Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Generalized Wilker-type inequalities with two parameters

Hong-Hu Chu[1], Zhen-Hang Yang[2], Yu-Ming Chu[2*] and Wen Zhang[3]

*Correspondence: chuyuming2005@126.com
[2]School of Mathematics and Computation Sciences, Hunan City University, Yiyang, 413000, China
Full list of author information is available at the end of the article

Abstract

In the article, we present certain $p, q \in \mathbb{R}$ such that the Wilker-type inequalities

$$\frac{2q}{p+2q}\left(\frac{\sin x}{x}\right)^p + \frac{p}{p+2q}\left(\frac{\tan x}{x}\right)^q > (<)1 \quad \text{and}$$
$$\left(\frac{\pi}{2}\right)^p\left(\frac{\sin x}{x}\right)^p + \left[1-\left(\frac{\pi}{2}\right)^p\right]\left(\frac{\tan x}{x}\right)^q > (<)1$$

hold for all $x \in (0, \pi/2)$.

MSC: 26D05; 33B10

Keywords: Wilker inequality; trigonometric function; Bernoulli number; monotonicity

1 Introduction

The well-known Wilker inequality $(\sin x/x)^2 + \tan x/x > 2$ for all $x \in (0, \pi/2)$ was proposed by Wilker [1] and proved by Sumner et al. [2].

Recently, the Wilker inequality has attracted the attention of many researchers. Many generalizations, improvements, and refinements of the Wilker inequality can be found in the literature [3–10].

Pinelis [11] and Sun and Zhu [12] proved that the inequalities

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 > \lambda x^3 \tan x \quad \text{and} \quad \left(\frac{y}{\sinh y}\right)^2 + \frac{y}{\tanh y} - 2 < \mu y^3 \sinh y$$

hold for all $x \in (0, \pi/2)$ and $y > 0$ if and only if $\lambda \le 8/45$ and $\mu \ge 2/45$.

Wu and Srivastava [13] provided polynomials $P_1(x)$ and $P_2(x)$ of degree $2n+3$ ($n \in \mathbb{N}$) with explicit expressions and coefficients concerning Bernoulli numbers such that the double inequality

$$P_1(x)\tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 < P_2(x)\tan x$$

holds for all $x \in (0, \pi/2)$.

Yang [14] proved that $p = 5/3$ and $q = \log 2/[2(\log\pi - \log 2)]$ are the best possible parameters such that the double inequality

$$\left(\frac{\sqrt{\cos^{2p} x + 8} + \cos^p x}{4}\right)^{1/p} < \frac{\sin x}{x} < \left(\frac{\sqrt{\cos^{2q} x + 8} + \cos^q x}{4}\right)^{1/q}$$

holds for all $x \in (0, \pi/2)$.

Very recently, Yang and Chu [15] proved that the Wilker-type inequality

$$\frac{2}{k+2}\left(\frac{\sin x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tan x}{x}\right)^{p} > (<)1$$

holds for any fixed $k \geq 1$ and all $x \in (0, \pi/2)$ if and only if $p > 0$ or $p \leq [\log 2 - \log(k + 2)]/[k(\log\pi - \log 2)]$ $(-12/[5(k+2)] \leq p < 0)$, and the hyperbolic version of Wilker-type inequality

$$\frac{2}{k+2}\left(\frac{\sinh x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tanh x}{x}\right)^{p} > (<)1$$

holds for any fixed $k \geq 1$ (< -2) and all $x \in (0, \infty)$ if and only if $p > 0$ or $p \leq -12/[5(k+2)]$ $(p < 0$ or $p \geq -12/[5(k+2)])$.

More results of the Wilker-type inequalities for hyperbolic, Bessel, circular, inverse trigonometric, inverse hyperbolic, lemniscate, generalized trigonometric, generalized hyperbolic, Jacobian elliptic and theta, and hyperbolic Fibonacci functions can be found in the literature [16–28].

The main purpose of the article is to establish the Wilker-type inequalities

$$\frac{2q}{p+2q}\left(\frac{\sin x}{x}\right)^{p} + \frac{p}{p+2q}\left(\frac{\sin x}{x}\right)^{q} > (<)1$$

and

$$\left(\frac{\pi}{2}\right)^{p}\left(\frac{\sin x}{x}\right)^{p} + \left[1 - \left(\frac{\pi}{2}\right)^{p}\right]\left(\frac{\tan x}{x}\right)^{q} > (<)1$$

for all $x \in (0, \pi/2)$ and certain $p, q \in \mathbb{R}$. Some complicated analytical computations are carried out using the computer algebra system Mathematica.

2 Lemmas

In order to prove our main results, we need several lemmas.

