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Topics on the spectral properties of degenerate non-self-adjoint differential operators

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Abstract

Let $(Pu)(t) = -\frac{d}{dt}(\omega^2(t)q(t)\frac{du(t)}{dt})$ be a degenerate non-self-adjoint operator defined on the space $H_\ell = L^2(0,1)^\ell$ with Dirichlet-type boundary conditions, where $\omega(t) \in C^1(0,1)$ is a positive function with further assumptions that will be specified later, and $q(t) \in C^2([0,1], \operatorname{End} C^\ell)$ is a matrix function. In this article, some spectral characteristics of the operator P are considered. We estimate the resolvent of P and then prove the limit argument theorem. Finally, we find a formula for the distribution of eigenvalues of the operator P acting on H_ℓ.

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Keywords: resolvent; asymptotic spectrum; eigenvalues; non-self-adjoint differential operators

1 Introduction

For more information, see papers [1–8]. In [1], the authors consider a certain matrix elliptic differential operator and some spectral characteristics of this operator. The spectral characteristics of non-self-adjoint elliptic differential operators are considered in [2, 3, 6–8]. In this paper, we generalize the operator in [4, 5] and consider its spectral properties by generalizing the distance function $\rho(t)$ to the function $\omega^2(t)$.

This paper consists of five sections. Section 1 is devoted to introduction and definitions. In Section 2, we consider the operator $(Pu)(t) = -\frac{d}{dt}(\omega^2(t)q(t)\frac{du(t)}{dt})$ on the one-dimensional space $H = L^2(0,1)$. Using different techniques, in Theorem 2.1 and Theorem 2.2, we prove estimates (2.3) and (2.4) (note that assumption (2.2) is not used in Theorem 2.2). We consider the operator P on the ℓ-dimensional space $H_\ell = L^2(0,1)^\ell$ and then prove Theorem 3.1 in Section 3. In Section 4, we prove the vanishing limit argument theorem, that is, we show that $\lim_{j\to\infty} \arg\lambda_j = 0$. Finally, in Section 5, we find the asymptotic distribution formula for the eigenvalue function $N(\tau) = \operatorname{card}\{j : |\lambda_j| \le \tau\}$ as $\tau \to +\infty$.

Formulation and notation: In this paper, $\mathcal{H}_\ell$ denotes the weighted Sobolev space $W^1_{2,\omega}(0,1)^\ell = W^1_{2,\omega}(0,1) \times \cdots \times W^1_{2,\omega}(0,1)$ (ℓ-times) of vector functions $u(t) = (u_1(t),\ldots, u_\ell(t))$ on $(0,1)$ with finite norm

$$|u|_+ = \left(\int_0^1 \omega^2(t)\left|\frac{du(t)}{dt}\right|^2_{C^\ell} dt + \int_0^1 |u(t)|^2_{C^\ell}\right)^{1/2}.$$

Here $|\frac{du(t)}{dt}|^2_{C^\ell}$ and $|u(t)|^2_{C^\ell}$ stand for the norm in the space C^ℓ (the above definition of the norm has been previously used in [1, 4, 5, 9]. Of course, this could also be done in matrix language at the cost of greater notational complexity). By $\dot{\mathcal{H}}_\ell$ we denote the closure of $C_0^\infty(0,1)$ in the space $\mathcal{H}_\ell$ with respect to the above norm, where $C_0^\infty(0,1)$ denotes the space of infinitely differentiable functions with compact support in $(0,1)$. Note that if $\ell = 1$, then $H = H_1$, $\mathcal{H} = \mathcal{H}_1$, and $\dot{\mathcal{H}} = \dot{\mathcal{H}}_1$. We now consider a non-self-adjoint differential operator of type $(Pu)(t) = -\frac{d}{dt}(\omega^2(t)q(t)\frac{du(t)}{dt})$ acting on the space $\mathcal{H}_\ell = L^2(0,1)^\ell$ with Dirichlet-type boundary conditions. Here $\omega(t) \in C^1(0,1)$ is a positive function that satisfies the following conditions:

$$c_1 t^\alpha (1-t)^\beta \le \left|\omega^2(t)\right| \le M, \tag{1.1}$$

$$\left|\left(\omega^2\right)'(t)\right| \le M t^{\frac{\alpha}{2}-1+\varepsilon_1}(1-t)^{\frac{\beta}{2}-1+\varepsilon_2}, \tag{1.2}$$

where $0 \le \alpha$, $0 \le \beta$, $\varepsilon_1 = 0$ if $\alpha \ne 1$ and $\varepsilon_1 > 0$ if $\alpha = 1$, and $\varepsilon_2 = 0$ if $\beta \ne 1$ and $\varepsilon_2 > 0$ if $\beta = 1$.

Suppose that $q(t) \in C^2([0,1], \operatorname{End} C^\ell)$ is a matrix function such that for each $\in [0,1]$, $q(t)$ has ℓ distinct nonzero simple eigenvalues $\mu_1(t), \dots, \mu_\ell(t)$ in the complex plane such that $\mu_j(t) \in C^2[0,1]$ for $j = 1, \dots, \ell$.

Note that in Section 4, the latter assumption enables us to diagonalize the matrix function $q(t)$ for each $t \in [0,1]$. Moreover, let $\Phi = \{z \in C : |\arg z| \le \varphi\}$, $\varphi \in (0,\pi)$ be some closed angle with vertex at zero. We now consider $\mu_1(t), \dots, \mu_\nu(t) \in R_+$ and $\mu_{\nu+1}(t), \dots, \mu_\ell(t) \in C \setminus \Phi$. In other words, for $t \in [0,1]$, the eigenvalues $\mu_j(t)$ are on the positive real line R_+ for $j = 1, \dots, \nu$ and are out of the closed angle Φ in the complex plane C for $j = \nu+1, \dots, \ell$.

