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p-Norm SDD tensors and eigenvalue localization

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Abstract

We present a new class of nonsingular tensors (p-norm strictly diagonally dominant tensors), which is a subclass of strong $\mathcal{H}$-tensors. As applications of the results, we give a new eigenvalue inclusion set, which is tighter than those provided by Li *et al.* (Linear Multilinear Algebra 64:727-736, 2016) in some case. Based on this set, we give a checkable sufficient condition for the positive (semi)definiteness of an even-order symmetric tensor.

Keywords: p-norm SDD tensor; strong $\mathcal{H}$-tensor; positive (semi)definiteness; eigenvalue localization

1 Introduction

Let $\mathbb{C}(\mathbb{R})$ denote the set of all complex (real) numbers, and $[n] := \{1, 2, \dots, n\}$. An mth-order n-dimensional complex (real) tensor, denoted by $\mathcal{A} \in \mathbb{C}^{[m,n]}(\mathbb{R}^{[m,n]})$, is a multidimensional array of n^m elements of the form

$$\mathcal{A} = (a_{i_1 i_2 \cdots i_m}), \quad a_{i_1 \cdots i_m} \in \mathbb{C}(\mathbb{R}), i_j \in [n], j \in [m].$$

When $m = 2$, $\mathcal{A}$ is an n-by-n matrix. A tensor $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called nonnegative if each its entry is nonnegative, and it is called symmetric [2, 3] if

$$a_{i_1 \cdots i_m} = a_{\pi(i_1 \cdots i_m)}, \quad \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of m indices. Moreover, an mth-order n-dimensional tensor $\mathcal{I} = (\delta_{i_1 i_2 \cdots i_m})$ is called the identity tensor [4] if

$$\delta_{i_1 i_2 \cdots i_m} = \begin{cases} 1 & \text{if } i_1 = i_2 = \cdots = i_m, \\ 0 & \text{otherwise.} \end{cases}$$

For an n-dimensional vector $x = (x_1, x_2, \dots, x_n)^T$, real or complex, we define the n-dimensional vector

$$\mathcal{A}x^{m-1} := \left(\sum_{i_2, \dots, i_m \in [n]} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1 \le i \le n},$$

and the n-dimensional vector

$$x^{[m-1]} := \left(x_i^{m-1}\right)_{1\le i\le n}.$$

The following definition related to eigenvalues of tensors was first introduced and studied by Qi [3] and Lim [5].

Definition 1 [3, 5] Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenvalue-eigenvector (or simply eigenpair) of $\mathcal{A}$ if satisfies the equation

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{1}$$

We call (λ, x) an H-eigenpair if they are both real.

In addition, the spectral radius of a tensor $\mathcal{A}$ is defined as

$$\rho(\mathcal{A}) = \max\left\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\right\}.$$

Definition 2 A tensor $\mathcal{A} \in \mathbb{C}^{[m,n]}$ is said to be nonsingular if zero is not an eigenvalue of $\mathcal{A}$. Otherwise, it is called singular.

Tensor eigenvalue problems have gained special attention in the realm of numerical multilinear algebra, and they have a wide range in practice; see [3, 4, 6–16]. For instance, we can use the smallest H-eigenvalues of tensors to determine their positive (semi)definiteness, that is, for an even-order real symmetric tensor $\mathcal{A}$, if its smallest H-eigenvalue is positive (nonnegative), then $\mathcal{A}$ is positive (semi)definite; consequently, the multivariate homogeneous polynomial $f(x)$ determined by $\mathcal{A}$ is positive (semi)definite [3].

Most often, it is difficult to compute the smallest H-eigenvalue. Therefore, we always try to give a distribution range of eigenvalues of a given tensor in the complex plane. In particular, if this range is in the right-half complex plane, which means that the smallest H-eigenvalue is positive, then the corresponding tensor is positive definite.

Qi [3] generalized the Geršgorin eigenvalue inclusion theorem from matrices to real symmetric tensors, which can be easily extended to generic tensors; see [4, 17].

Theorem 1 *Let* $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$. *Then*

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) = \bigcup_{i\in[n]} \Gamma_i(\mathcal{A}),$$

where $\sigma(\mathcal{A})$ *is the set of all the eigenvalues of* $\mathcal{A}$, *and*

$$\Gamma_i(\mathcal{A}) = \left\{z \in \mathbb{C} : |z - a_{ii\cdots i}| \le r_i(\mathcal{A})\right\}, \quad r_i(\mathcal{A}) = \sum_{\substack{i_2,\ldots,i_m\in[n],\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|.$$

Recently, as an extension of the theory in [18], Li *et al.* [1, 17, 19] proposed three new Brauer-type eigenvalue localization sets for tensors and showed tighter bounds than $\Gamma(\mathcal{A})$

of Theorem 1. We list the latest Brauer-type eigenvalue localization set as follows. For convenience, we denote

$$\Delta_i = \left\{(i_2, i_3, \ldots, i_m) : i_j = i \text{ for some } j \in \{2, 3, \ldots, m\}, \text{where } i, i_2, \ldots, i_m \in [n]\right\},$$
$$\overline{\Delta}_i = \left\{(i_2, i_3, \ldots, i_m) : i_j \neq i \text{ for any } j \in \{2, 3, \ldots, m\}, \text{where } i, i_2, \ldots, i_m \in [n]\right\},$$

and

$$r_i^{\Delta_i}(\mathcal{A}) = \sum_{\substack{(i_2,\ldots,i_m)\in\Delta_i,\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|, \qquad r_i^{\overline{\Delta}_i}(\mathcal{A}) = \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}_i} |a_{ii_2\cdots i_m}|.$$