Lemma 2.1 (See [29, 30]) *Let $-\infty < a < b < \infty$, $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ on (a, b). Then both of the functions*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \textit{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}$$

are increasing (decreasing) on (a, b) if $f'(x)/g'(x)$ is increasing (decreasing) on (a, b). If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2 (See [31]) *Let $A(t) = \sum_{k=0}^{\infty} a_k t^k$ and $B(t) = \sum_{k=0}^{\infty} b_k t^k$ be two real power series converging on $(-r, r)$ $(r > 0)$ with $b_k > 0$ for all k. If the nonconstant sequence $\{a_k/b_k\}$ is increasing (decreasing) for all k, then the function $t \mapsto A(t)/B(t)$ is strictly increasing (decreasing) on $(0, r)$.*

Lemma 2.3 (See [32]) *Let $n \in \mathbb{N}$, and B_n be the Bernoulli numbers. Then the power series formulas*

$$\frac{1}{\sin x} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!} |B_{2n}| x^{2n-1}, \qquad \cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1},$$
$$\frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{(2n-1)2^{2n}}{(2n)!} |B_{2n}| x^{2n-2},$$
$$\frac{\cos x}{\sin^3 x} = \frac{1}{x^3} - \sum_{n=1}^{\infty} \frac{n(2n+1)2^{2n+2}}{(2n+2)!} |B_{2n+2}| x^{2n-1}$$

hold for $x \in (-\pi, \pi)$, and the power series formulas

$$\tan x = \sum_{n=1}^{\infty} \frac{(2^{2n}-1)2^{2n}}{(2n)!} |B_{2n}| x^{2n-1}, \qquad \frac{1}{\cos^2 x} = \sum_{n=1}^{\infty} \frac{(2n-1)(2^{2n}-1)2^{2n}}{(2n)!} |B_{2n}| x^{2n-2}$$

hold for $x \in (-\pi/2, \pi/2)$.

Lemma 2.4 (See [33]) *Let B_n be the Bernoulli numbers. Then the double inequality*

$$\frac{2(2n)!}{(2\pi)^{2n}} < |B_{2n}| < \frac{2^{2n-1}}{2^{2n-1}-1} \frac{2(2n)!}{(2\pi)^{2n}}$$

holds for all $n \in \mathbb{N}$.

From Lemma 2.4 we immediately get the following:

Remark 2.1 Let B_n be the Bernoulli numbers. Then the double inequality

$$\frac{2^{2n-1}-1}{2^{2n-1}} \frac{(2\pi)^2}{2n(2n-1)} < \frac{|B_{2n-2}|}{|B_{2n}|} < \frac{2^{2n-3}}{2^{2n-3}-1} \frac{(2\pi)^2}{2n(2n-1)}$$

holds for all $n \in \mathbb{N}$ and $n \geq 1$.

Lemma 2.5 *Let $n \in \mathbb{N}$, B_n be the Bernoulli numbers, and a_n and b_n be respectively defined by*

$$a_n = 2^{2n} - 2n^2 - 3n - 2, \tag{2.1}$$
$$b_n = (2n-3)2^{2n} + 2n^2 + n + 4 - n(2n-1)\left(2^{2n-3}-1\right) \frac{|B_{2n-2}|}{|B_{2n}|}. \tag{2.2}$$

Then the sequence $\{b_n/a_n\}$ is strictly increasing for $n \geq 3$.

Proof Let $n \geq 3$ and

$$u_n = \frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n}. \tag{2.3}$$

Then from (2.1)-(2.3) and Remark 2.1 we get

$$\begin{aligned} u_n &= \frac{(2n-1)2^{2n+2} + 2n^2 + 5n + 7}{2^{2n+2} - 2n^2 - 7n - 7} - \frac{(n+1)(2n+1)(2^{2n-1} - 1)}{2^{2n+2} - 2n^2 - 7n - 7} \frac{|B_{2n}|}{|B_{2n+2}|} \\ &\quad - \frac{(2n-3)2^{2n} + 2n^2 + n + 4}{2^{2n} - 2n^2 - 3n - 2} + \frac{n(2n-1)(2^{2n-3} - 1)}{2^{2n} - 2n^2 - 3n - 2} \frac{|B_{2n-2}|}{|B_{2n}|} \\ &> \frac{2}{a_n a_{n+1}} \left[4 \times 2^{4n} - \left(6n^3 + 7n^2 + 5n + 11\right)2^{2n} - \left(2n^2 - 2n - 7\right)\right] \\ &\quad + \frac{\pi^2}{2^{2n+2}} \frac{(6n^2 + 5n - 39)2^{4n} + (20n^2 + 70n + 134)2^{2n} - (32n^2 + 112n + 112)}{a_n a_{n+1}}. \end{aligned} \tag{2.4}$$

Let

$$u_n^* = 4 \times 2^{4n} - \left(6n^3 + 7n^2 + 5n + 11\right)2^{2n} - \left(2n^2 - 2n - 7\right). \tag{2.5}$$

Then we clearly see that

$$u_3^* = 315 > 0, \tag{2.6}$$

$$u_{n+1}^* - 16u_n^* = \left(18n^3 + 3n^2 - 17n + 15\right)2^{2n+2} + \left(30n^2 - 34n - 105\right) > 0 \tag{2.7}$$

for $n \geq 3$.

It follows from (2.6) and (2.7) that

$$u_n^* > 0 \tag{2.8}$$

for all $n \geq 3$.

It is not difficult to verify that

$$a_n > 0 \tag{2.9}$$

and

$$\left(6n^2 + 5n - 39\right)2^{4n} + \left(20n^2 + 70n + 134\right)2^{2n} - \left(32n^2 + 112n + 112\right) > 0 \tag{2.10}$$

for all $n \geq 3$.

Therefore, Lemma 2.5 follows easily from (2.3)-(2.5) and (2.8)-(2.10). □

Lemma 2.6 *Let $n \in \mathbb{N}$, B_n be the Bernoulli numbers, u_n be defined by* (2.3), *and c_n and v_n be respectively defined by*

$$c_n = 2n\left(2^{2n} - 1\right) - 2n(2n-1)\left(2^{2n-3} - 1\right) \frac{|B_{2n-2}|}{|B_{2n}|}, \tag{2.11}$$

$$v_n = \frac{c_{n+1}}{a_{n+1}} - \frac{c_n}{a_n}. \tag{2.12}$$

Then $v_n > u_n$ *for all* $n \geq 3$.

