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Some Brunn-Minkowski type inequalities for L_p radial Blaschke-Minkowski homomorphisms

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Abstract
Schuster introduced radial Blaschke-Minkowski homomorphisms. Recently, they were generalized to L_p radial Blaschke-Minkowski homomorphisms by Wang et al. In this paper, we first establish Brunn-Minkowski type inequalities for some L_q radial sums of L_p radial Blaschke-Minkowski homomorphisms. Further, we consider monotonic inequalities for L_p radial Blaschke-Minkowski homomorphisms.

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1 Introduction

The setting for this paper is the Euclidean n-space $\mathbb{R}^n$. Let $\mathcal{S}_o^n$ denote the set of star bodies in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$. For the n-dimensional volume of body K, we write $V(K)$.

Intersection bodies first appeared in a paper by Busemann [1] and were explicitly defined and named by Lutwak in the important paper [2]. Intersection bodies have been becoming the central notion in the dual Brunn Minkowski theory (see, e.g., [2–16]). In 2006, Ludwig [14] characterized the intersection body operator, which is the only nontrivial $GL(n)$ contravariant radial valuation. Whereafter, Schuster [17] introduced radial Blaschke-Minkowski homomorphisms, which are more general intersection body operators:

Definition 1.1 A map $\Psi : \mathcal{S}_o^n \to \mathcal{S}_o^n$ is called a radial Blaschke-Minkowski homomorphism if it satisfies the following conditions:

(1) Ψ is continuous;
(2) for all $K, L \in \mathcal{S}_o^n$, $\Psi(K \widetilde{+}_{n-1} L) = \Psi K \widetilde{+} \Psi L$, that is, ΨK is a radial Blaschke-Minkowski sum, where $\widetilde{+}_{n-1}$ and $\widetilde{+}$ denote L_{n-1} and L_1 radial Minkowski addition, respectively;
(3) Ψ intertwines rotations, that is, $\Psi(\vartheta K) = \vartheta \Psi K$ for all $K \in \mathcal{S}_o^n$ and all $\vartheta \in SO(n)$.

Further, Schuster [17] showed that radial Blaschke-Minkowski homomorphisms satisfy the geometric inequalities of the Aleksandrov-Fenchel, Minkowski, and Brunn-

Minkowski types and established the following Brunn-Minkowski inequality for radial Blaschke-Minkowski homomorphisms of star bodies.

Theorem 1.A *If $K, L \in \mathcal{S}_o^n$, then*

$$V\big(\Psi(K \tilde{+} L)\big)^{\frac{1}{n(n-1)}} \leq V(\Psi K)^{\frac{1}{n(n-1)}} + V(\Psi L)^{\frac{1}{n(n-1)}}$$

with equality if and only if K and L are dilates.

In recent years, many inequalities for the radial Blaschke-Minkowski homomorphisms were established (see, e.g., [18–27]). Later, by associating the L_q harmonic radial sum with the L_q radial Blaschke sum of star bodies Wei et al. [23] gave the following Brunn-Minkowski type inequalities for radial Blaschke-Minkowski homomorphisms.

Theorem 1.B *If $K, L \in \mathcal{S}_o^n$ and real $q \geq 1$, then*

$$V\big(\Psi(K \check{+}_q L)\big)^{-\frac{q}{n(n-1)}} \geq V(\Psi K)^{-\frac{q}{n(n-1)}} + V(\Psi L)^{-\frac{q}{n(n-1)}}$$

with equality if and only if K and L are dilates, where $\check{+}_q$ is the L_q harmonic radial sum.

Theorem 1.C *If $K, L \in \mathcal{S}_o^n$ and real $n > q \geq 1$, then*

$$V\big(\Psi(K \hat{+}_q L)\big)^{\frac{n-q}{n(n-1)}} \leq V(\Psi K)^{\frac{n-q}{n(n-1)}} + V(\Psi L)^{\frac{n-q}{n(n-1)}}$$

with equality if and only if K and L are dilates, where $\hat{+}_q$ is the L_q radial Blaschke sum.

In 2011, Wang et al. [28] introduced the concept of an L_p radial Blaschke-Minkowski homomorphism.

Definition 1.2 Let K, L be star bodies, $p \in \mathbb{R}$, $p \neq 0$. A map $\Psi_p : \mathcal{S}_o^n \to \mathcal{S}_o^n$ is called an L_p radial Blaschke-Minkowski homomorphism if it satisfies the following conditions:

(1) Ψ_p is continuous with respect to radial metric;
(2) For all $K, L \in \mathcal{S}_o^n$, $\Psi_p(K \tilde{+}_{n-p} L) = \Psi_p K \tilde{+}_p \Psi_p L$, that is, $\Psi_p K$ is an L_p radial Blaschke-Minkowski sum, where $\tilde{+}_q$ denotes L_q radial Minkowski addition;
(3) Ψ_p is $SO(n)$ equivariant, that is, $\Psi_p(\vartheta K) = \vartheta \Psi_p K$ for all $K \in \mathcal{S}_o^n$ and all $\vartheta \in SO(n)$.

