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Weak Poincaré inequalities and hitting times for jump processes

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Abstract

In this paper, we get a criteria of weak Poincaré inequality by some integrability of hitting times for jump processes. In fact, integrability of hitting times on a subset F of state space E implies that the taboo process restricted on $E \setminus F$ is decay, from which we get a weak Poincaré inequality with absorbing (Dirichlet) boundary. Using it and a local Poincaré inequality, we obtain a weak Poincaré inequality by the decomposition method.

Keywords: weak Poincaré inequality; hitting times; jump process

1 Introduction and main results

During the recent years, a lot of progress has been made in the understanding of functional inequalities and their links with the convergence rates of Markov processes. As we know, the convergence rates of a Markov semigroup and the corresponding functional inequalities can be determined by each other. To describe the convergence rates slower than exponential, Röckner and Wang [1] introduced the following weak Poincaré inequality (WPI):

$$\operatorname{Var}_\mu(f) \le \alpha(r)D(f,f) + r\Phi(f), \quad f \in \mathcal{D}(D), r > 0, \tag{1.1}$$

where $\operatorname{Var}_\mu(f) = \|f - \mu(f)\|_2^2$, α is a nonnegative decreasing function on $(0,\infty)$, $D(f,f) = -\mu(fLf)$, L is the generator of a Markov semigroup P_t on $L^2(\mu)$, and $\Phi : L^2(\mu) \to (0,\infty)$ satisfies $\Phi(cf) = c^2\Phi(f)$ and $\Phi(P_tf) \le \Phi(f)$ for any $c \in \mathbb{R}$ and $f \in L^2(\mu)$. They proved that the L^2-convergence rate of a Markov semigroup and the corresponding weak Poincaré inequality can be determined by each other. We restate the results as follows. Let $(P_t)_{t\ge0}$ be the Markov semigroup determined by the Dirichlet form $(D, \mathcal{D}(D))$. Assume that (1.1) holds. Then

$$\left\|P_tf - \mu(f)\right\|_2^2 \le \xi(t)\left[\Phi(f) + \|f\|_2^2\right], \quad t > 0, \tag{1.2}$$

where $\xi(t) = \inf\{r > 0 : -\frac{1}{2}\alpha(r)\log(r) \le t\}$. Conversely, in the reversible case, (1.2) implies the weak Poincaré inequality (1.1) for $\alpha(r) = 2r\inf_{s>0} \frac{1}{s}\xi^{-1}(s\exp[1 - s/r])$. In this paper, we choose $\Phi(\cdot) = \|\cdot\|_{L^\infty(\mu)}^2 =: \|\cdot\|_\infty^2$.

Recently, much literature was devoted to the relationships of the Poincaré-type inequalities and Lyapunov conditions used in the 'Meyn-Tweedie' theory (see [2, 3]). Cattlaux *et al.* [4] went a step further by showing the equivalence between the (usual) Poincaré inequality, Lyapunov conditions, and the existence of exponential moments for hitting times for reversible diffusion processes. Now it is interesting to look at more general moments of hitting times and other processes. Mao and Xia [5] obtained a criterion for spectral gap by hitting times for jump processes via decomposition method. Cheng and Wang [6] studied the algebraic convergence rates for diffusion processes on Riemannian manifolds with boundary by using a Lyapunov condition. In the present paper, we use some integrability of hitting times to get a type of weak Poincaré inequality for jump processes, which can be used to study the convergence rates for jump processes in the sense of $\|P_t - \pi\|_{\infty\to 2}$.

Let $(q(x), q(x,dy))$ be the q-pair of a regular reversible q-process with transition kernel $P_t(x,dy)$ on a probability space $(E, \mathscr{E}, \mu)$. Denote by ${}_b\mathscr{E}$ the set of bounded functions on $\mathscr{E}$. For $f \in {}_b\mathscr{E}$, denote by L and P_t the generator and semigroup, respectively, where $Lf(x) = \int_E q(x,dy)(f(y) - f(x))$, and $P_tf(x) = \int_E f(y)P_t(x,dy)$. The Dirichlet form D is defined by $D(f,f) = \frac{1}{2}\int_{E\times E}\mu(dx)q(x,dy)(f(y) - f(x))^2$, where $f \in \mathcal{D}(D) = \{f \in L^2(\mu) : D(f,f) < \infty\}$. Now, we introduce two inequalities used in this paper.