Proof It follows from (2.1)-(2.3), (2.11), and (2.12) that

$$\begin{aligned} v_n - u_n = &-\frac{2[(6n^2+5n-2)2^{2n}+4n+5]}{a_n a_{n+1}} + \frac{n(2n-1)(2^{2n-3}-1)}{2^{2n}-2n^2-3n-2}\frac{|B_{2n-2}|}{|B_{2n}|} \\ &- \frac{(n+1)(2n+1)(2^{2n-1}-1)}{2^{2n+2}-2n^2-7n-7}\frac{|B_{2n}|}{|B_{2n+2}|}. \end{aligned} \tag{2.13}$$

From (2.13), Remark 2.1, and the inequality $\pi^2 > 9$ we get

$$v_3 - u_3 = \frac{8}{105}, \qquad v_4 - u_4 = \frac{104}{3,045}, \qquad v_5 - u_5 = \frac{15,496}{1,102,145}, \tag{2.14}$$

$$v_6 - u_6 = \frac{23,139,208}{4,326,527,205}, \qquad v_7 - u_7 = \frac{2,511,041,224}{1,319,700,084,885}, \tag{2.15}$$

$$\begin{aligned} &v_n - u_n \\ &\quad > -\frac{2[(6n^2+5n-2)2^{2n}+4n+5]}{a_n a_{n+1}} \\ &\qquad + \frac{n(2n-1)(2^{2n-3}-1)}{2^{2n}-2n^2-3n-2}\frac{2^{2n-1}-1}{2^{2n-1}}\frac{(2\pi)^2}{2n(2n-1)} \\ &\qquad - \frac{(n+1)(2n+1)(2^{2n-1}-1)}{2^{2n+2}-2n^2-7n-7}\frac{2^{2n-1}}{2^{2n-1}-1}\frac{(2\pi)^2}{(2n+1)(2n+2)} \\ &\quad = -\frac{2[(6n^2+5n-2)2^{2n}+4n+5]}{a_n a_{n+1}} \\ &\qquad + \frac{\pi^2}{2^{2n+2}}\frac{(6n^2+5n-39)2^{4n}+(20n^2+70n+134)2^{2n}-(32n^2+112n+112)}{a_n a_{n+1}} \\ &\quad > -\frac{2[(6n^2+5n-2)2^{2n}+4n+5]}{a_n a_{n+1}} \\ &\qquad + \frac{81}{2^{2n+2}}\frac{(6n^2+5n-39)2^{4n}+(20n^2+70n+134)2^{2n}-(32n^2+112n+112)}{a_n a_{n+1}} \\ &\quad = \frac{(6n^2+5n-335)2^{4n}+(180n^2+598n+1,166)2^{2n}-9(32n^2+112n+112)}{a_n a_{n+1}2^{2n+2}}. \end{aligned} \tag{2.16}$$

Note that

$$a_n > 0, \quad \big(180n^2+598n+1,166\big)2^{2n} - 9\big(32n^2+112n+112\big) > 0, \tag{2.17}$$

and

$$6n^2+5n-335 \geq 6\times 8^2 + 5\times 8 - 335 = 89 \tag{2.18}$$

for all $n \geq 8$.

Therefore, Lemma 2.6 follows easily from (2.14)-(2.18). □

Lemma 2.7 *Let $n \in \mathbb{N}$, and w_n be defined by*

$$w_n = 32 \times 2^{6n} - \left(48n^3 + 206n^2 + 165n + 2{,}183\right)2^{4n} + \left(3{,}284n^2 + 5{,}526n + 4{,}716\right)2^{2n} - \left(1{,}320n^2 + 1{,}980n + 1{,}320\right).$$

Then $w_n > 0$ for all $n \geq 5$.

Proof Let

$$w_n^* = 32 \times 4^n - \left(48n^3 + 206n^2 + 165n + 2{,}183\right).$$

Then we clearly see that

$$w_n = 2^{4n}w_n^* + \left(3{,}284n^2 + 5{,}526n + 4{,}716\right)2^{2n} - \left(1{,}320n^2 + 1{,}980n + 1{,}320\right), \tag{2.19}$$

$$w_5^* = 18{,}160 > 0, \tag{2.20}$$

$$w_{n+1}^* - 4w_n^* = 144n^3 + 474n^2 - 61n + 6{,}130 > 0 \tag{2.21}$$

for all $n \geq 5$.

Inequalities (2.20) and (2.21) lead to the conclusion that

$$w_n^* > 0 \tag{2.22}$$

for all $n \geq 5$.

Note that

$$\left(3{,}284n^2 + 5{,}526n + 4{,}716\right)2^{2n} - \left(1{,}320n^2 + 1{,}980n + 1{,}320\right) > 0 \tag{2.23}$$

for all $n \geq 5$.