Theorem 2 [1] *Let* $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$. *Then*

$$\sigma(\mathcal{A}) \subseteq \Omega(\mathcal{A}) = \left(\bigcup_{i\in[n]} \hat{\Omega}_i(\mathcal{A})\right) \cup \left(\bigcup_{\substack{i,j\in[n],\\ i\neq j}} \left(\hat{\Omega}_{i,j}(\mathcal{A}) \cap \Gamma_i(\mathcal{A})\right)\right),$$

where

$$\hat{\Omega}_i(\mathcal{A}) = \left\{z \in \mathbb{C} : |z - a_{i\cdots i}| \le r_i^{\Delta_i}(\mathcal{A})\right\}$$

and

$$\hat{\Omega}_{i,j}(\mathcal{A}) = \left\{z \in \mathbb{C} : \left(|z - a_{i\cdots i}| - r_i^{\Delta_i}(\mathcal{A})\right)\left(|z - a_{j\cdots j}| - r_j^{\overline{\Delta}_i}(\mathcal{A})\right) \le r_i^{\overline{\Delta}_i}(\mathcal{A}) r_j^{\Delta_i}(\mathcal{A})\right\}.$$

Li *et al.* [1] proved that the set $\Omega(\mathcal{A})$ in Theorem 2 is tighter than $\Theta(\mathcal{A})$ in [19] and $\mathcal{K}(\mathcal{A})$ in [17]; for details, see Theorem 2.3 in [1].

In this paper, we continue this research on the eigenvalue localization problem for tensors. A class of strictly diagonally dominant tensors that involve a parameter p in the interval $[1, \infty]$, denoted by p-norm SDD tensor, is introduced in Section 2. In Section 3, we discuss the relationships between p-norm SDD tensors and strong $\mathcal{H}$-tensors. A new eigenvalue inclusion set for tensors based on p-norm SDD tensors is obtained in Section 4, and numerical results show that the new set is tighter than $\Omega(\mathcal{A})$ in Theorem 2 in some case. Finally, in Section 5, we give a checkable sufficient condition for the positive (semi)definiteness of even-order symmetric tensors.

2 *p*-Norm SDD tensors

In this section, we propose a new class of nonsingular tensors, namely p-norm strictly diagonally dominant tensors. First, some notation and the definition of strictly diagonally dominant tensors are given.

Given a tensor $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$ and a real number $p \in [1, \infty]$, denote

$$r_i^p(\mathcal{A}) := \left(\sum_{\substack{i_2,\ldots,i_m\in[n],\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|^p\right)^{\frac{1}{p}} \quad \text{for all } i \in [n].$$

In particular, if $p = 1$, then $r_i^1(\mathcal{A}) = r_i(\mathcal{A})$ for all $i \in [n]$. If $p = \infty$, then $r_i^\infty(\mathcal{A}) = \max_{\substack{i_2,\dots,i_m\in[n],\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|$ for all $i \in [n]$. For a vector $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{C}^n$, the l_q-norm on $\mathbb{C}^n$ is

$$\|x\|_q := \left(\sum_{i\in[n]} |x_i|^q\right)^{\frac{1}{q}}.$$

Definition 3 [16] A tensor $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$ is diagonally dominant if

$$|a_{ii\cdots i}| \geq r_i(\mathcal{A}) \quad \text{for all } i \in [n], \tag{2}$$

and $\mathcal{A}$ is strictly diagonally dominant if the strict inequality holds in (2) for all i.

Remark 1 $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$ is strictly diagonally dominant if and only if

$$\max_{i\in[n]} \frac{r_i(\mathcal{A})}{|a_{ii\cdots i}|} < 1.$$

It is well known that strictly diagonally dominant tensors are nonsingular. An interesting problem arises: for a tensor $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$ satisfying

$$\max_{i\in[n]} \frac{r_i^p(\mathcal{A})}{|a_{ii\cdots i}|} < 1,$$

is $\mathcal{A}$ nonsingular or not? Certainly, when $p = 1$, $\mathcal{A}$ is a strictly diagonally dominant tensor, which means that $\mathcal{A}$ is nonsingular, but when $p > 1$, $\mathcal{A}$ may be singular as the following simple example shows.

Example 1 Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{[3,2]}$, where

$$a_{111} = a_{222} = -3, \quad \text{and the remaining} \quad a_{ijk} = 1.$$

Then, since $\mathcal{A}e^2 = 0$, where $e = (1,1,1)^T$, this implies $0 \in \sigma(\mathcal{A})$. However, for every $p > 1$, we have

$$\max_{i\in[2]} \frac{r_i^p(\mathcal{A})}{|a_{iii}|} = 3^{\frac{1-p}{p}} < 1.$$

Therefore, something needs to be added in order to obtain a nonsingular $\mathcal{A}$ for a real number $p \in (1,\infty]$. We provide an answer further, but we first introduce a class of strictly diagonally dominant tensors that involve a parameter p in the interval $[1,\infty]$.