Taking $F \in \mathscr{E}$ such that $\mu(F) > 0$, we have the inequality

$$\mu\bigl(f^2\bigr) \le \alpha_F(r)D(f,f) + r\|f\|_\infty^2, \quad f \in \mathcal{D}(D), f|_F = 0, r > 0, \tag{1.3}$$

which is said to be a weak Poincaré inequality with Dirichlet boundary (WPID).

A local Poincaré inequality (LPI) restricted on F is satisfied if

$$\mathrm{Var}_{\mu_F}(f) \le CD_F(f,f), \quad f \in \mathcal{D}(D_F), \tag{1.4}$$

where $D_F(f,f) = \frac{1}{2}\int_{F\times F}\mu_F(dx)q(x,dy)(f(y) - f(x))^2$ and $\mu_F(\cdot) = \frac{\mu(\cdot\cap F)}{\mu(F)}$. Define

$$\lambda_1(F) = \inf\bigl\{D_F(f,f) : \mu_F(f) = 0, \mu_F\bigl(f^2\bigr) = 1\bigr\}. \tag{1.5}$$

Then $\lambda_1(F)^{-1}$ is the smallest constant such that local Poincaré inequality restricted on F holds.

This paper is organized as follows. In Section 2, we prove that some integrability of hitting times on a subset of state space is sufficient for the WPID (1.3). In Section 3, we show how to use LPI and WPID to get WPI.

The following is the main result of this paper.

Theorem 1.1 *Fix $F \in \mathscr{E}$ with $\mu(F) > 0$. Assume that a local Poincaré inequality restricted on F is satisfied for a reversible q-processes with q-pair $(q(x), q(x,dy))$. Let $\tau_F := \inf\{t \ge 0 : X_t \in F\}$ be the hitting time of F. If there exist a decreasing function $\xi : [0,\infty) \to (0,\infty)$ such that $\xi(t) \to 0$ as $t \to \infty$, $\mathbb{E}_\mu\xi(\tau_F)^{-1} =: c < \infty$, and $M_F := \sup_{x\in F} q(x,F^c) < \infty$, then we have the weak Poincaré inequality*

$$\mathrm{Var}_\mu(f) \le \alpha(r)D(f,f) + r\|f\|_\infty^2, \quad f \in \mathcal{D}(D), r > 0, \tag{1.6}$$

for $\alpha(r) \le (1 + \frac{M_F}{\lambda_1(F)})\xi^{-1}(\frac{r}{4c}) + \frac{1}{\lambda_1(F)}$.

In this theorem, we get a criterion of weak Poincaré inequality by integrability of hitting times and a local Poincaré inequality on a fixed subset for jump processes. Since the subset is fixed, it is different from the criterion of WPI in Theorem 4.3.1 of [7].

2 The hitting time and weak Poincaré inequality with Dirichlet boundary

In this section, we prove that some integrability of hitting times on a subset F implies that the taboo process restricted on $E \setminus F$ is decay, by which we get a weak Poincaré inequality with Dirichlet boundary.

Denote ${}_F P_t(x,H) = \mathbb{P}_x[X_t \in H, t < \tau_F]$. Then ${}_F P_t(x,dy)$ satisfies the backward Kolmogorov equation (see [8])

$$ {}_F P_t(x,H) = \int_0^t e^{-q(x)(t-s)} \int \frac{q(x,dy \setminus F)}{q(x)} {}_F P_s(y,H)\,ds + e^{-q(x)t} 1_F(x), \quad x \in F^c. $$

So the q-pair of ${}_F P_t(x,dy)$ is $({}_F q(x), {}_F q(x,dy))$ with ${}_F q(x,H) = q(x,H \setminus F)$ and ${}_F q(x) = q(x)$ for $x \in F^c$ and 0 otherwise; ${}_F P_t$ is a Markov semigroup,

$$ {}_F P_t f(x) = \int_{F^c} {}_F P_t(x,dy) f(y), \quad f \in L^2(F^c, \mu). $$

Its generator ${}_F L$ is defined as

$$ {}_F L f(x) = \int_{F^c} q(x,dy) f(y) - q(x) f(x), \quad f \in L^2(F^c, \mu). $$

Here $(\cdot,\cdot)$ denotes the inner product in $L^2(F^c,\mu)$. We can prove that the generator ${}_F L$ is self-adjoint in $L^2(F^c,\mu)$. Then ${}_F P_t$ is reversible with respect to μ. We also define the Dirichlet form

$$ \begin{aligned} {}_F D(f,f) :&= -({}_F L f, f) \\ &= \int_{F^c} \mu(dx) q(x) f^2(x) - \int_{F^c \times F^c} \mu(dx) q(x,dy) f(x) f(y). \end{aligned} \tag{2.1} $$