Therefore, Lemma 2.7 follows from (2.19), (2.22), and (2.23). □

Lemma 2.8 *Let $n \in \mathbb{N}$, and u_n and v_n be defined by* (2.3) *and* (2.12), *respectively. Then $v_3 = 37u_3/35$ and $v_n < 37u_n/35$ for all $n \geq 4$.*

Proof It follows from (2.1)-(2.3), (2.11), and (2.12) that

$$\begin{aligned} &35v_n - 37u_n \\ &\quad = -\frac{2[8 \times 2^{4n} - (12n^3 - 196n^2 - 165n + 92)2^{2n} - (4n^2 - 144n - 189)]}{(2^{2n} - 2n^2 - 3n - 2)(2^{2n+2} - 2n^2 - 7n - 7)} \\ &\quad\quad - \frac{33(2n+1)(n+1)(2^{2n-1} - 1)}{2^{2n+2} - 2n^2 - 7n - 7}\frac{|B_{2n}|}{|B_{2n+2}|} + \frac{33n(2n-1)(2^{2n-3} - 1)}{2^{2n} - 2n^2 - 3n - 2}\frac{|B_{2n-2}|}{|B_{2n}|}. \end{aligned} \tag{2.24}$$

From Remark 2.1, (2.24), and the inequality $\pi^2 < 10$ we get

$$35v_3 - 37u_3 = 0, \qquad 35v_4 - 37u_4 = -\frac{288}{145}, \tag{2.25}$$

$$
\begin{aligned}
&35v_n - 37u_n \\
&\quad < -\frac{2[8\times 2^{4n} - (12n^3 - 196n^2 - 165n + 92)2^{2n} - (4n^2 - 144n - 189)]}{(2^{2n} - 2n^2 - 3n - 2)(2^{2n+2} - 2n^2 - 7n - 7)} \\
&\qquad - \frac{33(2n+1)(n+1)(2^{2n-1} - 1)}{2^{2n+2} - 2n^2 - 7n - 7}\frac{2^{2n+1} - 1}{2^{2n+1}}\frac{(2\pi)^2}{(2n+1)(2n+2)} \\
&\qquad + \frac{33n(2n-1)(2^{2n-3} - 1)}{2^{2n} - 2n^2 - 3n - 2}\frac{2^{2n-3}}{2^{2n-3} - 1}\frac{(2\pi)^2}{2n(2n+1)} \\
&\quad = -\frac{2[8\times 2^{4n} - (12n^3 - 196n^2 - 165n + 92)2^{2n} - (4n^2 - 144n - 189)]}{a_n a_{n+1}} \\
&\qquad + \frac{33\pi^2}{4}\frac{(6n^2 + 5n + 11)2^{4n} - 2(10n^2 + 15n + 12)2^{2n} + 8n^2 + 12n + 8}{a_n a_{n+1}2^{2n}} \\
&\quad < -\frac{2[8\times 2^{4n} - (12n^3 - 196n^2 - 165n + 92)2^{2n} - (4n^2 - 144n - 189)]}{a_n a_{n+1}} \\
&\qquad + \frac{33\times 10}{4}\frac{(6n^2 + 5n + 11)2^{4n} - 2(10n^2 + 15n + 12)2^{2n} + 8n^2 + 12n + 8}{a_n a_{n+1}2^{2n}} \\
&\quad = -\frac{w_n}{a_n a_{n+1}2^{2n+1}},
\end{aligned}
\tag{2.26}
$$

where w_n is given in Lemma 2.7.

Therefore, Lemma 2.8 follows easily from Lemma 2.7, (2.25), and (2.26). □

Let

$$A(x) = (x - \sin x\cos x)(\sin x - x\cos x)^2\cos x, \tag{2.27}$$

$$B(x) = (\sin x - x\cos x)(x - \sin x\cos x)^2, \tag{2.28}$$

$$
\begin{aligned}
C(x) &= x\bigl(x\sin x - 2x^2\cos x + \sin^2 x\cos x\bigr)\sin^2 x \\
&= x^3\sin^2 x\cos x\left(\frac{\sin^2 x}{x^2} + \frac{\tan x}{x} - 2\right).
\end{aligned}
\tag{2.29}
$$

Then from the Wilker inequality and Lemma 2.3 we clearly see that

$$A(x) > 0, \qquad B(x) > 0, \qquad C(x) > 0$$

for all $x \in (0, \pi/2)$ and

$$\frac{A(x)}{\sin^3 x\cos^2 x} = \sum_{n=3}^{\infty}\frac{2^{2n}|B_{2n}|a_n}{(2n)!}x^{2n}, \qquad \frac{B(x)}{\sin^3 x\cos^2 x} = \sum_{n=3}^{\infty}\frac{2^{2n}|B_{2n}|b_n}{(2n)!}x^{2n}, \tag{2.30}$$

$$\frac{C(x)}{\sin^3 x\cos^2 x} = \sum_{n=3}^{\infty}\frac{2^{2n}|B_{2n}|c_n}{(2n)!}x^{2n}, \tag{2.31}$$

where a_n, b_n, and c_n are respectively given by (2.1), (2.2), and (2.11).

Lemma 2.9 *Let $q \in \mathbb{R}$, $A(x)$, $B(x)$, and $C(x)$ be respectively given by* (2.27)-(2.29), *and $f(x)$: $(0, \pi/2) \to \mathbb{R}$ be defined as*

$$f(x) = \frac{qB(x) + C(x)}{A(x)}. \tag{2.32}$$

Then the following statements are true:

(1) *if* $q = -1$, *then* $f(x)$ *is strictly increasing from* $(0, \pi/2)$ *onto* $(2q + 12/5, 3 - \pi^2/4)$;

(2) *if* $q > -1$, *then* $f(x)$ *is strictly increasing from* $(0, \pi/2)$ *onto* $(2q + 12/5, \infty)$;

(3) *if* $q \leq -37/35$, *then* $f(x)$ *is strictly decreasing from* $(0, \pi/2)$ *onto* $(-\infty, 2q + 12/5)$.