Meanwhile, they [28] studied the Busemann-Petty type problem for L_p radial Blaschke-Minkowski homomorphisms. These results are generalized to a large class of L_p radial valuations.

The main goal of this paper is to establish Brunn-Minkowski type inequalities for the L_q radial Minkowski sum, L_q harmonic radial sum, L_q radial Blaschke sum, and L_q harmonic Blaschke sum of L_p radial Blaschke-Minkowski homomorphisms. First, we obtain the following Brunn-Minkowski type inequality for an L_q radial Minkowski sum.

Theorem 1.1 *Let $K, L \in \mathcal{S}_o^n$ and $p, q \in \mathbb{R}$, $p, q \neq 0$.*

(i) *If $p > 0$ and $0 < q < n - p$, then*

$$V\big(\Psi_p(K\widetilde{+}_qL)\big)^{\frac{pq}{n(n-p)}} \leq V(\Psi_pK)^{\frac{pq}{n(n-p)}} + V(\Psi_pL)^{\frac{pq}{n(n-p)}}; \tag{1.1}$$

(ii) *If $q > n - p > 0 > p$ or $q < n - p < 0$ or $q < 0 < n - p$ and $p > 0$, then*

$$V\big(\Psi_p(K\widetilde{+}_qL)\big)^{\frac{pq}{n(n-p)}} \geq V(\Psi_pK)^{\frac{pq}{n(n-p)}} + V(\Psi_pL)^{\frac{pq}{n(n-p)}}. \tag{1.2}$$

Equality holds in each inequality if and only if K and L are dilates.

Taking $p = q = 1$ in Theorem 1.1, by (1.1) we obtain Theorem 1.A. As applications of Theorem 1.1, in Section 3, we give Brunn-Minkowski type inequalities for the L_q harmonic radial sum and L_q radial Blaschke sum of L_p radial Blaschke-Minkowski homomorphisms, that is, Theorem 3.1 and Theorem 3.2. Taking $p = 1$ in Theorems 3.1 and 3.2, we easily get Theorems 1.B and 1.C, respectively.

Further, a Brunn-Minkowski type inequality for the L_q harmonic Blaschke sum of L_p radial Blaschke-Minkowski homomorphisms can be given as follows.

Theorem 1.2 *Let $K, L \in \mathcal{S}_o^n$, $p, q \in \mathbb{R}$, $p \neq 0$, $q \neq -n$.*

(i) *If $0 < p < -q < n$, then*

$$\frac{V(\Psi_p(K \mp_q L))^{\frac{p(n+q)}{n(n-p)}}}{V(K \mp_q L)} \leq \frac{V(\Psi_pK)^{\frac{p(n+q)}{n(n-p)}}}{V(K)} + \frac{V(\Psi_pL)^{\frac{p(n+q)}{n(n-p)}}}{V(L)}; \tag{1.3}$$

(ii) *If $-q < p < 0$, then*

$$\frac{V(\Psi_p(K \mp_q L))^{\frac{p(n+q)}{n(n-p)}}}{V(K \mp_q L)} \geq \frac{V(\Psi_pK)^{\frac{p(n+q)}{n(n-p)}}}{V(K)} + \frac{V(\Psi_pL)^{\frac{p(n+q)}{n(n-p)}}}{V(L)}. \tag{1.4}$$

Equality holds in each inequality if and only if K and L are dilates.

Here $K \mp_q L$ denotes the L_q harmonic Blaschke sum of K and L.

In 2006, Haberl and Ludwig [29] defined the L_p-intersection bodies as follows: For $K \in \mathcal{S}_o^n$, real $p < 1$, $p \neq 0$, the L_p-intersection body I_pK of K is the origin-symmetric star body whose radial function is defined by

$$\rho_{I_pK}^p(u) = \int_K |u \cdot x|^{-p}\,dx = \frac{1}{n-p}\int_{S^{n-1}} |u \cdot v|^{-p}\rho_K^{n-p}(v)\,dS(v) \tag{1.5}$$

for all $u \in \mathcal{S}^{n-1}$. For the studies of L_p-intersection bodies, also see [30–35].

According to Definition 1.2 and (1.5), we easily see that the L_p-intersection body operator I_p is a particular L_p radial Blaschke-Minkowski homomorphism. So from Theorems 1.1 and 1.2 we have the following results.