Lemma 2.1 *If there exist a decreasing function $\xi : [0,\infty) \to (0,\infty)$ such that $\xi(t) \to 0$ as $t \to \infty$ and $\mathbb{E}_\mu \xi(\tau_F)^{-1} =: c < \infty$, then WPID holds for $\alpha_F(r) \le \xi^{-1}(\frac{r}{c})$.*

Proof

$$ \begin{aligned} \left|(f, {}_F P_t f)\right| &= \left| \int_{F^c} \mu(dx) f(x) {}_F P_t f(x) \right| \\ &\le \|f\|^2_{L^\infty(F^c,\mu)} \int \mu(dx) \mathbb{P}_x[\tau_F > t] \\ &\le \|f\|^2_{L^\infty(F^c,\mu)} \int \mu(dx) \mathbb{E}_x \xi(\tau_F)^{-1} \xi(t) \\ &= c\|f\|^2_{L^\infty(F^c,\mu)} \xi(t), \end{aligned} \tag{2.2} $$

where the last inequality holds since $\xi(t)$ is decreasing.

Since ${}_F P_t$ is reversible with respect to μ, by (2.2) we have

$$ \|{}_F P_{t/2} f\|_2^2 \le c\|f\|^2_{L^\infty(F^c,\mu)} \xi(t). $$

Set $\eta(t) = c\xi(2t)$. Then

$$\|_F P_t f\|_2^2 \le \|f\|_{L^\infty(F^c,\mu)}^2 \eta(t).$$

By [9] this implies the nonergodic weak Poincaré inequality

$$\|f\|_{L^2(F^c,\mu)}^2 \le {}_F\alpha(r)_F D(f,f) + r\|f\|_{L^\infty(F^c,\mu)}^2, \quad r > 0, \tag{2.3}$$

for

$${}_F\alpha(r) \le 2\eta^{-1}(r) = \xi^{-1}\left(\frac{r}{c}\right).$$

For every g with $g|_F = 0$, let $f = g|_{F^c}$. Thus,

$$\|f\|_{L^2(F^c,\mu)} = \|g\|_2, \qquad \|f\|_{L^\infty(F^c,\mu)} = \|g\|_\infty.$$

Besides, by (2.1) we have

$$\begin{aligned} D(g,g) &:= -(Lg,g) \\ &= \int_E \mu(dx)q(x)g^2(x) - \int_{E\times E} \mu(dx)q(x,dy)g(x)g(y) \\ &= \int_{F^c} \mu(dx)q(x)g^2(x) - \int_{F^c\times F^c} \mu(dx)q(x,dy)g(x)g(y) \\ &= {}_F D(f,f). \end{aligned}$$

By (2.3) it follows that

$$\mu\left(g^2\right) \le {}_F\alpha(r)D(g,g) + r\|g\|_\infty^2, \quad g \in \mathcal{D}(D), g|_F = 0, r > 0.$$

Thus, (1.3) holds for $\alpha_F(r) \le {}_F\alpha(r) \le \xi^{-1}(\frac{r}{c})$. □

3 The relationship of WPID and WPI

In this section, we show how to use the local Poincaré inequality and weak Poincaré inequality with absorbing boundary to obtain a weak Poincaré inequality.

3.1 The relationship of WPID and WPI when F is a singleton

First, we consider the case that F is a singleton. Assume that $F = \{\theta\}$ and $\mu(\theta) > 0$. For convenience, we denote $\mu(\theta) =: \mu_\theta$ and $f(\theta) =: f_\theta$.

Lemma 3.1 *For any $r > 0$, define*

$$\alpha(r) := \inf\left\{C > 0 \mid \mathrm{Var}_\mu(f) \le CD(f,f) + r\|f\|_\infty^2, f \in \mathcal{D}(D)\right\}$$

and

$$\alpha_\theta(r) := \inf\left\{C_\theta > 0 \mid \mu\left(f^2\right) \le C_\theta D(f,f) + r\|f\|_\infty^2, f \in \mathcal{D}(D), f(\theta) = 0\right\}.$$

Then

$$\alpha_\theta\left(\frac{r}{\mu_\theta}\right)\mu_\theta \le \alpha(r) \le \alpha_\theta\left(\frac{r}{4}\right). \tag{3.1}$$