Proof Let a_n, b_n, c_n, u_n, and v_n be respectively defined by (2.1)-(2.3), (2.11), and (2.12). Then from (2.30)-(2.32) and Lemma 2.5 we have

$$f(x) = \frac{\sum_{n=3}^{\infty}(qb_n + c_n)x^{2n}}{\sum_{n=3}^{\infty} a_n x^{2n}}, \tag{2.33}$$

$$\frac{qb_{n+1} + c_{n+1}}{a_{n+1}} - \frac{qb_n + c_n}{a_n} = qu_n + v_n, \tag{2.34}$$

$$u_n = \frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n} > 0 \tag{2.35}$$

for all $n \geq 3$.

Note that

$$f\left(0^+\right) = \frac{qb_3 + c_3}{a_3} = 2q + \frac{12}{5}, \tag{2.36}$$

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} \frac{C(x) - B(x)}{A(x)} = 3 - \frac{\pi^2}{4} \quad (q = -1), \tag{2.37}$$

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} \frac{qB(x) + C(x)}{A(x)} = +\infty \quad (q > -1), \tag{2.38}$$

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} \frac{qB(x) + C(x)}{A(x)} = -\infty \quad (q < -1). \tag{2.39}$$

We divide the proof into two cases.

Case 1 $q \geq -1$. Then it follows from (2.34) and (2.35), together with Lemma 2.6, that

$$\frac{qb_{n+1} + c_{n+1}}{a_{n+1}} - \frac{qb_n + c_n}{a_n} \geq v_n - u_n > 0 \tag{2.40}$$

for $n \geq 3$.

Therefore, parts (1) and (2) follow from (2.33), (2.36)-(2.38), (2.40), and Lemma 2.2.

Case 2 $q \leq -37/35$. Then (2.34) and (2.35), together with Lemma 2.8, lead to

$$\frac{qb_4 + c_4}{a_4} - \frac{qb_3 + c_3}{a_3} \leq v_3 - \frac{37}{35}u_3 = 0, \tag{2.41}$$

$$\frac{qb_{n+1} + c_{n+1}}{a_{n+1}} - \frac{qb_n + c_n}{a_n} \leq v_n - \frac{37}{35}u_n < 0 \tag{2.42}$$

for $n \geq 4$.

Therefore, part (3) follows from (2.33), (2.36), (2.39), (2.41), (2.42), and Lemma 2.2. □

Let $p, q \in \mathbb{R}$, $x \in (0, \pi/2)$, and the functions $x \to S_p(x)$, $x \to T_q(x)$, and $x \to W_{p,q}(x)$ be respectively defined by

$$S_p(x) = \frac{1 - (\frac{\sin x}{x})^p}{p} \quad (p \neq 0), \qquad S_0(x) = \lim_{p \to 0} S_p(x) = \log \frac{x}{\sin x}, \tag{2.43}$$

$$T_q(x)=\frac{(\frac{\tan x}{x})^q-1}{q}\quad (q\neq 0),\qquad T_0(x)=\lim_{q\to 0}T_q(x)=\log\frac{\tan x}{x}, \tag{2.44}$$

and

$$W_{p,q}(x)=\frac{S_p(x)}{T_q(x)}.$$

Then we clearly see that

$$S_p(0^+)=T_q(0^+)=0,$$

$$W_{p,q}(x)=\frac{S_p(x)}{T_q(x)}=\frac{S_p(x)-S_p(0^+)}{T_q(x)-T_q(0^+)}=\begin{cases}\frac{q}{p}\frac{1-(\frac{\sin x}{x})^p}{(\frac{\tan x}{x})^q-1}, & pq\neq 0,\\ \frac{1}{p}\frac{1-(\frac{\sin x}{x})^p}{\log\frac{\tan x}{x}}, & p\neq 0, q=0,\\ q\frac{\log\frac{x}{\sin x}}{(\frac{\tan x}{x})^q-1}, & p=0, q\neq 0,\\ \frac{\log(\frac{x}{\sin x})}{\log(\frac{\tan x}{x})}, & p=q=0,\end{cases} \tag{2.45}$$

$$W_{p,q}(0^+)=\frac{1}{2}, \tag{2.46}$$

$$W_{p,q}\left(\frac{\pi}{2}^-\right)=\frac{q}{p}\left[\left(\frac{2}{\pi}\right)^p-1\right]\quad (p\neq 0, q<0), \tag{2.47}$$

$$W_{0,q}\left(\frac{\pi}{2}^-\right)=\lim_{p\to 0}W_{p,q}\left(\frac{\pi}{2}^-\right)=q\log\frac{2}{\pi}\quad (q<0).$$

Lemma 2.10 *Let $x\in(0,\pi/2)$, and $W_{p,q}(x)$ be defined by* (2.45). *Then the following statements are true*:

(1) *$W_{p,q}(x)$ is strictly decreasing on $(0,\pi/2)$ if $q\geq -1$ and $p+2q+12/5\geq 0$;*

(2) *$W_{p,q}(x)$ is strictly increasing on $(0,\pi/2)$ if $-37/35<q\leq -1$ and $p\leq \pi^2/4-3$;*