In all remaining sections, we need to extend the domain of operator P to the closed domain

$$D(P) = \left\{u \in \dot{\mathcal{H}}_\ell \cap W^2_{2,\mathrm{loc}}(0,1)^\ell : \omega u' \in H_\ell, \frac{d}{dt}\left(\omega^2 q \frac{du}{dt}\right) \in H_\ell\right\}.$$

Here the closed domain refers to the following sesquilinear form:

$$t[u,v] = \int_0^1 \omega^2(t)q(t)u'(t)\overline{v'(t)}\,dt$$

connected with P by $t[u,v] = \langle Pu, v\rangle$ (for more explanation, see the representation theorems in Chapter 6 of [10]). In this article, $W^2_{2,\mathrm{loc}}(0,1)^\ell \times \cdots \times W^2_{2,\mathrm{loc}}(0,1)$ (ℓ-times), where $W^2_{2,\mathrm{loc}}(0,1)$ is the space of the functions $u(t)$ $(0 < t < 1)$ satisfying

$$\sum_{i=0}^{2} \int_\varepsilon^{1-\varepsilon} \left|u^{(i)}(t)\right|^2 dt < \infty \quad \forall \varepsilon \in \left(0, \frac{1}{2}\right).$$

2 On the resolvent estimate of the differential operator P on $H = L^2(0,1)$

In this section, we need to reduce the operator $(Pu)(t) = -\frac{d}{dt}(\omega^2(t)q(t)\frac{du(t)}{dt})$ acting on the ℓ-dimensional space $H_\ell = L^2(0,1)^\ell$ to the operator $(Pu)(t) = -\frac{d}{dt}(\omega^2(t)\mu(t)\frac{du(t)}{dt})$ acting on the one-dimensional space $H = L^2(0,1)$. In fact, the matrix $q(t)$ exchanges to a one-dimensional function $\mu(t)$ satisfying the following conditions:

$$\mu(t) \in C^2[0,1], \qquad \mu(t) \in C \setminus \Phi \quad \forall t \in [0,1]. \tag{2.1}$$

Since we work on the small oscillation of the argument of the function $\mu(t)$, as in [2], without loss of generality, we can assume that the oscillation of this function in the interval $[0,1]$ does not exceed $\frac{\pi}{8}$, that is,

$$\left|\arg\left\{\mu(t_1)\mu^{-1}(t_2)\right\}\right| \leq \frac{\pi}{8} \quad \left(\forall t_1, t_2 \in [0,1]\right). \tag{2.2}$$

Theorem 2.1 *Let* $(Pu)(t) = -\frac{d}{dt}(\omega^2(t)\mu(t)\frac{du(t)}{dt})$ *be a differential operator acting on the space* $H = L_2(0,1)$. *If* (2.1) *and* (2.2) *are satisfied, then for sufficiently large in modulus* $\lambda \in \Phi$, *the inverse operator* $(P - \lambda I)^{-1}$ *exists and is continuous in the space* $H = L_2(0,1)$, *and the following estimates hold*:

$$\left\|(P - \lambda I)^{-1}\right\| \leq M|\lambda|^{-1} \quad \left(\lambda \in \Phi, |\lambda| > C\right), \tag{2.3}$$

$$\left\|\omega(t)\frac{d}{dt}(P - \lambda I)^{-1}\right\| \leq M|\lambda|^{-\frac{1}{2}} \quad \left(\lambda \in \Phi, |\lambda| > C\right), \tag{2.4}$$

where $M, C > 0$ *are sufficiently large numbers depending on* Φ.

Proof *Step* 1. In this step, we prove assertion (2.3). We need to extend the domain of P to the closed set

$$D(P) = \left\{u \in \dot{\mathcal{H}} \cap W^2_{2,\mathrm{loc}}(0,1) : \omega u' \in H, \left(\omega^2(t)\mu u'\right)' \in H\right\}.$$

For all $\lambda = |\lambda|e^{i\alpha} \in \Phi$ and $\mu = |\mu|e^{i\beta} \in C \setminus \Phi$, we can choose $\gamma \in (-\pi, \pi]$ such that $\cos(\gamma + \alpha) < 0$ and $\cos(\gamma + \beta) > 0$. Now we define c' as follows:

$$c' = \min\left\{\frac{-\operatorname{Re}\{e^{i\gamma}\lambda\}}{|\lambda|}, \operatorname{Re}\left\{e^{i\gamma}\mu\right\}\right\}.$$

So we have:

$$c' \leq \operatorname{Re}\left\{e^{i\gamma}\mu(t)\right\}, \qquad c'|\lambda| \leq -\operatorname{Re}\left\{e^{i\gamma}\lambda\right\}, \quad c' > 0, t \in [0,1], \lambda \in \Phi. \tag{2.5}$$

For $u \in D(P)$, by integrating two sides of $c' \leq \operatorname{Re}\{e^{i\gamma}\mu(t)\}$ we have

$$c' \int_0^1 \omega^2(t)|u'(t)|^2\,dt \leq \operatorname{Re}\int_0^1 e^{i\gamma}\omega^2(t)\mu(t)|u'(t)|^2\,dt = \operatorname{Re}\{e^{i\gamma}(Pu,u)\}.$$

Here the symbol $(\,,)$ denotes the inner product in H.

By multiplying the inequality $c'|\lambda| \leq -\operatorname{Re}\{e^{i\gamma}\lambda\}$ by $\int_0^1 |u(t)|^2\,dt = (u,u) = \|u\|^2 > 0$ we have

$$c'|\lambda| \int_0^1 \left|u(t)\right|^2 dt \leq -\operatorname{Re}\left\{e^{i\gamma}\lambda\right\}(u,u).$$

For $c' = \frac{1}{M}$, from the above inequalities we have:

$$\begin{aligned}\int_0^1 \omega^2(t)\left|u'(t)\right|^2 dt + |\lambda| \int_0^1 \left|u(t)\right|^2 dt &\leq M\operatorname{Re}\left\{e^{i\gamma}(Pu,u) - e^{i\gamma}\lambda(u,u)\right\} \\ &= M\operatorname{Re}\left\{e^{i\gamma}(P - \lambda I)u, u\right\}\end{aligned}$$

$$\le M\left\|e^{i\gamma}\right\|\|u\|\left\|(P-\lambda I)u\right\|$$
$$= M\|u\|\left\|(P-\lambda I)u\right\|. \tag{2.6}$$