Corollary 1.1 *For $K, L \in \mathcal{S}_o^n$, $p, q \in \mathbb{R}$, $q \neq 0$, $p < 1$, and $p \neq 0$, we have:*

(i) *If $p > 0$ and $0 < q < n - p$, then*

$$V\big(I_p(K\widetilde{+}_qL)\big)^{\frac{pq}{n(n-p)}} \leq V(I_pK)^{\frac{pq}{n(n-p)}} + V(I_pL)^{\frac{pq}{n(n-p)}};$$

(ii) *If $p < 0$ and $q > n - p$, or $p > 0$ and $q < 0$, then*

$$V\big(I_p(K\widetilde{+}_q L)\big)^{\frac{pq}{n(n-p)}} \geq V(I_pK)^{\frac{pq}{n(n-p)}} + V(I_pL)^{\frac{pq}{n(n-p)}}.$$

Equality holds in each inequality if and only if K and L are dilates.

Corollary 1.2 *For $K, L \in \mathcal{S}_o^n$, $p, q \in \mathbb{R}$, $q \neq -n$, $p < 1$, $p \neq 0$, we have*:

(i) *If $0 < p < -q$, then*

$$\frac{V(I_p(K \mp_q L))^{\frac{p(n+q)}{n(n-p)}}}{V(K \mp_q L)} \leq \frac{V(I_pK)^{\frac{p(n+q)}{n(n-p)}}}{V(K)} + \frac{V(I_pL)^{\frac{p(n+q)}{n(n-p)}}}{V(L)};$$

(ii) *If $0 > p > -q$, then*

$$\frac{V(I_p(K \mp_q L))^{\frac{p(n+q)}{n(n-p)}}}{V(K \mp_q L)} \geq \frac{V(I_pK)^{\frac{p(n+q)}{n(n-p)}}}{V(K)} + \frac{V(I_pL)^{\frac{p(n+q)}{n(n-p)}}}{V(L)}.$$

Equality holds in each inequality if and only if K and L are dilates.

The proofs of Theorems 1.1 and 1.2 are completed in Section 3. Besides, in Section 4, we establish two monotonic inequalities for L_p radial Blaschke-Minkowski homomorphisms.

2 Background materials

If K is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, then its radial function $\rho_K = \rho(K,\cdot) : R^n\backslash\{0\} \to [0,\infty)$ is defined as (see [4])

$$\rho(K,u) = \max\{\lambda \geq 0 : \lambda u \in K\}$$

for all $u \in S^{n-1}$. If $\rho(K,\cdot)$ is positive and continuous, K is called a star body.

2.1 L_p radial Minkowski combination and L_p dual mixed volume

For $K, L \in \mathcal{S}_o^n$, real $p \neq 0$, and $\lambda, \mu \geq 0$ (not both 0), the L_p radial Minkowski combination $\lambda \cdot K\widetilde{+}_p\mu \cdot L \in \mathcal{S}_o^n$ of K and L is defined by (see [30])

$$\rho(\lambda \cdot K\widetilde{+}_p\mu \cdot L, \cdot)^p = \lambda\rho(K,\cdot)^p + \mu\rho(L,\cdot)^p. \tag{2.1}$$

If $p = 1$ in (2.1), then $\lambda \cdot K\widetilde{+}\mu \cdot L$ is called the radial Minkowski combination of K and L.

For $K, L \in \mathcal{S}_o^n$, real $p \neq 0$, and $\varepsilon > 0$, the L_p dual mixed volume $\widetilde{V}_p(K,L)$ of K and L is defined by (see [30])

$$\frac{n}{p}\widetilde{V}_p(K,L) = \lim_{\varepsilon\longrightarrow 0^+} \frac{V(K\widetilde{+}_p\varepsilon \cdot L) - V(K)}{\varepsilon}.$$

This definition and the polar coordinate formula for volume give the following integral representation of the L_p dual mixed volume (see [30]):

$$\widetilde{V}_p(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-p}(u)\rho_L^p(u)\,du, \tag{2.2}$$

where the integration is with respect to the spherical Lebesgue measure on S^{n-1}.

From (2.2) it follows immediately that, for each $K \in \mathcal{S}_o^n$,

$$\widetilde{V}_p(K,K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^n(u)\,du = V(K). \tag{2.3}$$

As an application of the Hölder inequality, we get the L_p dual Minkowski inequality for L_p dual mixed volume (see [30]).

Lemma 2.1 *For $K, L \in \mathcal{S}_o^n$, if $0 < p < n$, then*

$$\widetilde{V}_p(K,L) \leq V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}; \tag{2.4}$$

if $p < 0$ or $p > n$, then

$$\widetilde{V}_p(K,L) \geq V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}. \tag{2.5}$$

Equality holds in each inequality if and only if K and L are dilates.

