_n\| = 0. \tag{2.61}$$

By (2.53) and (2.61) we determine

$$\lim_{n\to\infty} \|Ax_n - By_n\| = 0. \tag{2.62}$$

Next, following the same argument as the final proof of Theorem 2.1, we get the conclusion of Theorem 2.2. □

Remark 2.4 Suppose that $\{\rho_n\}_{n\in\mathbb{N}}$ satisfy the following inequality:

$$0 < \kappa \leq \rho_n \leq \frac{\delta}{\max\{\sqrt{2}\cdot\|A\|^2, \sqrt{2\cdot\|A\|^2\cdot\|B\|^2+\|B\|^3}, \|B\|^2\}}.$$

Then $\{\rho_n\}_{n\in\mathbb{N}}$ satisfy the conditions in Remark 2.3 and Theorem 2.2.

3 Applications of the split equality problem

3.1 The split feasibility problem

Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a linear and bounded operator with adjoint operator A^*. The following problem is the split feasibility problem in Hilbert spaces, which was first introduced by Censor and Elfving [11]:

$$\text{(SFP)} \quad \text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in C \text{ and } A\bar{x} \in Q.$$

Here, let $\Omega_1 := \{x \in C : Ax \in Q\}$ be the solution set of problem (SFP). It is worth noting that this problem is a particular case of the split equality problem when $H_3 = H_2$ and B is the identity mapping on H_2. For additional details, one can refer to [6, 11–24] and related literature.

By Algorithm 2.2, we get the following algorithm to study problem (SFP).

Algorithm 3.1 For given $x_n \in H_1$ and $y_n \in H_2$, find the approximate solution by the following iterative process.

Step 1. For $n \in \mathbb{N}$, let u_n and v_n be defined by

$$\begin{cases} u_n = P_C[x_n - \rho_n A^*(Ax_n - y_n)], \\ v_n = P_Q[y_n + \rho_n(Au_n - y_n)], \end{cases}$$

where $\rho_n > 0$ satisfies

$$\begin{aligned} &\rho_n^2\left(\left\|A^*(Ax_n - y_n) - A^*(Au_n - v_n)\right\|^2 + \left\|(Au_n - y_n) - B^*(Au_n - v_n)\right\|^2\right) \\ &\quad \leq \delta\|x_n - u_n\|^2 + \delta\|y_n - v_n\|^2, \quad 0 < \delta < 1. \end{aligned} \tag{3.1}$$

Step 2. If $x_n = u_n$ and $y_n = v_n$, then (x_n, y_n) is a solution of problem (SFP) and stop. Otherwise, go to Step 3.

Step 3. Compute the next iterate (x_{n+1}, y_{n+1}) as follows:

$$\begin{cases} D_{(n,1)} := x_n - u_n + \rho_n[A^*(Au_n - v_n) - A^*(Ax_n - y_n)], \\ D_{(n,2)} := y_n - v_n - \rho_n[(Au_n - v_n) - (Au_n - y_n)], \\ \alpha_n := \frac{\langle x_n - u_n, D_{(n,1)}\rangle + \langle y_n - v_n, D_{(n,2)}\rangle}{\|D_{(n,1)}\|^2 + \|D_{(n,2)}\|^2}, \\ x_{n+1} = P_C[x_n - \alpha_n D_{(n,1)}], \\ y_{n+1} = P_Q[y_n - \alpha_n D_{(n,2)}]. \end{cases}$$

Next, update $n := n + 1$ and go to Step 1.

We get the following convergence theorem for the split feasibility problem by using Theorem 2.2.

Theorem 3.1 *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a linear and bounded operator with adjoint operator A^*. Choose $\delta \in (0,1)$. Let Ω_1 be the solution set of the split feasibility problem and suppose that $\Omega_1 \neq \emptyset$. Let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence in $(0, 1/\max\{\|A\|^2, 1\})$ such that* (3.1) *hold and assume that* $\liminf_{n\to\infty} \rho_n(1 - \rho_n\|A\|^2) > 0$ *or* $\liminf_{n\to\infty} \rho_n(1 - \rho_n) > 0s$. *Then, for the sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ in Algorithm* 3.1, *there exists $\bar{x} \in \Omega_1$ such that $x_n \rightharpoonup \bar{x}$ as $n \to \infty$.*

3.2 Linear inverse problem

In this subsection, we study an inverse problem by our algorithms and convergence theorems. Let H_1 and H_2 be real Hilbert spaces. Let C be a nonempty closed convex subset of H_1, and $A : H_1 \to H_2$ be a linear and bounded operator with adjoint operator A^*. Given $b \in H_2$. Then we consider the following inverse problem in this section:

(IV) Find $\bar{x} \in C$ such that $A\bar{x} = b$.

This is a particular case of the split equality problem if $H_2 = H_3$, $Q = \{b\}$, and $B(x) = x$ for all $x \in H_2$. Next, take any $(x_1, y_1) \in H_1 \times H_2$ with $y_1 = b$. Then, by Algorithm 2.2 we get the following algorithm to study problem (IV).

Algorithm 3.2 For given$x_n \in H_1$, find the approximate solution by the following iterative process.

Step 1. Compute the next iterate u_n as follows:

$$u_n = P_C\big[x_n - \rho_n A^*(Ax_n - b)\big],$$

where $\rho_n > 0$ satisfies

$$\rho_n^2 \cdot \big\|A^*(Ax_n) - A^*(Au_n)\big\|^2 \le \delta\|x_n - u_n\|^2, \quad 0 < \delta < 1. \tag{3.2}$$

Step 2. If $x_n = u_n$, then x_n is a solution of problem (IV) and stop. Otherwise, go to Step 3.
Step 3. Compute the next iterate x_{n+1} as follows:

$$\begin{cases} D_n := x_n - u_n + \rho_n[A^*(Au_n) - A^*(Ax_n)], \\ \alpha_n := \frac{\langle x_n - u_n, D_n\rangle}{\|D_n\|^2}, \\ x_{n+1} = P_C[x_n - \alpha_n D_n]. \end{cases}$$

Next, update $n := n + 1$ and go to Step 1.