Since $\int_0^1 \omega^2(t)|u'(t)|^2\,dt > 0$, we have

$$|\lambda|\int_0^1 \left|u(t)\right|^2 dt \le M\|u\|\left\|(P-\lambda I)u\right\| \tag{2.7}$$

or

$$|\lambda|\|u\| \le M\left\|(P-\lambda I)u\right\|.$$

The above relation ensures that the operator $(P-\lambda I)$ is one-to-one, which implies that $\ker(P-\lambda I)=0$. Therefore, the inverse operator $(P-\lambda I)^{-1}$ exists, and its continuity follows from the proof of estimate (2.3) of Theorem 2.1. To prove (2.3), we set $u=(P-\lambda I)^{-1}f$, $f\in H$. By (2.7) we have:

$$|\lambda|\int_0^1 \left|(P-\lambda I)^{-1}f\right|^2 dt \le M\left\|(P-\lambda I)^{-1}f\right\|\left\|(P-\lambda I)(P-\lambda I)^{-1}f\right\|.$$

Since $(P-\lambda I)(P-\lambda I)^{-1}f = I(f) = f$, it follows that

$$|\lambda|\int_0^1 \left|(P-\lambda I)^{-1}f\right|^2 dt \le M\left\|(P-\lambda I)^{-1}f\right\||f|.$$

Therefore,

$$|\lambda|\left\|(P-\lambda I)^{-1}f\right\|^2 \le M\left\|(P-\lambda I)^{-1}(f)\right\||f|,$$

which implies $|\lambda|\|(P-\lambda I)^{-1}(f)\| \le M|f|$. Since $\lambda\neq 0$, we have $\|(P-\lambda I)^{-1}(f)\| \le M|\lambda|^{-1}|f|$. The final result is

$$\left\|(P-\lambda I)^{-1}\right\| \le M|\lambda|^{-1}.$$

This estimate completes the proof of assertion (2.3).

Step 2. In this step, we prove inequality (2.4). Since $|\lambda|\int_0^1 |u(t)|^2\,dt > 0$, from (2.6) we have

$$\int_0^1 \omega^2(t)\left|u'(t)\right|^2 dt \le M\|u\|\left\|(P-\lambda I)u\right\|.$$

By setting $u=(P-\lambda I)^{-1}f, f\in H$, in the last inequality we have

$$\int_0^1 \omega^2(t)\left|\frac{d}{dt}(P-\lambda I)^{-1}f(t)\right|^2 dt \le M\left\|(P-\lambda I)^{-1}f\right\|\left\|(P-\lambda I)(P-\lambda I)^{-1}f\right\|.$$

Since $(P-\lambda I)(P-\lambda I)^{-1}f = f$, we get

$$\int_0^1 \omega^2(t)\left|\frac{d}{dt}(P-\lambda I)^{-1}f(t)\right|^2 dt \le M\left\|(P-\lambda I)^{-1}f\right\|\|f\|,$$

and by (2.3) we have $\|(P-\lambda I)^{-1}f\| \leq M\|f\||\lambda|^{-1}$, so

$$\int_0^1 \omega^2(t)\left|\frac{d}{dt}(P-\lambda I)^{-1}f(t)\right|^2 dt \leq MM|\lambda|^{-1}\|f\|^2.$$

Therefore,

$$\int_0^1 \omega^2(t)\left|\frac{d}{dt}(P-\lambda I)^{-1}f(t)\right|^2 dt \leq M^2|\lambda|^{-1}\|f\|^2,$$

that is,

$$\left\|\omega(t)\frac{d}{dt}(P-\lambda I)^{-1}f(t)\right\|^2 \leq M^2|\lambda|^{-1}\|f\|^2.$$

So

$$\left\|\omega(t)\frac{d}{dt}(P-\lambda I)^{-1}f(t)\right\| \leq M|\lambda|^{-\frac{1}{2}}. \qquad \square$$

Now we claim that in spite of dropping assumption (2.2), assertions (2.3) and (2.4) of Theorem 2.1 are valid.

Theorem 2.2 *Let* $(Pu)(t) = -\frac{d}{dt}(\omega^2(t)\mu(t)\frac{du(t)}{dt})$ *be a differential operator acting on the space* $H = L_2(0,1)$. *If* (2.1) *is satisfied, then for sufficiently large in modulus* $\lambda \in \Phi$, *the inverse operator* $(P-\lambda I)^{-1}$ *exists and is continuous, and the following estimates hold*:

$$\left\|(P-\lambda I)^{-1}\right\| \leq M_\Phi|\lambda|^{-1} \quad \left(\lambda \in \Phi, |\lambda| > C_\Phi\right), \tag{2.8}$$

$$\left\|\omega(t)\frac{d}{dt}(P-\lambda I)^{-1}\right\| \leq M'_\Phi|\lambda|^{-\frac{1}{2}} \quad \left(\lambda \in \Phi, |\lambda| > C_\Phi\right), \tag{2.9}$$

where $M_\Phi, M'_\Phi, C_\Phi > 0$ *are sufficiently large numbers depending on* Φ.

Proof *Step* 1. In this step, we prove assertion (2.8). We need to construct nonnegative functions $\varphi_{(1)}(t), \dots, \varphi_{(\rho)}(t)$ and new functions $\mu_{(1)}(t), \dots, \mu_{(\rho)}(t)$ with the following properties:

$$\begin{aligned}
&\mu_{(1)}(t), \dots, \mu_{(\rho)}(t), \varphi_{(1)}(t), \dots, \varphi_{(\rho)}(t) \in C^\infty[0,1], \\
&0 \leq \varphi_{(j)}(t), \quad \varphi_{(1)}^2(t) + \cdots + \varphi_{(\rho)}^2(t) \equiv 1 \quad (0 \leq t \leq 1), \\
&\frac{d}{dt}\varphi_{(j)}(t) \in C_0^\infty(0,1), \qquad \mu_{(j)}(t) = \mu(t) \quad \forall t \in \operatorname{supp}\varphi_{(j)}, \\
&\mu_{(j)}(t) \in C \setminus \Phi \quad \forall t \in [0,1], j = 1, \dots, \rho, \\
&\left|\arg\left\{\mu_{(j)}(t_1)\mu_{(j)}^{-1}(t_2)\right\}\right| \leq \frac{\pi}{8} \quad (\forall t_1, t_2 \in \operatorname{supp}\varphi_{(j)}), j = 1, \dots, \rho.
\end{aligned}$$