Proof (a) For the upper bound of $\alpha(r)$, noticing that $\mathrm{Var}_\mu(f) = \inf_c \mu((f-c)^2)$ and $\|f - f_\theta\|_\infty \le 2\|f\|_\infty$, for any $f \in \mathcal{D}(D)$, we have

$$\begin{aligned}\mathrm{Var}_\mu(f) &\le \mu\big((f-f_\theta)^2\big)\\ &\le \alpha_\theta(r)D(f-f_\theta, f-f_\theta) + r\|f-f_\theta\|_\infty^2\\ &\le \alpha_\theta(r)D(f,f) + 4r\|f\|_\infty^2.\end{aligned}$$

Then, by the definition of $\alpha(r)$ we have

$$\alpha(r) \le \alpha_\theta\left(\frac{r}{4}\right). \tag{3.2}$$

(b) On the other hand, for any $f \in L^2(E)$ with $f_\theta = 0$, we have

$$\begin{aligned}\mathrm{Var}_\mu(f) &= \mu\big(f^2\big) - \mu(f)^2 = \mu\big(f^2\big) - \mu(f\mathbf{1}_{\{\theta\}^c})^2\\ &\ge \mu\big(f^2\big) - \mu\big(f^2\big)\mu\big(\{\theta\}^c\big) = \mu\big(f^2\big)\mu_\theta.\end{aligned}$$

Thus,

$$\mu\big(f^2\big)\mu_\theta \le \mathrm{Var}_\mu(f) \le \alpha(r)D(f,f) + r\|f\|_\infty^2$$

and

$$\mu\big(f^2\big) \le \frac{1}{\mu_\theta}\alpha(r)D(f,f) + \frac{r}{\mu_\theta}\|f\|_\infty^2.$$

Then, by the definition of $\alpha_\theta(r)$ we have

$$\alpha_\theta\left(\frac{r}{\mu_\theta}\right) \le \frac{1}{\mu_\theta}\alpha(r). \tag{3.3}$$

Thus, by (3.2) and (3.3) we have $\alpha_\theta(\frac{r}{\mu_\theta})\mu_\theta \le \alpha(r) \le \alpha_\theta(\frac{r}{4})$. □

3.2 The relationship of WPI and WPID for general *F*

For a general set F, we consider the relationships of $\alpha(r)$ and $\alpha_F(r)$ by the decomposition method and the conclusions given in Section 3.1.

Thinking of F as a point θ, we construct another state space $\hat{E}$ and q-process with q-pair $(\hat{q}(x), \hat{q}(x,dy))$ as follows: $\hat{E} = \{\theta\} \cup F^c$ and $\hat{\mathscr{E}} = \sigma(\{\theta\}, \mathscr{E} \cap F^c)$. For any $x \in F^c$ and $H \in \hat{\mathscr{E}}$, define

$$\hat{q}(x,H) = \begin{cases} q(x,F) + q(x, H\setminus\{\theta\}), & \theta \in H,\\ q(x,H), & \theta \notin H,\end{cases}$$

and, for $x = \theta$ and $H \in \hat{\mathscr{E}}$,

$$\hat{q}(\theta, H) = \frac{1}{\mu(F)} \int_F \mu(dy) q\big(y, H \setminus \{\theta\}\big),$$

and, for $x \in E$, $\hat{q}(x) = \hat{q}(x, \hat{E})$. Then $(\hat{q}(x), \hat{q}(x, dy))$ is a reversible q-pair on $(\hat{E}, \hat{\mathscr{E}})$ with respect to the probability measure $\hat{\mu}$, where, for any $H \in \hat{\mathscr{E}}$,

$$\hat{\mu}(H) = \begin{cases} \mu(F) + \mu(H \setminus \{\theta\}), & \theta \in H, \\ \mu(H), & \theta \notin H. \end{cases}$$

Denote

$$\hat{\alpha}_\theta(r) := \inf\big\{\hat{C}_\theta \mid \hat{\mu}\big(\hat{f}^2\big) \le \hat{C}_\theta \hat{D}(\hat{f}, \hat{f}) + r\|\hat{f}\|_\infty^2, \hat{f} \in \mathcal{D}(\hat{D}), \hat{f}(\theta) = 0\big\}$$

and

$$\hat{\alpha}(r) := \inf\big\{\hat{C} \mid \operatorname{Var}\mu(\hat{f}) \le \hat{C}\hat{D}(\hat{f}, \hat{f}) + r\|\hat{f}\|_\infty^2, \hat{f} \in \mathcal{D}(\hat{D})\big\},$$

where $\hat{D}(\hat{f}, \hat{f}) = \frac{1}{2}\int_{\hat{E}\times\hat{E}} \hat{\mu}(dx)\hat{q}(x, dy)(\hat{f}(y) - \hat{f}(x))^2$ is the Dirichlet form of the q-pair $(\hat{q}(x), \hat{q}(x, dy))$ in $L^2(\hat{E}, \hat{\mathscr{E}})$.