2.2 L_q harmonic radial sum, L_q radial Blaschke sum, and L_q harmonic Blaschke sum

The notion of L_q harmonic radial sum can be introduced as follows: For $K, L \in S_o^n$, real $q \geq 1$, the L_q harmonic radial sum $K \check{+}_q L \in \mathcal{S}_o^n$ of K and L is defined by (see [36])

$$\rho(K \check{+}_q L, \cdot)^{-q} = \rho(K, \cdot)^{-q} + \rho(L, \cdot)^{-q}. \tag{2.6}$$

If $q = 1$, then $K \check{+} L$ is the harmonic radial sum of K and L (see [4]).

The notion of radial Blaschke sum was given by Lutwak [2]. For $K, L \in \mathcal{S}_o^n$, $n \geq 2$, the radial Blaschke sum $K \hat{+} L \in \mathcal{S}_o^n$ of K and L is defined by

$$\rho(K \hat{+} L, \cdot)^{n-1} = \rho(K, \cdot)^{n-1} + \rho(L, \cdot)^{n-1}.$$

In 2015, Wang and Wang [37] introduced the notion of L_q radial Blaschke sum as follows: For $K, L \in \mathcal{S}_o^n$, $q \in \mathbb{R}$, and $n > q > 0$, the L_q radial Blaschke sum $K \hat{+}_q L \in \mathcal{S}_o^n$ of K and L is defined by

$$\rho(K \hat{+}_q L, \cdot)^{n-q} = \rho(K, \cdot)^{n-q} + \rho(L, \cdot)^{n-q}. \tag{2.7}$$

The harmonic Blaschke sum was introduced by Lutwak [38]. For $K, L \in \mathcal{S}_o^n$, the harmonic Blaschke sum $K \mp L \in \mathcal{S}_o^n$ of K and L is defined by

$$\frac{\rho(K \mp L, \cdot)^{n+1}}{V(K \mp L)} = \frac{\rho(K, \cdot)^{n+1}}{V(K)} + \frac{\rho(L, \cdot)^{n+1}}{V(L)}.$$

Based on this definition, Feng and Wang [39] defined the L_q harmonic Blaschke sum as follows: For $K, L \in \mathcal{S}_o^n$ and real $q \neq -n$, the L_q harmonic Blaschke sum $K \mp_q L \in \mathcal{S}_o^n$ of K and L is given by

$$\frac{\rho(K \mp_q L, \cdot)^{n+q}}{V(K \mp_q L)} = \frac{\rho(K, \cdot)^{n+q}}{V(K)} + \frac{\rho(L, \cdot)^{n+q}}{V(L)}. \tag{2.8}$$

3 Brunn-Minkowski type inequalities for L_p radial Blaschke-Minkowski homomorphisms

Theorems 1.1 and 1.2 show Brunn-Minkowski type inequalities for the L_q radial Minkowski sum and L_q harmonic Blaschke sum of L_p radial Blaschke-Minkowski homomorphisms. In this section, we prove Theorems 1.1 and 1.2. As applications of Theorem 1.1, we yet give two Brunn-Minkowski type inequalities for both L_q harmonic radial sum and L_q radial Blaschke sum of L_p radial Blaschke-Minkowski homomorphisms. In order to prove Theorem 1.1, the following lemmas shall be needed.

Lemma 3.1 ([28]) *Let $\Psi_p : \mathcal{S}_o^n \to \mathcal{S}_o^n$ be an L_p radial Blaschke-Minkowski homomorphism with real $p \neq 0$. Then, for $K, L \in \mathcal{S}_o^n$,*

$$\widetilde{V}_p(K, \Psi_p L) = \widetilde{V}_p(L, \Psi_p K). \tag{3.1}$$

Lemma 3.2 *Let $K, L \in \mathcal{S}_o^n$, $p, q \in \mathbb{R}$, $p, q \neq 0$.*

(i) *If $\frac{n-p}{q} > 1$, then, for any $Q \in \mathcal{S}_o^n$,*

$$\widetilde{V}_p(K \widetilde{+}_q L, Q)^{\frac{q}{n-p}} \leq \widetilde{V}_p(K, Q)^{\frac{q}{n-p}} + \widetilde{V}_p(L, Q)^{\frac{q}{n-p}}. \tag{3.2}$$

(ii) *If $\frac{n-p}{q} < 1$, then, for any $Q \in \mathcal{S}_o^n$,*

$$\widetilde{V}_p(K \widetilde{+}_q L, Q)^{\frac{q}{n-p}} \geq \widetilde{V}_p(K, Q)^{\frac{q}{n-p}} + \widetilde{V}_p(L, Q)^{\frac{q}{n-p}}. \tag{3.3}$$

Equality holds in each inequality if and only if K and L are dilates.