Considering Theorem 2.1, by (2.3) and (2.4) set

$$(A_{(j)}v)(t) = -\frac{d}{dt}\left(\omega^2(t)\mu_{(j)}(t)\frac{dv(t)}{dt}\right),$$

acting on the space $H = L_2(0,1)$, where

$$D(A_{(j)}) = \{\nu \in \dot{\mathcal{H}} \cap W^2_{2,\mathrm{loc}}(0,1) : (\omega^2(t)\mu_{(j)}\nu')' \in H\}.$$

By Theorem 2.1 the operator $(A_{(j)} - \lambda I)$ has a continuous inverse for $0 \neq \lambda \in \Phi$ and satisfies

$$\|(A_{(j)} - \lambda I)^{-1}\| \leq M_1|\lambda|^{-1}, \qquad \left\|\omega(t)\frac{d}{dt}(A_{(j)} - \lambda I)^{-1}\right\| \leq M_1|\lambda|^{-\frac{1}{2}}. \tag{2.10}$$

Let us introduce the operator

$$T(\lambda) = \sum_{J=1}^{\rho} \varphi_{(j)}(A_{(j)} - \lambda I)^{-1}\varphi_{(j)} \tag{2.11}$$

in the space $\mathcal{H}$. Here $\varphi_{(j)}$ is the operator of multiplication by the function $\varphi_{(j)}(t)$. Consequently, it is easily verified that

$$(P - \lambda I)T(\lambda)\nu = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = -\sum_{j=1}^{\rho}\left(\omega^2(t)\mu(\varphi_{(j)})'_t\right)'_t(A_{(j)} - \lambda I)^{-1}\varphi_{(j)}\nu,$$

$$I_2 = -\sum_{j=1}^{\rho}\omega^2(t)\mu(\varphi_{(j)})'_t\frac{d}{dt}(A_{(j)} - \lambda I)^{-1}\varphi_{(j)}\nu,$$

$$I_3 = -\sum_{j=1}^{\rho}\varphi_{(j)}\left(\omega^2(t)\mu\frac{d}{dt}(A_{(j)} - \lambda I)^{-1}\varphi_{(j)}\nu\right)'_t,$$

$$I_4 = -\lambda\sum_{j=1}^{\rho}\varphi_{(j)}(A_{(j)} - \lambda I)^{-1}\varphi_{(j)}\nu.$$

Since $\mu_{(j)}(t) = \mu(t)$ $\forall t \in \operatorname{supp}\varphi_{(j)}$, we can replace $\mu(t)$ by $\mu_{(j)}(t)$ in I_3. Then using $\sum_{j=1}^{\rho}\varphi^2_{(j)}(t) \equiv 1$, it follows that

$$\begin{aligned} I_3 + I_4 &= -\sum_{j=1}^{\rho}\varphi_{(j)}\left[\left(\omega^2(t)\mu\frac{d}{dt}(A_{(j)} - \lambda I)^{-1}\varphi_{(j)}\nu\right)'_t + \lambda(A_{(j)} - \lambda I)^{-1}\varphi_{(j)}\nu\right] \\ &= \sum_{j=1}^{\rho}\varphi_{(j)}(A_{(j)} - \lambda I)(A_{(j)} - \lambda I)^{-1}\varphi_{(j)}\nu = \sum_{j=1}^{\rho}\varphi^2_{(j)}\nu = \nu. \end{aligned}$$

Now if we suppose that $I_1 + I_2 = G(\lambda)\nu$, then $(P - \lambda I)T(\lambda)\nu = \nu + G(\lambda)\nu$, so

$$(P - \lambda I)T(\lambda) = I + G(\lambda). \tag{2.12}$$

Since $\varphi'_{(j)_t} \in C^\infty(0,1)$, by (2.10) we can estimate I_1, I_2 as follows:

$$\|I_1\| \leq M\sum_{j=1}^{\rho}\left|(A_{(j)} - \lambda I)^{-1}\varphi_{(j)}\nu\right| \leq M|\lambda|^{-\frac{1}{2}}\|\nu\|,$$

$$\|I_2\| \le M\sum_{j=1}^{\rho}\left|\frac{d}{dt}(A_{(j)}-\lambda I)^{-1}\varphi_{(j)}\nu\right| \le M'|\lambda|^{-\frac{1}{2}}\|\nu\|.$$

Using these estimates, we have

$$\|G(\lambda)\| \le \|I_1\| + \|I_2\| \le M|\lambda|^{-1}\|\nu\| + M'^{|\lambda|^{-\frac{1}{2}}\|\nu\|}.$$

For sufficiently large number $C_\Phi > 0$ such that $|\lambda| > C_\Phi$, we have $|\lambda|^{-1} \le |\lambda|^{-\frac{1}{2}}$, so

$$\|G(\lambda)\| \le M''^{|\lambda|^{-\frac{1}{2}}}. \tag{2.13}$$

By choosing suitable λ we conclude that $\|G(\lambda)\| \le \frac{1}{2} < 1$. Using this and the well-known theorem in the operator theory, we conclude that $I + G(\lambda)$ is invertible. So by (2.12) we have that $(P-\lambda I)T(\lambda)$ is eversible and

$$\big(T(\lambda)\big)^{-1}(P-\lambda I)^{-1} = \big(I + G(\lambda)\big)^{-1}. \tag{2.14}$$

We add $+I$ and $-I$ to the right side of the (2.14) and consider $F(\lambda) = (I + G(\lambda))^{-1} - I$:

$$\big(T(\lambda)\big)^{-1}(P-\lambda I)^{-1} = F(\lambda) + I. \tag{2.15}$$

In view of $\|G(\lambda)\| \le \frac{1}{2} < 1$ and (2.13), by applying the geometric series for $F(\lambda)$ we have

$$\|F(\lambda)\| \le \sum_{i=2}^{+\infty}\|G^k(\lambda)\| \le \|G(\lambda)\|^2\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right) \le 2\big(M''|\lambda|^{-\frac{1}{2}}\big)^2,$$

so

$$\|F(\lambda)\| \le M'''|\lambda|^{-1}.$$

By (2.10) and (2.11) we have

$$\|T(\lambda)\| = \left\|\sum_{j=1}^{\rho}\varphi_{(j)}(A_{(j)}-\lambda I)^{-1}\varphi_{(j)}\right\| \le K_1\|(A_{(j)}-\lambda I)^{-1}\|,$$

that is,

$$\|T(\lambda)\| \le K_1M_1|\lambda|^{-1} = K_2|\lambda|^{-1}. \tag{2.16}$$

Now by (2.15) and (2.16) we have

$$\|(P-\lambda I)^{-1}\| = \|T(\lambda)\|\,\|\big(I+F(\lambda)\big)\| \le K_2|\lambda|^{-1}\big(1+M'''|\lambda|^{-1}\big) \le K_2|\lambda|^{-1}+K_2|\lambda|^{-1}M'''|\lambda|^{-1}.$$