(3) *$W_{p,q}(x)$ is strictly increasing on $(0,\pi/2)$ if $q\leq -37/35$ and $p+2q+12/5\leq 0$.*

Proof Let $pq\neq 0$ and $x\in(0,\pi/2)$. Then (2.43) and (2.44) lead to

$$\begin{aligned}\left[\frac{S_p'(x)}{T_q'(x)}\right]'&=\left[\frac{\sin x-x\cos x}{x-\sin x\cos x}\left(\frac{\sin x}{x}\right)^{p-q}\cos^{q+1}x\right]'\\&=-\frac{x^{q-p-1}\sin^{p-q-1}x\cos^q x}{(x-\sin x\cos x)^2}A(x)\big[f(x)+p\big],\end{aligned} \tag{2.48}$$

where $A(x)$ and $f(x)$ are respectively given by (2.27) and (2.32).

(1) If $q\geq -1$ and $p+2q+12/5\geq 0$, then from Lemma 2.9(1) and (2) and from (2.48) we have

$$\left[\frac{S_p'(x)}{T_q'(x)}\right]'<-\frac{x^{q-p-1}\sin^{p-q-1}x\cos^q x}{(x-\sin x\cos x)^2}A(x)\left(p+2q+\frac{12}{5}\right)\leq 0 \tag{2.49}$$

for $x\in(0,\pi/2)$.

Therefore, Lemma 2.10(1) follows easily from (2.45) and (2.49) together with Lemma 2.1.

(2) If $-37/35 < q \leq -1$ and $p \leq \pi^2/4 - 3$, then (2.48) and Lemma 2.9(1) lead to

$$\left[\frac{S_p'(x)}{T_q'(x)}\right]' \geq -\frac{x^{q-p-1}\sin^{p-q-1}x\cos^q x}{(x-\sin x\cos x)^2}A(x)\left[p+\frac{C(x)-B(x)}{A(x)}\right] > -\frac{x^{q-p-1}\sin^{p-q-1}x\cos^q x}{(x-\sin x\cos x)^2}A(x)\left(p+3-\frac{\pi^2}{4}\right) \geq 0 \tag{2.50}$$

for $x \in (0, \pi/2)$.

Therefore, Lemma 2.10(2) follows from (2.45) and (2.50) together with Lemma 2.1.

(3) If $q \leq -37/35$ and $p + 2q + 12/5 \leq 0$, then Lemma 2.9(3) and (2.48) lead to the conclusion that

$$\left[\frac{S_p'(x)}{T_q'(x)}\right]' > -\frac{x^{q-p-1}\sin^{p-q-1}x\cos^q x}{(x-\sin x\cos x)^2}A(x)\left(p+2q+\frac{12}{5}\right) \geq 0 \tag{2.51}$$

for $x \in (0, \pi/2)$.

Therefore, Lemma 2.10(3) follows from (2.45) and (2.51) together with Lemma 2.1. □

Remark 2.2 It is not difficult to verify that (2.48) is also true if $pq = 0$.

3 Main results

Let

$$E_1 = \left\{(p,q)\middle| q \geq -1, p + 2q + \frac{12}{5} \geq 0\right\}, \tag{3.1}$$

$$E_2 = \left\{(p,q)\middle| -\frac{37}{35} < q \leq -1, p \leq \frac{\pi^2}{4} - 3\right\}, \tag{3.2}$$

$$E_3 = \left\{(p,q)\middle| q \leq -\frac{37}{35}, p + 2q + \frac{12}{5} \leq 0\right\}, \tag{3.3}$$

$$D_1 = \{(p,q)|pq(p+2q) > 0\}, \qquad D_2 = \{(p,q)|pq(p+2q) < 0\}, \tag{3.4}$$

$$D_3 = \{(p,q)|p > 0, q < 0\}, \qquad D_4 = \{(p,q)|p < 0, q < 0\}, \tag{3.5}$$

$$G_1 = E_1 \cap D_1, \qquad G_2 = E_2 \cup E_3 \cap D_2, \tag{3.6}$$

$$G_3 = E_1 \cap D_2, \qquad G_4 = E_2 \cup E_3 \cap D_1, \tag{3.7}$$

$$G_5 = E_1 \cap D_3, \qquad G_6 = E_2 \cup E_3 \cap D_4, \tag{3.8}$$

$$G_7 = E_1 \cap D_4, \qquad G_8 = E_2 \cup E_3 \cap D_3. \tag{3.9}$$

Then (3.1)-(3.9) lead to

$$G_1 = \{(p,q)|p > 0, q > 0\} \cup \{(p,q)|0 < p < -2q, q \geq -1\} \cup \left\{(p,q)\middle| q > 0, -\frac{12}{5} \leq p + 2q < 0\right\}, \tag{3.10}$$

$$G_2 = G_6 = \left\{(p,q)\middle| p \leq \frac{\pi^2}{4} - 3, q \leq -1\right\} \cup \left\{(p,q)\middle| \frac{\pi^2}{4} - 3 < p < 0, q \leq -\frac{37}{35}, p + 2q + \frac{12}{5} \leq 0\right\}, \tag{3.11}$$

$$G_3 = \{(p,q)|p < 0, p + 2q > 0\} \cup \{(p,q)| -1 \le q < 0, p + 2q > 0\} \cup \left\{(p,q)| -1 \le q < 0, -2q - \frac{12}{5} \le p < 0\right\}, \tag{3.12}$$

$$G_4 = G_8 = \left\{(p,q)\middle| 0 < p \le -2q - \frac{12}{5}\right\}, \tag{3.13}$$

$$G_5 = \{(p,q)|p > 0, -1 \le q < 0\}, \tag{3.14}$$

$$G_7 = \left\{(p,q)\middle| -1 \le q < 0, -2q - \frac{12}{5} \le p < 0\right\}. \tag{3.15}$$