Proof From (2.1) and the Minkowski integral inequality, which enforces the condition $\frac{n-p}{q} > 1$, it follows that, for any $Q \in \mathcal{S}_o^n$,

$$\begin{aligned}
\widetilde{V}_p(K \widetilde{+}_q L, Q)^{\frac{q}{n-p}} &= \left[\frac{1}{n}\int_{S^{n-1}} \rho(K \widetilde{+}_q L, u)^{n-p} \rho(Q, u)^p \, du\right]^{\frac{q}{n-p}} \\
&= \left\{\frac{1}{n}\int_{S^{n-1}} \left[\rho(K, u)^q + \rho(L, u)^q\right]^{\frac{n-p}{q}} \rho(Q, u)^p \, du\right\}^{\frac{q}{n-p}} \\
&\leq \left[\frac{1}{n}\int_{S^{n-1}} \rho(K, u)^{n-p} \rho(Q, u)^p \, du\right]^{\frac{q}{n-p}} \\
&\quad + \left[\frac{1}{n}\int_{S^{n-1}} \rho(L, u)^{n-p} \rho(Q, u)^p \, du\right]^{\frac{q}{n-p}} \\
&= \widetilde{V}_p(K, Q)^{\frac{q}{n-p}} + \widetilde{V}_p(L, Q)^{\frac{q}{n-p}};
\end{aligned}$$

this is just (3.2). According to the condition of equality in the Minkowski integral inequality, we see that equality holds in (3.2) if and only if $\rho(K, \cdot)$ and $\rho(L, \cdot)$ are positively proportional, that is, equality holds in (3.2) if and only if K and L are dilates.

Similarly, again using (2.1) and the Minkowski integral inequality, which now enforces the condition $\frac{n-p}{q} < 1$, we obtain inequality (3.3) with the equality condition. □

Proof of Theorem 1.1 (i) If $p > 0$ and $0 < q < n - p$, then $\frac{n-p}{q} > 1$ and $0 < p < n$. Thus, by (3.1), (3.2), and (2.4) we have, for any $N \in \mathcal{S}_o^n$,

$$\begin{aligned}
\widetilde{V}_p\bigl(N, \Psi_p(K\widetilde{+}_q L)\bigr)^{\frac{q}{n-p}} &= \widetilde{V}_p(K\widetilde{+}_q L, \Psi_p N)^{\frac{q}{n-p}} \\
&\le \widetilde{V}_p(K, \Psi_p N)^{\frac{q}{n-p}} + \widetilde{V}_p(L, \Psi_p N)^{\frac{q}{n-p}} \qquad (3.4)\\
&= \widetilde{V}_p(N, \Psi_p K)^{\frac{q}{n-p}} + \widetilde{V}_p(N, \Psi_p L)^{\frac{q}{n-p}} \\
&\le V(N)^{\frac{q}{n}}\bigl[V(\Psi_p K)^{\frac{pq}{n(n-p)}} + V(\Psi_p L)^{\frac{pq}{n(n-p)}}\bigr]. \qquad (3.5)
\end{aligned}$$

Setting $N = \Psi_p(K\widetilde{+}_q L)$, by (2.3) we obtain

$$V\bigl(\Psi_p(K\widetilde{+}_q L)\bigr)^{\frac{pq}{n(n-p)}} \le V(\Psi_p K)^{\frac{pq}{n(n-p)}} + V(\Psi_p L)^{\frac{pq}{n(n-p)}}.$$

This gives inequality (1.1).

By the equality conditions of (3.4) and (3.5) we know that equality in (1.1) holds if and only if K, L, $\Psi_p K$, $\Psi_p L$, and $\Psi_p(K\widetilde{+}_q L)$ all are dilates. But if K and L are dilates, then $\Psi_p(K\widetilde{+}_q L)$, $\Psi_p K$, and $\Psi_p L$ all are dilates. Thus, equality in (1.1) holds if and only if K and L are dilates.

(ii) For $q > n - p > 0 > p$ or $q < n - p < 0$, we know that $0 < \frac{n-p}{q} < 1$, $p < 0$ or $p > n$ (for $q < 0 < n - p$ and $p > 0$, we get $\frac{n-p}{q} < 0$ and $0 < p < n$). From this, using (3.1), (3.3), and (2.5) (or (2.4)), we have, for any $N \in \mathcal{S}_o^n$,

$$\begin{aligned}
\widetilde{V}_p\bigl(N, \Psi_p(K\widetilde{+}_q L)\bigr)^{\frac{q}{n-p}} &= \widetilde{V}_p(K\widetilde{+}_q L, \Psi_p N)^{\frac{q}{n-p}} \\
&\ge \widetilde{V}_p(K, \Psi_p N)^{\frac{q}{n-p}} + \widetilde{V}_p(L, \Psi_p N)^{\frac{q}{n-p}} \\
&= \widetilde{V}_p(N, \Psi_p K)^{\frac{q}{n-p}} + \widetilde{V}_p(N, \Psi_p L)^{\frac{q}{n-p}} \\
&\ge V(N)^{\frac{q}{n}}\bigl[V(\Psi_p K)^{\frac{pq}{n(n-p)}} + V(\Psi_p L)^{\frac{pq}{n(n-p)}}\bigr].
\end{aligned}$$