As before, for sufficiently large number $C_\Phi > 0$ such that $|\lambda| > C_\Phi$, we have $|\lambda|^{-1}|\lambda|^{-1} = |\lambda|^{-2} \le |\lambda|^{-1}$, so

$$\|(P-\lambda I)^{-1}\| \le K_2|\lambda|^{-1} + K_2|\lambda|^{-1}M'''|\lambda|^{-1} = M_\Phi|\lambda|^{-1} \quad \big(|\lambda| \ge C_\Phi, \lambda \in \Phi\big).$$

Step 2. To prove assertion (2.9), by replacing $\omega(t)\frac{d}{dt}(A_{(j)}-\lambda I)^{-1}$ instead of $(A_{(j)}-\lambda I)^{-1}$ in the $T(\lambda)$ and using the previous calculations, we have

$$\left\| T'(\lambda)=\sum_{j=1}^{\rho}\varphi_{(j)}\frac{d}{dt}(A_{(j)}-\lambda I)^{-1}\varphi_{(j)}\right\| \le K_1'\left\|\omega(t)\frac{d}{dt}(A_{(j)}-\lambda I)^{-1}\right\|,$$

that is,

$$\left\|T'(\lambda)\right\| \le K_1'M_1|\lambda|^{-\frac{1}{2}} = K_2'|\lambda|^{-\frac{1}{2}}.$$

By similar calculations we have

$$\omega(t)\frac{d}{dt}(P-\lambda I)^{-1} = T'(\lambda)\bigl(I+F'(\lambda)\bigr),$$

where $\|F'(\lambda)\| \le M_1'''|\lambda|^{-1}$, and hence

$$\left\|\omega(t)\frac{d}{dt}(P-\lambda I)^{-1}\right\| = \left\|T'(\lambda)\right\|\left\|\bigl(I+F'(\lambda)\bigr)\right\|.$$

Therefore,

$$\left\|\omega(t)\frac{d}{dt}(P-\lambda I)^{-1}\right\| \le K_2'|\lambda|^{-\frac{1}{2}}\bigl(1+M_1'''|\lambda|^{-1}\bigr) = K_2'|\lambda|^{-\frac{1}{2}} + K_2'|\lambda|^{-\frac{1}{2}}M_1'''|\lambda|^{-1}$$

since $|\lambda|^{-\frac{1}{2}}|\lambda|^{-1} \le |\lambda|^{-\frac{1}{2}}$, and, consequently,

$$\left\|\omega(t)\frac{d}{dt}(P-\lambda I)^{-1}\right\| \le K_2'|\lambda|^{-\frac{1}{2}} + K_2'M_1'''|\lambda|^{-\frac{1}{2}} = M_\Phi'|\lambda|^{-\frac{1}{2}}.$$ □

3 On the resolvent estimate of the operator on $H_\ell = L^2(0,1)^\ell$

In this section, we consider the operator P on the ℓ-dimensional space $H_\ell = L^2(0,1)^\ell$. In fact, the conditions in this section are more general than in the previous one.

Theorem 3.1 *Let* $(Pu)(t) = -\frac{d}{dt}(\omega^2(t)q(t)\frac{du(t)}{dt})$ *be an operator acting on the space* $H_\ell = L^2(0,1)^\ell$ *with Dirichlet-type boundary conditions. Here* $\omega(t)\in C^1(0,1)$ *is a positive function that satisfies conditions* (1.1) *and* (1.2). *Let* $\Phi=\{z\in C:|\arg z|\le\varphi\}$, *where* $\varphi\in(0,\pi)$ *is a closed angle with vertex at zero. Let* $q(t)\in C^2([0,1],\mathrm{End}\,C^\ell)$ *be such that the matrix function* $q(t)$ *has* ℓ *distinct nonzero simple eigenvalues* $\mu_j(t)\in C^2[0,1]$ $(1\le j\le\ell)$ *for each* $t\in[0,1]$ *that are arranged in the complex plane* C *in the following way*: $\mu_1(t),\dots,\mu_\nu(t)\in R_+$, $\mu_{\nu+1}(t),\dots,\mu_\ell(t)\in C\setminus\Phi$. *Then for sufficiently large in modulus* $\lambda\in\Phi$, *the inverse operator* $(P-\lambda I)^{-1}$ *exists and is continuous in the space* $H_\ell = L^2(0,1)^\ell$, *and the following estimates hold*:

$$\left\|(P-\lambda I)^{-1}\right\| \le M|\lambda|^{-1} \quad \bigl(\lambda\in\Phi, |\lambda|>C\bigr), \tag{3.1}$$

$$\left\|\omega(t)\frac{d}{dt}(P-\lambda I)^{-1}\right\| \le M_2'|\lambda|^{-\frac{1}{2}} \quad \bigl(\lambda\in\Phi, |\lambda|>C\bigr), \tag{3.2}$$

where the $M, C>0$ *are sufficiently large numbers depending on* Φ.