It is easy to see that $\hat{\alpha}_\theta(r) = \alpha_F(r)$. By Lemma 3.1 we have a nice relationship $\hat{\alpha}_\theta(\frac{r}{\hat{\mu}_\theta})\hat{\mu}_\theta \le \hat{\alpha}(r) \le \hat{\alpha}_\theta(\frac{r}{4})$. So we obtain the following lemma.

Lemma 3.2

$$\alpha_F\left(\frac{r}{\mu(F)}\right)\mu(F) \le \hat{\alpha}(r) \le \alpha_F\left(\frac{r}{4}\right). \tag{3.4}$$

Next, we first prove the relationships between $\hat{\alpha}(r)$ and $\alpha(r)$.

Lemma 3.3

$$\hat{\alpha}(r) \le \alpha(r) \le \left(1 + \frac{M_F}{\lambda_1(F)}\right)\hat{\alpha}(r) + \frac{1}{\lambda_1(F)}. \tag{3.5}$$

Proof (a) For the lower bound, given an arbitrary function $\hat{f}$ on $\hat{E}$, set

$$f(x) = \begin{cases} \hat{f}(\theta), & x \in F, \\ \hat{f}(x), & x \in F^c. \end{cases}$$

By the definition of $\alpha(r)$, for this function f, we have

$$\operatorname{Var}_\mu(f) \le \alpha(r) D(f, f) + r\|f\|_{L^\infty(E,\mu)}^2.$$

By a simple calculation we get $\mu(f) = \hat{\mu}(\hat{f})$, $\mu(f^2) = \hat{\mu}(\hat{f}^2)$, and $D(f, f) = \hat{D}(\hat{f}, \hat{f})$. Noticing that $\|f\|_{L^\infty(E,\mu)} = \|\hat{f}\|_{L^\infty(\hat{E},\hat{\mu})}$, we have

$$\operatorname{Var}_{\hat{\mu}}(\hat{f}) \le \alpha(r)\hat{D}(\hat{f}, \hat{f}) + r\|\hat{f}\|_{L^\infty(\hat{E},\hat{\mu})}^2.$$

We obtain

$$\hat{\alpha}(r) \le \alpha(r). \tag{3.6}$$

(b) On the other hand, for any function f on E, we set

$$g(x) = \begin{cases} \mu_F(f), & x \in F, \\ f(x), & x \in F^c, \end{cases}$$

and

$$\hat{f}(x) = \begin{cases} \mu_F(f), & x = \theta, \\ f(x), & x \in F^c. \end{cases}$$

It is easy to see that $\mu(g) = \hat{\mu}(\hat{f}) = \mu(f)$. Thus, we have

$$\begin{aligned} \mathrm{Var}_\mu(f) &= \mu\big((f - \mu(f))^2\big) \\ &= \mu\big((f - g)^2\big) + \mu\big((g - \mu(f))^2\big) \\ &= \mu(F)\,\mathrm{Var}_{\mu_F}(f) + \mathrm{Var}_{\hat{\mu}}(\hat{f}) \\ &\le \frac{\mu(F)}{\lambda_1(F)} D_F(f,f) + \hat{\alpha}(r)\hat{D}(\hat{f},\hat{f}) + r\|\hat{f}\|^2_{L^\infty(\hat{E},\hat{\mu})} \\ &= \frac{1}{2\lambda_1(F)} \int_{F\times F} \mu(dx)q(x,dy)\big(f(y) - f(x)\big)^2 \\ &\quad + \hat{\alpha}(r)\bigg(\int_{F^c} \mu(dx)q(x,F)\big(f(x) - \mu_F(f)\big)^2 \\ &\quad + \frac{1}{2}\int_{F^c\times F^c} \mu(dx)q(x,dy)\big(f(y) - f(x)\big)^2\bigg) + r\|f\|^2_{L^\infty(E,\mu)}. \end{aligned} \tag{3.7}$$