Setting $N = \Psi_p(K\widetilde{+}_q L)$ and using (2.3), we have

$$V\bigl(\Psi_p(K\widetilde{+}_q L)\bigr)^{\frac{pq}{n(n-p)}} \ge V(\Psi_p K)^{\frac{pq}{n(n-p)}} + V(\Psi_p L)^{\frac{pq}{n(n-p)}}.$$

This yields (1.2), and equality holds in (1.2) if and only if K and L are dilates. □

As an application of Theorem 1.1, from the L_q harmonic radial sum (2.6) we obtain the following:

Theorem 3.1 *Let $K, L \in \mathcal{S}_o^n$, $p, q \in \mathbb{R}$, $p \neq 0$, $q \ge 1$. If $-q < n - p < 0$ or $0 < p < n$, then*

$$V\bigl(\Psi_p(K\check{+}_q L)\bigr)^{-\frac{pq}{n(n-p)}} \ge V(\Psi_p K)^{-\frac{pq}{n(n-p)}} + V(\Psi_p L)^{-\frac{pq}{n(n-p)}}, \tag{3.6}$$

where equality holds if and only if K and L are dilates.

Proof By (2.1) and (2.6) we see that, for $q \geq 1$, $K\check{+}_q L = K\widetilde{+}_{-q} L$. Hence, if $-q < n-p < 0$, then (3.6) is true by (1.2); if $0 < p < n$, then since $q \geq 1$, we have $-q < 0 < n-p$, which, together with (1.2), shows that (3.6) also holds. □

Similarly, as another application of Theorem 1.1, by the L_q radial Blaschke sum (2.7) we have the following:

Theorem 3.2 *Let $K, L \in \mathcal{S}_o^n$, $p, q \in \mathbb{R}$, $p \neq 0$, $0 < q < n$.*
(i) *If $n > q > p > 0$, then*

$$V\big(\Psi_p(K\hat{+}_q L)\big)^{\frac{p(n-q)}{n(n-p)}} \leq V(\Psi_p K)^{\frac{p(n-q)}{n(n-p)}} + V(\Psi_p L)^{\frac{p(n-q)}{n(n-p)}} \tag{3.7}$$

with equality if and only if K and L are dilates.

Proof From (2.1) and (2.7) we know that, for $0 < q < n$, $K\hat{+}_q L = K\widetilde{+}_{n-q} L$. Thus, if $n > q > p > 0$, then $0 < n-q < n-p$ and $p > 0$. This, together with (1.1), yields (3.7). □

The proof of Theorem 1.2 requires the following lemma.

Lemma 3.3 *Let $K, L \in \mathcal{S}_o^n$, $p, q \in \mathbb{R}$, $p \neq 0$, $q \neq -n$.*
(i) *If $\frac{n-p}{n+q} > 1$, then, for any $Q \in \mathcal{S}_o^n$,*

$$\frac{\widetilde{V}_p(K \mp_q L, Q)^{\frac{n+q}{n-p}}}{V(K \mp_q L)} \leq \frac{\widetilde{V}_p(K, Q)^{\frac{n+q}{n-p}}}{V(K)} + \frac{\widetilde{V}_p(L, Q)^{\frac{n+q}{n-p}}}{V(L)}. \tag{3.8}$$

(ii) *If $\frac{n-p}{n+q} < 1$, then, for any $Q \in \mathcal{S}_o^n$,*

$$\frac{\widetilde{V}_p(K \mp_q L, Q)^{\frac{n+q}{n-p}}}{V(K \mp_q L)} \geq \frac{\widetilde{V}_p(K, Q)^{\frac{n+q}{n-p}}}{V(K)} + \frac{\widetilde{V}_p(L, Q)^{\frac{n+q}{n-p}}}{V(L)}. \tag{3.9}$$

Equality holds in each inequality if and only if K and L are dilates.