Theorem 3.1 *Let G_1, G_2, G_3, and G_4 be respectively defined by* (3.10)-(3.13). *Then the Wilker-type inequality*

$$\frac{2q}{p+2q}\left(\frac{\sin x}{x}\right)^p + \frac{p}{p+2q}\left(\frac{\tan x}{x}\right)^q > 1 \tag{3.16}$$

holds for all $x \in (0, \pi/2)$ if $(p,q) \in G_1 \cup G_2$, and inequality (3.16) *is reversed if $(p,q) \in G_3 \cup G_4$.*

Proof Let $W_{p,q}(x)$ be defined by (2.45). We only prove that inequality (3.16) holds for all $x \in (0, \pi/2)$ if $(p,q) \in G_1 \cup G_2$; the reversed inequality for $(p,q) \in G_3 \cup G_4$ can be proved by a completely similar method.

We divide the proof into two cases.

Case 1 $(p,q) \in G_1$. Then (3.1), (3.4), and (3.6) lead to

$$q \ge -1, \quad p + 2q + \frac{12}{5} \ge 0, \tag{3.17}$$

$$pq(p+2q) > 0. \tag{3.18}$$

It follows from (2.45), (2.46), Lemma 2.10(1), and (3.17) that

$$w_{p,q}(x) = \frac{q}{p}\frac{1 - (\frac{\sin x}{x})^p}{(\frac{\tan x}{x})^q - 1} < \frac{1}{2} \tag{3.19}$$

for $x \in (0, \pi/2)$.

Therefore, inequality (3.16) follows easily from (3.18) and (3.19).

Case 2 $(p,q) \in G_2$. Then from (2.45), (2.46), Lemma 2.10(2) and (3), (3.2)-(3.4), and (3.6) we clearly see that

$$w_{p,q}(x) = \frac{q}{p}\frac{1 - (\frac{\sin x}{x})^p}{(\frac{\tan x}{x})^q - 1} > \frac{1}{2} \tag{3.20}$$

and

$$pq(p+2q) < 0. \tag{3.21}$$

Therefore, inequality (3.16) follows from (3.20) and (3.21). □

Theorem 3.2 *Let G_5, G_6, G_7, and G_8 be respectively defined by* (3.11) *and* (3.13)-(3.15). *Then the Wilker-type inequality*

$$\left(\frac{\pi}{2}\right)^p\left(\frac{\sin x}{x}\right)^p+\left[1-\left(\frac{\pi}{2}\right)^p\right]\left(\frac{\tan x}{x}\right)^q<1 \tag{3.22}$$

holds for all $x \in (0,\pi/2)$ if $(p,q) \in G_5 \cup G_6$, and inequality (3.22) *is reversed if $(p,q) \in G_7 \cup G_8$.*

Proof Let $W_{p,q}(x)$ be defined by (2.45). We only prove that inequality (3.22) holds for all $x \in (0,\pi/2)$ if $(p,q) \in G_5 \cup G_6$; the reversed inequality for $(p,q) \in G_7 \cup G_8$ can be proved by a completely similar method.

We divide the proof into two cases.

Case 1 $(p,q) \in G_5$. Then from (2.45), (2.47), Lemma 2.10(1), (3.1), (3.5), and (3.8) we clearly see that

$$w_{p,q}(x)=\frac{q}{p}\frac{1-(\frac{\sin x}{x})^p}{(\frac{\tan x}{x})^q-1}>\frac{q}{p}\left[\left(\frac{2}{\pi}\right)^p-1\right] \tag{3.23}$$

and

$$p>0. \tag{3.24}$$

Therefore, inequality (3.22) follows easily from (3.23) and (3.24).

Case 2 $(p,q) \in G_6$. Then (2.45), (2.47), Lemma 2.10(2) and (3), (3.2), (3.3), (3.5), and (3.8) lead to the conclusion that

$$w_{p,q}(x)=\frac{q}{p}\frac{1-(\frac{\sin x}{x})^p}{(\frac{\tan x}{x})^q-1}<\frac{q}{p}\left[\left(\frac{2}{\pi}\right)^p-1\right] \tag{3.25}$$

and

$$p<0. \tag{3.26}$$

Therefore, inequality (3.22) follows easily from (3.25) and (3.26). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]School of Civil Engineering and Architecture, Changsha University of Sciences & Technology, Changsha, 410114, China. [2]School of Mathematics and Computation Sciences, Hunan City University, Yiyang, 413000, China. [3]Albert Einstein College of Medicine, Yeshiva University, New York, NY 10033, United States.

Acknowledgements
The research was supported by the Natural Science Foundation of China under Grants 11371125, 61374086, and 11401191 and the Natural Science Foundation of Zhejiang Province under Grant LY13A010004.