Proof Step 1. In this step, we proof assertion (3.1). First, note that the conditions we consider on the eigenvalues $\mu_j(t)$ of the matrix function $q(t)$ guarantee that we can convert the matrix to the diagonal form

$$q(t) = U(t)\Lambda(t)U^{-1}(t), \quad \text{where } U(t), U^{-1}(t) \in C^2\bigl([0,1], \operatorname{End} C^\ell\bigr)$$

and $\Lambda(t) = \operatorname{diag}\{\mu_1(t), \dots, \mu_\ell(t)\}$. Consider the space $H_\ell = H \oplus \cdots \oplus H$ (ℓ-times) and let us introduce the operator

$$B(\lambda) = \operatorname{diag}\bigl\{(P_1 - \lambda I)^{-1}, \dots, (P_1 - \lambda I)^{-1}\bigr\} \tag{3.3}$$

acting on H_ℓ, where $(P_j v)(t) = -\frac{d}{dt}(\omega^2 \mu_j \frac{dv(t)}{dt})$ and

$$D(P_j) = \left\{ v \in \dot{\mathcal{H}} \cap W^2_{2,\mathrm{loc}}(0,1); \omega u' \in H, \frac{d}{dt}\left(\omega^2 \mu_j \frac{du}{dt}\right) \in H \right\}.$$

According to the results that obtained in Section 2, the operator $B(\lambda)$ exists and is continuous for sufficiently large modulus of $\lambda \in \Phi$. Consider the operator $\Gamma(\lambda) = UB(\lambda)U^{-1}$, where $(Uu)(t) = U(t)u(t)$ ($u \in H_\ell$). Consequently, it follows that

$$(P - \lambda I)\Gamma(\lambda)u = -\frac{d}{dt}\left(\omega^2 q(t)\frac{d}{dt}\bigl(U(t)B(\lambda)U^{-1}(t)u(t)\bigr)\right) = T_1 + T_2 + T_3,$$

where

$$\begin{aligned}
T_1 &= -\frac{d}{dt}\left(\omega^2 q(t)U(t)\frac{d}{dt}B(\lambda)U^{-1}(t)u(t)\right) = -\frac{d}{dt}\left(\omega^2 U(t)\Lambda(t)\frac{d}{dt}B(\lambda)U^{-1}u(t)\right) \\
&= -U\frac{d}{dt}\left(\omega^2\Lambda(t)\frac{d}{dt}B(\lambda)U^{-1}u\right) - U'(t)\omega^2\Lambda\frac{d}{dt}B(\lambda)U^{-1}u \\
&= \lambda UB(\lambda)U^{-1}u - U'(t)\omega^2\Lambda\frac{d}{dt}B(\lambda)U^{-1}u + UU^{-1}u, \\
T_2 &= -\frac{d}{dt}\bigl(\omega^2 qU'B(\lambda)U^{-1}u\bigr),
\end{aligned}$$

and

$$T_3 = -\lambda U(t)B(\lambda)U^{-1}u.$$

Using (2.3) and (2.4), have $(P - \lambda I)\Gamma(\lambda) = I + T_1^0 + T_2^0$, where $T_2^0 = (\omega^2)' qU'B(\lambda)U^{-1}$ and $\|T_1^0\| \le M|\lambda|^{-\frac{1}{2}}$.

Now by the Hardy-type inequality we estimate the operator T_2^0 as follows:

$$\begin{aligned}
&\int_0^1 t^{-1+\varepsilon'}(1-t)^{-1+\varepsilon' 2}\bigl|y(t)\bigr|^2\,dt \\
&\quad \le M\bigl(\varepsilon_1', \varepsilon_2'\bigr)\int_0^1 \bigl|y(t)\bigr|^2\,dt \\
&\qquad + M\bigl(\varepsilon_1', \varepsilon_2'\bigr)\int_0^1 t^{1+\varepsilon_1'}(1-t)^{1+\varepsilon_2'}\bigl|y'(t)\bigr|^2\,dt \quad \forall y \in \dot{\mathcal{H}}, \varepsilon_1', \varepsilon_2' \neq 0.
\end{aligned}$$

Since $|q(t)U'(t)|_{C^\ell\to C^\ell}\le M$, by (1.2) we have the following inequality:

$$\begin{aligned}
&\int_0^1 \bigl|\bigl(\omega^2(t)\bigr)'\bigr|^2\bigl|\bigl(B(\lambda)u(t)\bigr)\bigr|_{C^\ell}^2\,dt\\
&\quad\le M_2\int_0^1 t^{\alpha-2+2\varepsilon_1'}(1-t)^{\beta-2+2\varepsilon'2}\bigl|W'(t)\bigr|_{C^\ell}^2\,dt\\
&\quad\le M_3\int_0^1 \bigl|t^\alpha(1-t)^\beta\omega^{-2}(t)\bigr|\bigl|\omega^2\bigl(\bigl(B(\lambda)u\bigr)(t)\bigr)_t'\bigr|_{C^\ell}^2\,dt\\
&\qquad+M\bigl|\bigl(B(\lambda)u\bigr)\bigr|_{H_\ell}^2,\quad W=B(\lambda)u.
\end{aligned}$$

Now by (1.1) and estimate (2.3) it follows

$$\begin{aligned}
&\int_0^1 \bigl|\bigl(\omega^2(t)\bigr)'\bigr|^2\bigl|\bigl(B(\lambda)u\bigr)(t)\bigr|_{C^\ell}^2\,dt\\
&\quad\le M\int_0^1 \omega^2\bigl|\bigl(\bigl(B(\lambda)u\bigr)(t)\bigr)_t'\bigr|_{C^\ell}^2\,dt+M\bigl|\bigl(B(\lambda)u\bigr)\bigr|_{H_\ell}^2\\
&\quad\le M'|\lambda|^{-1}|u|_{H_\ell}^2\quad\bigl(\lambda\in\Phi,|\lambda|>C\bigr).
\end{aligned}$$

Then $\|T_2^0\|\le M'|\lambda|^{-1/2}$ for sufficiently large in modulus of $\lambda\in\Phi$; consequently,

$$(P-\lambda I)\Gamma(\lambda)=I+F(\lambda),\qquad \bigl\|F(\lambda)\bigr\|\le M|\lambda|^{-\frac{1}{2}}\quad\bigl(\lambda\in\Phi,|\lambda|>C\bigr).\tag{3.4}$$

Proceeding as at the end of Section 2 (*e.g.*, see (2.12)) from $\|F(\lambda)\|\le M|\lambda|^{-\frac{1}{2}}$ it easily follows that $I+F(\lambda)$ is inversible and then that $(A-\lambda I)\Gamma(\lambda)$ is inversible, that is,

$$\bigl((P-\lambda I)\Gamma(\lambda)\bigr)^{-1}=\bigl(I+F(\lambda)\bigr)^{-1}.$$

Then by adding $-I$ and $+I$ to the last relation we have

$$\bigl(I+F(\lambda)\bigr)^{-1}=\bigl(I+F(\lambda)\bigr)^{-1}-I+I.$$

Since $\|F(\lambda)\|\le M|\lambda|^{-\frac{1}{2}}$, in a calculation as in Section 2, take $y(\lambda)=(i+F(\lambda))^{-1}-I$. Then $y(\lambda)$ satisfies

$$\bigl\|y(\lambda)\bigr\|\le M|\lambda|^{-1}\quad\bigl(\lambda\in\Phi,|\lambda|>C\bigr).\tag{3.5}$$