Proof By (2.2), (2.8), and the Minkowski integral inequality, which enforces the condition $\frac{n-p}{n+q} > 1$, we have, for any $Q \in \mathcal{S}_o^n$,

$$\begin{aligned}
\frac{\widetilde{V}_p(K \mp_q L, Q)^{\frac{n+q}{n-p}}}{V(K \mp_q L)} &= \frac{[\frac{1}{n}\int_{S^{n-1}} \rho(K \mp_q L, u)^{n-p} \rho(Q, u)^p \, du]^{\frac{n+q}{n-p}}}{V(K \mp_q L)} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \left(\frac{\rho(K, u)^{n+q}}{V(K)} + \frac{\rho(L, u)^{n+q}}{V(L)}\right)^{\frac{n-p}{n+q}} \rho(Q, u)^p \, du\right]^{\frac{n+q}{n-p}} \\
&\leq \frac{1}{V(K)}\left[\frac{1}{n}\int_{S^{n-1}} \rho(K, u)^{n-p} \rho(Q, u)^p \, du\right]^{\frac{n+q}{n-p}} \\
&\quad + \frac{1}{V(L)}\left[\frac{1}{n}\int_{S^{n-1}} \rho(L, u)^{n-p} \rho(Q, u)^p \, du\right]^{\frac{n+q}{n-p}} \\
&= \frac{\widetilde{V}_p(K, Q)^{\frac{n+q}{n-p}}}{V(K)} + \frac{\widetilde{V}_p(L, Q)^{\frac{n+q}{n-p}}}{V(L)}
\end{aligned}$$

with equality if and only if K and L are dilates. This inequality gives (3.8).

Similarly, again using (2.2) and the Minkowski integral inequality, which now enforces the condition $\frac{n-p}{n+q} < 1$, we obtain inequality (3.9) with the equality condition. □

Proof of Theorem 1.2 (i) For $K, L \in \mathcal{S}_o^n$, since $q \neq -n$, if $0 < p < -q < n$, then $\frac{n-p}{n+q} > 1$ and $0 < p < n$. So by (3.1), (3.8), and (2.4) we have, for any $N \in \mathcal{S}_o^n$,

$$\begin{aligned}\frac{\widetilde{V}_p(N, \Psi_p(K \mp_q L))^{\frac{n+q}{n-p}}}{V(K \mp_q L)} &= \frac{\widetilde{V}_p(K \mp_q L, \Psi_p N)^{\frac{n+q}{n-p}}}{V(K \mp_q L)} \\ &\leq \frac{\widetilde{V}_p(K, \Psi_p N)^{\frac{n+q}{n-p}}}{V(K)} + \frac{\widetilde{V}_p(L, \Psi_p N)^{\frac{n+q}{n-p}}}{V(L)} \\ &= \frac{\widetilde{V}_p(N, \Psi_p K)^{\frac{n+q}{n-p}}}{V(K)} + \frac{\widetilde{V}_p(N, \Psi_p L)^{\frac{n+q}{n-p}}}{V(L)} \\ &\leq V(N)^{\frac{n+q}{n}} \left[\frac{V(\Psi_p K)^{\frac{p(n+q)}{n(n-p)}}}{V(K)} + \frac{V(\Psi_p L)^{\frac{p(n+q)}{n(n-p)}}}{V(L)} \right].\end{aligned} \tag{3.10}$$

Setting $N = \Psi_p(K \mp_q L)$ in (3.10), by (2.3) we get

$$\frac{V(\Psi_p(K \mp_q L))^{\frac{p(n+q)}{n(n-p)}}}{V(K \mp_q L)} \leq \frac{V(\Psi_p K)^{\frac{p(n+q)}{n(n-p)}}}{V(K)} + \frac{V(\Psi_p L)^{\frac{p(n+q)}{n(n-p)}}}{V(L)},$$

and equality holds if and only if K and L are dilates. Therefore, inequality (1.3) is obtained.

(ii) If $-q < p < 0$, then $0 < \frac{n-p}{n+q} < 1$. This, together with (3.1), (3.9), and (2.5), yields

$$\frac{V(\Psi_p(K \mp_q L))^{\frac{p(n+q)}{n(n-p)}}}{V(K \mp_q L)} \geq \frac{V(\Psi_p K)^{\frac{p(n+q)}{n(n-p)}}}{V(K)} + \frac{V(\Psi_p L)^{\frac{p(n+q)}{n(n-p)}}}{V(L)},$$

and equality holds if and only if K and L are dilates. This is just inequality (1.4). □

4 Monotonic inequalities for the L_p radial Blaschke-Minkowski homomorphisms

In this section, we establish monotonic inequalities for the L_p radial Blaschke-Minkowski homomorphisms.