For each $x \in F^c$ with $q(x,F) > 0$, we set

$$\mu_F^x(dy) = \frac{q(x,dy)}{q(x,F)}.$$

Let $1 \le p,q \le \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ to be determined later. Then we have

$$\begin{aligned} &\int_{F^c} \mu(dx)q(x,F)\big(f(x) - \mu_F(f)\big)^2 \\ &\quad \le \int_{F^c} \mu(dx)q(x,F)\big[p\big(f(x) - \mu_F^x(f)\big)^2 + q\big(\mu_F^x(f) - \mu_F(f)\big)^2\big] \\ &\quad \le \int_{F^c} \mu(dx)q(x,F)\bigg[p\int_F \mu_F^x(dy)\big(f(x) - f(y)\big)^2 + q\int_F \mu_F^x(dy)\big(f(y) - \mu_F(f)\big)^2\bigg] \\ &\quad \le p\int_{F\times F^c} \mu(dx)q(x,dy)\big(f(y) - f(x)\big)^2 \\ &\qquad + \frac{qM_F}{2\lambda_1(F)}\int_{F\times F} \mu(dx)q(x,dy)\big(f(y) - f(x)\big)^2. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we have

$$\begin{aligned}
\mathrm{Var}_\mu(f) \le{}& \frac{1}{2\lambda_1(F)}\int_{F\times F}\mu(dx)q(x,dy)\big(f(y)-f(x)\big)^2 \\
&+\hat{\alpha}(r)\bigg\{\frac{1}{2}\int_{F^c\times F^c}\mu(dx)q(x,dy)\big(f(y)-f(x)\big)^2 \\
&+p\int_{F\times F^c}\mu(dx)q(x,dy)\big(f(y)-f(x)\big)^2 \\
&+\frac{qM_F}{2\lambda_1(F)}\int_{F\times F}\mu(dx)q(x,dy)\big(f(y)-f(x)\big)^2\bigg\}+r\|f\|_\infty^2 \\
\le{}& \max\bigg\{p\hat{\alpha}(r),\frac{1+qM_F\hat{\alpha}(r)}{\lambda_1(F)}\bigg\}D(f,f)+r\|f\|_\infty^2.
\end{aligned}$$

Recall that $p^{-1}+q^{-1}=1$. Let p_0 be the solution to the equation

$$p\hat{\alpha}(r)=\frac{1+\frac{p}{p-1}M_F\hat{\alpha}(r)}{\lambda_1(F)}.$$

Then we have

$$p_0=\frac{(\lambda_1(F)+M_F)\hat{\alpha}(r)+1+\sqrt{((\lambda_1(F)+M_F)\hat{\alpha}(r)+1)^2-4\lambda_1(F)\hat{\alpha}(r)}}{2\lambda_1(F)\hat{\alpha}(r)}.$$

So we obtain $\mathrm{Var}_\mu(f)\le\alpha(r)D(f,f)+r\|f\|_\infty^2, f\in\mathcal{D}(D), r>0$, where

$$\begin{aligned}
\alpha(r)&\le p_0\hat{\alpha}(r) \\
&=\frac{(\lambda_1(F)+M_F)\hat{\alpha}(r)+1+\sqrt{((\lambda_1(F)+M_F)\hat{\alpha}(r)+1)^2-4\lambda_1(F)\hat{\alpha}(r)}}{2\lambda_1(F)} \\
&\le\frac{(\lambda_1(F)+M_F)\hat{\alpha}(r)+1}{\lambda_1(F)}.
\end{aligned}\tag{3.9}$$

Combining (3.6) and (3.9), we have

$$\hat{\alpha}(r)\le\alpha(r)\le\bigg(1+\frac{M_F}{\lambda_1(F)}\bigg)\hat{\alpha}(r)+\frac{1}{\lambda_1(F)}.$$ □

So by (3.4) we get

$$\alpha_F\bigg(\frac{r}{\mu(F)}\bigg)\mu(F)\le\alpha(r)\le\bigg(1+\frac{M_F}{\lambda_1(F)}\bigg)\alpha_F\bigg(\frac{r}{4}\bigg)+\frac{1}{\lambda_1(F)}.\tag{3.10}$$

Recall that by Lemma 2.1 we have $\alpha_F(r)\le\xi^{-1}(\frac{r}{c})$. Then Theorem 1.1 follows from Lemma 2.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this work equally and read and approved the final manuscript.

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