Consequently,

$$(P-\lambda I)^{-1}=\Gamma(\lambda)\bigl(I+y(\lambda)\bigr)\tag{3.6}$$

since

$$\Gamma(\lambda)=UB(\lambda)U^{-1},\quad B(\lambda)=\operatorname{diag}\bigl\{(P_1-\lambda I)^{-1},\dots,(P_\ell-\lambda I)^{-1}\bigr\}.\tag{3.7}$$

Put $P_j = A_j, j = 1, \dots, \ell$, as in (2.10). By (3.5)-(3.7) we have $\|(P_j - \lambda I)^{-1}\| \le M|\lambda|^{-1}, j = 1, \dots, \ell$, and it follows that $\|\Gamma(\lambda)\| \le M|\lambda|^{-1}$, so

$$\begin{aligned}\|(P-\lambda I)^{-1}\| &\le \|\Gamma(\lambda)\| \|(I+y(\lambda))\| \\ &\le M|\lambda|^{-1}(1+M|\lambda|^{-1}) \le M|\lambda|^{-1}.\end{aligned}$$

Step 2. Now we prove estimate (2.9). Since $\|\omega\frac{d}{dt}(P_j - \lambda I)^{-1}\| \le M|\lambda|^{-\frac{1}{2}}$, $j = 1, \dots, \ell$, for $\Gamma_1(\lambda)$, we can get the corresponding estimate $\|\Gamma_1(\lambda)\| \le M_1|\lambda|^{-\frac{1}{2}}$, and this implies

$$\left\|\omega\frac{d}{dt}(P-\lambda I)^{-1}\right\| \le \|\Gamma_1(\lambda)\| \|(I+y_1(\lambda))\|.$$

Since $\|y_1(\lambda)\| \le M_1'|\lambda|^{-1}$, we have $\|\omega\frac{d}{dt}(P_j - \lambda I)^{-1}\|\|\omega\frac{d}{dt}(P_j - \lambda I)^{-1}\| \le M|\lambda|^{-\frac{1}{2}}(1 + M_1'|\lambda|^{-1})$, which implies $\|\omega\frac{d}{dt}(P_j - \lambda I)^{-1}\| \le M|\lambda|^{-\frac{1}{2}}$ for ($\lambda \in \Phi$, $|\lambda| \ge C$), so that the proof of the fundamental Theorem 3.1 in the general case $H_\ell = L^2(0,1)^\ell$ is completed. □

4 Vanishing limit arguments

Denote by $\lambda_1, \lambda_2, \dots$ the eigenvalues of P that belong to the angle

$$\Phi = \{z \in C : |\arg z| < \varphi\}, \quad \varphi \in (0, \pi).$$

We want to find the limit of the sequence $\{\lambda_j\}$ as $j \to \infty$.

Theorem 4.1 *Suppose that for every closed angle $S \subset \Phi \setminus R_+$ with origin at zero, there exists a number $C(S) > 0$ such that for all $\lambda \in S$ with $|\lambda| \ge C(S)$, the inverse operator $(P - \lambda I)^{-1}$ exists and is continuous. Then*

$$\arg \lambda_j \to 0 \quad (j \to \infty).$$

Proof Assume the set $K = \{\arg \lambda_j : j = 1, 2, \dots\}$ has a nonzero limit point $\varphi_1 \in [-\varphi, +\varphi]$. Then there exists a subsequence $\{\lambda_{j_k}\}$ such that

$$\lim_{k\to\infty} \arg \lambda_{j_k} = \varphi_1.$$

We consider the closed sector $S \subset \overline{\Phi} \setminus R_+$ such that the ray

$$\Gamma = \{z \in C; \arg z = \varphi_1\} \subset S.$$

By the definition of the limit and our assumption there exists $N_1 \in N$ such that $\forall k > N_1$, $\lambda_{j_k} \in S$. Since $\lambda_n \to \infty$ ($n \to \infty$) for $C(S) > 0$, there exists $N_2 \in N$ such that $|\lambda_{j_k}| \ge C(S)$ for $k > N_2$.

Now if $k > \max(N_1, N_2)$, then $\lambda_{j_k} \in S$ and $|\lambda_{j_k}| \ge C(S)$, so that the latter condition implies that $(P - \lambda_{j_k} I)^{-1}$ exists and is continuous, which in turn by definition implies that λ_{j_k} is not an eigenvalue of P (since by Theorem 2.1 this λ_{j_k} is the resolvent of P, that is, cannot be an eigenvalue of P), so $\lambda_{j_k} \notin \Phi$, which is a contradiction. Hence, we must have

$$\arg \lambda_j \to 0 \quad (j \to \infty).$$

□

5 On the asymptotic distribution of eigenvalues of the differential operator P on H_ℓ

Theorem 5.1 *Let* $\Lambda = \frac{1}{\pi}\sum_{j=1}^{\nu}\int_0^1 \omega^{-1}(t)\mu_j^{-\frac{1}{2}}(t)\,dt$ *and* $N(\tau) = \operatorname{card}\{j : |\lambda_j| \le \tau\}$ *be a distribution function. Then we have the following asymptotic formula*:

$$N(\tau) \sim \Lambda\tau^{1/2} \quad as \ \to +\infty.$$

Proof We know that for an arbitrary kernel operator T_1 and an arbitrary bounded operator T_2,

$$\operatorname{tr} T_1T_2 = \operatorname{tr} T_2T_1, \qquad |T_1T_2|_1 \le |T_1|_1\|T_2\|$$

(see [10]).