Theorem 4.1 *Let $\Phi_p : \mathcal{S}_o^n \to \mathcal{S}_o^n$ be an L_p radial Blaschke-Minkowski homomorphism, $p \neq 0$, $K, L \in \mathcal{S}_o^n$, and $\Phi_p K \subseteq \Phi_p L$. If $p > 0$, then, for any $Q \in \Phi_p \mathcal{S}_o^n$,*

$$\widetilde{V}_p(K, Q) \leq \widetilde{V}_p(L, Q); \tag{4.1}$$

if $p < 0$, then, for any $Q \in \Phi_p \mathcal{S}_o^n$,

$$\widetilde{V}_p(K, Q) \geq \widetilde{V}_p(L, Q). \tag{4.2}$$

Equality holds in (4.1) *or* (4.2) *if and only if $\Phi_p K = \Phi_p L$.*

Proof Since $\Phi_p K \subseteq \Phi_p L$, by (2.2) we know that, for $p > 0$ and any $N \in \mathcal{S}_o^n$,

$$\widetilde{V}_p(N, \Phi_p K) \leq \widetilde{V}_p(N, \Phi_p L). \tag{4.3}$$

This, together with (3.1), gives

$$\widetilde{V}_p(K, \Phi_p N) \le \widetilde{V}_p(L, \Phi_p N).$$

Let $Q = \Phi_p N$. Then $Q \in \Phi_p \mathcal{S}_o^n$ and $\widetilde{V}_p(K, Q) \le \widetilde{V}_p(L, Q)$. From the equality condition for (4.3), we see that equality holds in (4.1) if and only if $\Phi_p K = \Phi_p L$.

Similarly, if $p < 0$ and $\Phi_p K \subseteq \Phi_p L$, by (2.2) we easily obtain that, for any $Q \in \Phi_p \mathcal{S}_o^n$, $\widetilde{V}_p(K, Q) \ge \widetilde{V}_p(L, Q)$, and equality holds if and only if $\Phi_p K = \Phi_p L$. □

For $0 < p < n$, let $K \in \Phi_p \mathcal{S}_o^n$ in (4.1) (for $p > n$, let $L \in \Phi_p \mathcal{S}_o^n$ in (4.1); for $p < 0$, let $K \in \Phi_p \mathcal{S}_o^n$ in (4.2)). Using inequality (2.4) (or inequality (2.5)), we may get a positive form of Busemann-Petty type problem for the L_p radial Blaschke-Minkowski homomorphisms given by Wang et al. [28].

Corollary 4.1 *Let $\Phi_p : \mathcal{S}_o^n \to \mathcal{S}_o^n$ be an L_p radial Blaschke-Minkowski homomorphism, $p \neq 0$, $K, L \in \mathcal{S}_o^n$, and $\Phi_p K \subseteq \Phi_p L$. If $n > p > 0$ and $K \in \Phi_p \mathcal{S}_o^n$, then*

$$\Phi_p K \subseteq \Phi_p L \quad \Longrightarrow \quad V(K) \le V(L),$$

and $V(K) = V(L)$ if and only if $K = L$.

If $p > n$ and $L \in \Phi_p \mathcal{S}_o^n$ or $p < 0$ and $K \in \Phi_p \mathcal{S}_o^n$, then

$$\Phi_p K \subseteq \Phi_p L \quad \Longrightarrow \quad V(K) \ge V(L),$$

and $V(K) = V(L)$ if and only if $K = L$.

Theorem 4.2 *Let $\Phi_p : \mathcal{S}_o^n \to \mathcal{S}_o^n$ be an L_p radial Blaschke-Minkowski homomorphism, $p \neq 0$. If $K, L \in \mathcal{S}_o^n$, and for any $Q \in \mathcal{S}_o^n$,*

$$\widetilde{V}_p(K, Q) \le \widetilde{V}_p(L, Q), \tag{4.4}$$

then, for $p > 0$,

$$V(\Phi_p K) \le V(\Phi_p L), \tag{4.5}$$

and, for $p < 0$,

$$V(\Phi_p K) \ge V(\Phi_p L). \tag{4.6}$$

Equality holds in (4.5) *or* (4.6) *only if $K = L$.*

Proof For $0 < p < n$, let $Q = \Phi_p \Phi_p K$ in (4.4). Then

$$\widetilde{V}_p(K, \Phi_p \Phi_p K) \le \widetilde{V}_p(L, \Phi_p \Phi_p K).$$

Using inequality (2.4) and equalities (3.1) and (2.3), we have

$$V(\Phi_p K) = \widetilde{V}_p(\Phi_p K, \Phi_p K) \le \widetilde{V}_p(\Phi_p K, \Phi_p L) \le V(\Phi_p K)^{\frac{n-p}{n}} V(\Phi_p L)^{\frac{p}{n}},$$

which yields (4.5), and equality holds only if $K = L$.

Similarly, for $p > n$ (or $p < 0$), let $Q = \Phi_p \Phi_p L$ in (4.4). By inequality (2.5) and equalities (3.1) and (2.3), we can obtain (4.5) (or (4.6)). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

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