Received: 26 April 2016 Accepted: 9 July 2016 Published online: 28 July 2016

References

1. Wilker, JB: Problem E3306. Am. Math. Mon. **96**(1), 55 (1989)
2. Sumner, JS, Jagers, AA, Vowe, M, Anglesio, J: Inequalities involving trigonometric functions. Am. Math. Mon. **98**(3), 264-267 (1991)
3. Wu, S-H, Srivastava, HM: A weighted and exponential generalization of Wilker's inequality and its applications. Integral Transforms Spec. Funct. **18**(7-8), 529-535 (2007)
4. Wu, S-H: On extension and refinement of Wilker's inequality. Rocky Mt. J. Math. **39**(2), 683-687 (2009)
5. Neuman, E, Sándor, J: On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities. Math. Inequal. Appl. **13**(4), 715-723 (2010)
6. Neuman, E: On Wilker and Huygens type inequalities. Math. Inequal. Appl. **15**(2), 271-279 (2012)
7. Sándor, J: The Huygens and Wilker-type inequalities as inequalities for means of two arguments. Adv. Stud. Contemp. Math. **22**(4), 487-498 (2012)
8. Chen, C-P, Sándor, J: Inequality chains for Wilker, Huygens and Lazarević type inequalities. J. Math. Inequal. **8**(1), 55-67 (2014)
9. Wu, S-H, Yue, H-P, Deng, Y-P, Chu, Y-M: Several improvements of Mitrinović-Adamović and Lazarević's inequalities with applications to the sharpening of Wilker-type inequalities. J. Nonlinear Sci. Appl. **9**(4), 1755-1765 (2016)
10. Wu, S-H, Li, S-G, Bencze, M: Sharpened versions of Mitrinović-Adamović, Lazarević and Wilker's inequalities for trigonometric and hyperbolic functions. J. Nonlinear Sci. Appl. **9**(5), 2688-2696 (2016)
11. Pinelis, I: L'Hospital rules for monotonicity and the Wilker-Anglesio inequality. Am. Math. Mon. **111**(10), 905-999 (2004)
12. Sun, Z-J, Zhu, L: On Wilker-type inequalities. ISRN Math. Anal. **2011**, Article ID 681702 (2011)
13. Wu, S-H, Srivastava, HM: A further refinement of Wilker's inequality. Integral Transforms Spec. Funct. **19**(9-10), 757-765 (2008)
14. Yang, Z-H: The sharp inequalities related to Wilker type. Math. Inequal. Appl. **17**(3), 1015-1026 (2014)
15. Yang, Z-H, Chu, Y-M: Sharp Wilker-type inequalities with applications. J. Inequal. Appl. **2014**, Article ID 166 (2014)
16. Zhu, L: On Wilker-type inequalities. Math. Inequal. Appl. **10**(4), 727-731 (2007)
17. Baricz, Á, Sándor, J: Extensions of the generalized Wilker inequality to Bessel functions. J. Math. Inequal. **2**(3), 397-406 (2008)
18. Zhu, L: Some new Wilker-type inequalities for circular and hyperbolic functions. Abstr. Appl. Anal. **2009**, Article ID 485842 (2009)
19. Sándor, J: On some Wilker and Huygens type trigonometric-hyperbolic inequalities. Proc. Jangjeon Math. Soc. **15**(2), 145-153 (2012)
20. Chen, C-P, Cheung, W-S: Wilker- and Huygens-type inequalities and solution to Oppenheim's problem. Integral Transforms Spec. Funct. **23**(5), 325-336 (2012)
21. Chen, C-P: Wilker and Huygens type inequalities for the lemniscate functions. J. Math. Inequal. **6**(4), 673-684 (2012)
22. Wu, S-H, Debnath, L: Wilker-type inequalities for hyperbolic functions. Appl. Math. Lett. **25**(5), 837-842 (2012)
23. Chen, C-P: Wilker and Huygens type inequalities for the lemniscate functions II. Math. Inequal. Appl. **16**(2), 577-586 (2013)
24. Chen, C-P: Sharp Wilker- and Huygens-type inequalities for inverse trigonometric and inverse hyperbolic functions. Integral Transforms Spec. Funct. **23**(12), 865-873 (2012)
25. Neuman, E: Wilker and Huygens-type inequalities for generalized trigonometric and for generalized hyperbolic functions. Appl. Math. Comput. **230**, 211-217 (2014)
26. Neuman, E: Wilker- and Huygens-type inequalities for Jacobian elliptic and theta functions. Integral Transforms Spec. Funct. **25**(3), 240-248 (2014)
27. Mortici, C: A subtly analysis of Wilker inequality. Appl. Math. Comput. **231**, 516-520 (2014)
28. Bahşi, M: Wilker-type inequalities for hyperbolic Fibonacci functions. J. Inequal. Appl. **2016**, Article ID 146 (2016)
29. Anderson, GD, Vamanamurthy, MK, Vuorinen, M: Conformal Invariants, Inequalities, and Quasiconformal Maps. Wiley, New York (1997)
30. Anderson, GD, Qiu, S-L, Vamanamurthy, MK, Vuorinen, M: Generalized elliptic integrals and modular equations. Pac. J. Math. **192**(1), 1-37 (2000)
31. Biernacki, M, Krzyż, J: On the monotonity of certain functionals in the theory of analytic functions. Ann. Univ. Mariae Curie-Skłodowska, Sect. A **9**(1), 135-147 (1955)
32. Editorial Committee of Mathematics Handbook: Mathematics Handbook. Peoples' Education Press, Beijing (1979) (in Chinese)
33. Abramowitz, M, Stegun, IA: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. U.S. Government Printing Office, Washington (1964)