Using these relations, from (3.3) and (3.4) we get

$$\operatorname{tr}(P-\lambda I)^{-1} = \sum_{J=1}^{+\infty}\frac{1}{\lambda_j-\lambda} = \operatorname{tr}\Gamma(\lambda) + O(1)\bigl|\Gamma(\lambda)\bigr|_1\bigl\|y(\lambda)\bigr\|,$$

$$\operatorname{tr}\Gamma(\lambda) = \operatorname{tr} UB(\lambda)U^{-1} = \operatorname{tr} B(\lambda) = \sum_{j=1}^{\ell}\operatorname{tr}(P_j-\lambda I)^{-1},$$

$$\bigl|\Gamma(\lambda)\bigr|_1 \le \|U\|\bigl\|U^{-1}\bigr\|\bigl|B(\lambda)\bigr|_1 \le M|\lambda|^{-1} \quad \bigl(\lambda\in\Phi, |\lambda| > C\bigr).$$

Now if $\lambda\to\infty$, then

$$\sum_{j=1}^{+\infty}\frac{1}{\lambda_j-\lambda} = \sum_{j=1}^{\ell}\operatorname{tr}(P_j-\lambda I)^{-1} + O\bigl(|\lambda|^{-1}\bigr). \tag{5.1}$$

Let $\lambda_1, \lambda_2, \ldots$ and $\lambda_{j,1}, \lambda_{j,2}, \ldots$ be the sequences of the eigenvalues of the operators P and P_j (for $j = \nu+1, \ldots, l$), respectively. So

$$\sum_{j=1}^{+\infty}\frac{1}{\lambda_j-\lambda} = \sum_{k=1}^{\ell}\sum_{j=1}^{+\infty}\frac{1}{\lambda_{j,k}-\lambda} + O\bigl(|\lambda|^{-1}\bigr), \quad \lambda\to\infty, \lambda\in\Phi. \tag{5.2}$$

If $\Psi = \{z\in C : |\arg z| < \psi\}$, $\psi\in(0,\varphi)$, then we can take the index j_0 such that, for $j > j_0$,

$$|\arg\lambda_j| < \psi, \qquad |\arg\lambda_{j,k}| > \varphi \quad \text{for } k = \nu+1, \ldots, \ell. \tag{5.3}$$

Suppose that the eigenvalues of the operators P and P_j satisfy the following conditions:

$$\lambda_1(t), \ldots, \lambda_\nu \in R_+ \quad \text{and} \quad \lambda_{\nu+1}(t), \ldots, \lambda_\ell(t) \in C\setminus\Phi;$$

$$\lambda_{j,1}, \ldots, \lambda_{j,\nu} \in R_+ \quad \text{and} \quad \lambda_{j,\nu+1}, \ldots, \lambda_{j,\ell} \in C\setminus\Phi.$$

So we can change the sum $\sum_{k=1}^{\ell}$ to $\sum_{k=1}^{\nu}$ and the sum $\sum_{j=1}^{+\infty}$ to $\sum_{j=j_0}^{+\infty}$ in (5.2):

$$\sum_{j=j_0}^{+\infty}\frac{1}{\lambda-\lambda_j} = \sum_{k=1}^{\nu}\sum_{j=j_0}^{+\infty}\frac{1}{\lambda-\lambda_{j,k}} + O\bigl(|\lambda|^{-1}\bigr), \quad \lambda\to+\infty, \lambda\in\partial\Phi.$$

Multiplying both sides of the last equation by $\frac{1}{\lambda+\tau}$, $\tau > 1$, using the contour integral method (see [9], Chapter 4), and from (5.3) integrating with respect to $\lambda \in \partial\Psi$, we get that

$$\sum_{j=j_0}^{+\infty} \frac{1}{\lambda + \lambda_j} = \sum_{k=1}^{\nu} \sum_{j=j_0}^{+\infty} \frac{1}{\lambda + \lambda_{j,k}} + O\bigl(|\lambda|^{-1}\bigr), \quad \lambda \to +\infty.$$

Replace λ by η in the previous relation, we have

$$\sum_{j=j_0}^{+\infty} \frac{1}{\eta + \lambda_j} = \sum_{k=1}^{\nu} \sum_{j=1}^{+\infty} \frac{1}{\eta + \lambda_{j,k}} + O\bigl(\eta^{-1}\bigr), \quad \eta \to +\infty$$

(remark that here, as before, by applying the contour integral method we can change the negative sign to the positive sign in the denominator of the last relation). We now use the countable discreteness of the eigenvalues of the operator P, and then we can change the above series to the following integral:

$$\int_0^{+\infty} \frac{dN(\tau)}{\eta + \tau} = \sum_{k=1}^{\nu} \int_0^{+\infty} \frac{dN_k(\tau)}{\eta + \tau} + O\bigl(\eta^{-1}\bigr), \quad \eta \to +\infty, \tag{5.4}$$

where

$$N(\tau) = \operatorname{card}\bigl\{j : \bigl|\lambda_j'\bigr| \le \tau\bigr\},$$
$$N_k(\tau) = \operatorname{card}\{j : \lambda_{j,k} \le \tau\}, \quad k = 1, \dots, \nu.$$

By considering the previous conditions concerning the functions $\omega^2(t)$, $\mu_k(t)$ and by applying the theorems of Chapter 7 of [10] we can derive the following formula for the functions $N_k(\tau)$, $k = 1, \dots, \nu$:

$$N_k(\tau) \sim c_k \tau^{\frac{1}{2}}, \quad \tau \to +\infty,$$

where

$$c_k = \frac{1}{\pi} \int_0^1 \omega^{-1}(t) \mu_k^{-\frac{1}{2}}(t)\, dt, \quad k = 1, \dots, \nu.$$

By (5.4) we have

$$\int_0^{+\infty} \frac{dN(\tau)}{x + \tau} \sim \sum_{j=1}^{\nu} \int_0^{+\infty} \frac{dN(\tau)}{x + \tau}, \quad x \to +\infty.$$

Now applying for this last relation the Shakalikov multiray Tauberian theorem (see [6]), we get

$$N(\tau) \sim \sum_{k=1}^{\nu} N_k(\tau), \quad \tau \to \infty,$$

where $N_k(\tau) \sim c_k \tau^{\frac{1}{2}}$ with c_k as before. Consequently, the following asymptotic formula is valid:

$$N(\tau) \sim \Lambda \tau^{\frac{1}{2}}, \quad \tau \to +\infty,$$

where

$$\Lambda = \sum_{k=1}^{\nu} c_k = \frac{1}{\pi} \sum_{j=1}^{\nu} \int_0^1 \omega^{-1}(t) \mu_j^{-\frac{1}{2}}(t)\, dt.$$

This completes the proof of the theorem. □

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The authors declare that they have no competing interests.

Authors' contributions
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