We get the following convergence theorem for the linear inverse problem by using Theorem 2.2.

Theorem 3.2 *Let H_1 and H_2 be real Hilbert spaces. Let C be a nonempty closed convex subset of H_1, and $A : H_1 \to H_2$ be a linear and bounded operator with adjoint operator A^*. Given $b \in H_2$ and $\delta \in (0,1)$. Let Ω_2 be the solution set of* (IV) *and suppose that $\Omega_2 \neq \emptyset$.*

Let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence in $(0, 1/\max\{\|A\|^2, 1\})$ such that (3.2) *holds and assume that $\liminf_{n\to\infty}\rho_n(1-\rho_n\|A\|^2) > 0$ or $\liminf_{n\to\infty}\rho_n(1-\rho_n) > 0$. Then, for the sequence $\{x_n\}_{n\in\mathbb{N}}$ in Algorithm* 3.2, *there exists $\bar{x} \in \Omega_2$ such that $x_n \rightharpoonup \bar{x}$ as $n \to \infty$.*

Remark 3.1 By Algorithm 2.1 and Theorem 2.1, we can get the related algorithms and convergence theorems for the split feasibility problem and the inverse problems.

4 Numerical results

All codes were written in R language (version 3.2.4 (2016-03-10), the R Foundation for Statistical Computing Platform: x86-64-w64-mingw32/x64).

Example 4.1 Let $H_1 = H_2 = H_3 = \mathbb{R}^2$, $C := \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, $Q := \{x = (u, v) \in \mathbb{R}^2 : (u-6)^2 + (v-8)^2 \leq 25\}$,

$$A := \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, \qquad B := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then problem (SEP) has a unique solution $(\bar{x}, \bar{y}) \in \mathbb{R}^2 \times \mathbb{R}^2$, where $\bar{x} := (\bar{x}_1, \bar{x}_2)$, $\bar{y} := (\bar{y}_1, \bar{y}_2)$. Indeed, $\bar{x}_1 = 0.6$, $\bar{x}_2 = 0.8$, $\bar{y}_1 = 3$, $\bar{y}_2 = 4$. Let $\varepsilon > 0$ and the algorithm stop if $\|x_n - \bar{x}\| + \|y_n - \bar{y}\| < \varepsilon$.

In Table 1, setting $\varepsilon = 10^{-1}$, $x_1 = (10, 10)^T$, $y_1 = (1, 1)^T$, and $\rho_n = 0.01$ for all $n \in \mathbb{N}$, we get the numerical results.

In Table 2, setting $\varepsilon = 10^{-1}$, $x_1 = (5, 5)^T$, $y_1 = (1, 1)^T$, and $\rho_n = 0.01$ for all $n \in \mathbb{N}$, we get the numerical results.

In Table 3, setting $\varepsilon = 4 \times 10^{-2}$, $x_1 = (-12, -50)^T$, $y_1 = (-40, 20)^T$, and $\rho_n = 0.01$ for all $n \in \mathbb{N}$, we get the numerical results.

Table 1 $\varepsilon = 10^{-1}$, $x_1 = (10, 10)^T$, $y_1 = (1, 1)^T$, $\rho_n = 0.01$

Algorithm	Time (s)	Iteration	Approximate solution (x_n^1, x_n^2)	Approximate solution (y_n^1, y_n^2)
Algorithm 2.1	0.01	196	(0.6114674, 0.7912309)	(3.0850778, 3.9920504)
Algorithm 2.2	0.00	122	(0.5970952, 0.8020906)	(3.0421971, 4.0866474)
(ACQA)	1.94	58,324	(0.6132467, 0.7898914)	(3.0670840, 3.9505550)
(PLA)	2.57	78,654	(0.6132467, 0.7898914)	(3.0670840, 3.9505550)

Table 2 $\varepsilon = 10^{-1}$, $x_1 = (5, 5)^T$, $y_1 = (1, 1)^T$, $\rho_n = 0.01$

Algorithm	Time (s)	Iteration	Approximate solution (x_n^1, x_n^2)	Approximate solution (y_n^1, y_n^2)
Algorithm 2.1	0.82	11,168	(0.6132467, 0.7898915)	(3.067084, 3.950555)
Algorithm 2.2	0.02	205	(0.6077392, 0.7940725)	(3.0847143, 4.0304899)
(ACQA)	1.94	58,324	(0.6132467, 0.7898914)	(3.067084, 3.950555)
(PLA)	2.28	71,521	(0.6132467, 0.7898915)	(3.067084, 3.950555)

Table 3 $\varepsilon = 4 \times 10^{-2}$, $x_1 = (12, -50)^T$, $y_1 = (-40, 20)^T$, $\rho_n = 0.01$

Algorithm	Time (s)	Iteration	Approximate solution (x_n^1, x_n^2)	Approximate solution (y_n^1, y_n^2)
Algorithm 2.1	0.07	527	(0.5988387, 0.8008379)	(3.0167366, 4.0343372)
Algorithm 2.2	45.89	474,754	(0.5946535, 0.8039821)	(2.973400, 4.020089)
(ACQA)	20.44	579,771	(0.5946535, 0.8039821)	(2.973400, 4.020089)
(PLA)	22.55	585,380	(0.5946536, 0.8039821)	(2.973400, 4.020089)

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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