Definition 4 Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$ and $p \in [1,\infty]$, $\mathcal{A}$ is called a p-norm strictly diagonally dominant tensor (or, shortly, p-norm SDD tensor) if

$$\left\|\delta_p(\mathcal{A})\right\|_q < 1, \tag{3}$$

where

$$\delta_p(\mathcal{A}) := (\delta_1, \delta_2, \ldots, \delta_n)^T, \qquad \delta_i := \left(\frac{r_i^p(\mathcal{A})}{|a_{i\cdots i}|}\right)^{\frac{1}{m-1}} \quad \text{for all } i \in [n],$$

and q is Hölder's complement of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 2 Definition 4 extends the concept of SDD(p) matrix given in [20] to tensors. Clearly, the SDD(p) matrix is a 2nd-order p-norm SDD tensor.

Remark 3 Taking $p = 1$, $\mathcal{A}$ is a 1-norm SDD tensor if and only if

$$\left\|\delta_1(\mathcal{A})\right\|_\infty = \max_{i\in[n]}\left(\frac{r_i(\mathcal{A})}{|a_{ii\cdots i}|}\right)^{\frac{1}{m-1}} < 1,$$

that is,

$$\max_{i\in[n]} \frac{r_i(\mathcal{A})}{|a_{ii\cdots i}|} < 1,$$

which is equivalent to the fact that $\mathcal{A}$ is a strictly diagonally dominant tensor. The other extreme case is $p = \infty$. $\mathcal{A}$ is a ∞-norm SDD tensor if and only if

$$\left\|\delta_\infty(\mathcal{A})\right\|_1 = \sum_{i\in[n]}\left(\frac{\max_{i_2,\ldots,i_m\in[n], \, \delta_{ii_2\cdots i_m}=0} |a_{ii_2\cdots i_m}|}{|a_{ii\cdots i}|}\right)^{\frac{1}{m-1}} < 1.$$

The p-norm SDD tensors can also be characterized in the following way.

Proposition 1 *Let $\mathcal{A} \in \mathbb{C}^{[m,n]}$ and $p \in [1,\infty]$. Then $\mathcal{A}$ is a p-norm SDD tensor if and only if there exists an entrywise positive vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ such that $\|x\|_q \le 1$, where q is Hölder's complement of p such that*

$$x_i^{m-1}|a_{i\cdots i}| > r_i^p(\mathcal{A}) \quad \textit{for all } i \in [n]. \tag{4}$$

Proof Necessity. Suppose that $\mathcal{A}$ is a p-norm SDD tensor. It follows from inequality (3) of Definition 4 that there exists a sufficiently small $\varepsilon > 0$ such that, for $x_i := \delta_i + \varepsilon > 0$, where $i \in [n]$, $\|x\|_q \le 1$. Thus, $x_i^{m-1} > \delta_i^{m-1} = \frac{r_i^p(\mathcal{A})}{|a_{i\cdots i}|}$, which implies inequality (4).

Sufficiency. Suppose that there exists an entrywise positive vector $x > 0$ such that $\|x\|_q \le 1$ and inequality (4) holds. By inequality (4) we have

$$x_i^{m-1} > \frac{r_i^p(\mathcal{A})}{|a_{i\cdots i}|} = \delta_i^{m-1} \quad \text{for all } i \in [n],$$

which implies $x_i > \delta_i$ for all $i \in [n]$, which, together with $\|x\|_q \le 1$, yields

$$\left\|\delta_p(\mathcal{A})\right\|_q < \|x\|_q \le 1.$$

Thus, $\mathcal{A}$ is a p-norm SDD tensor. The proof is completed. □

The following result proves the nonsingular of p-norm SDD tensors.

Theorem 3 *Let $\mathcal{A} \in \mathbb{C}^{[m,n]}$ be a p-norm SDD tensor. Then $\mathcal{A}$ is nonsingular.*

Proof Suppose that $\mathcal{A}$ is singular, that is, $0 \in \sigma(\mathcal{A})$. It follows from equality (1) that there exists $y \in \mathbb{C}^n \setminus \{0\}$, such that

$$\mathcal{A}y^{m-1} = 0. \tag{5}$$

Without the loss of generality, we can assume that $\|y\|_q = 1$. Then, equality (5) yields

$$a_{i\cdots i}y_i^{m-1} = -\sum_{\substack{i_2,\ldots,i_m \in [n], \\ \delta_{ii_2\cdots i_m}=0}} a_{ii_2\cdots i_m} y_{i_2}\cdots y_{i_m} \quad \text{for all } i \in [n],$$

which implies that

$$|a_{i\cdots i}||y_i|^{m-1} = \left|\sum_{\substack{i_2,\ldots,i_m \in [n], \\ \delta_{ii_2\cdots i_m}=0}} a_{ii_2\cdots i_m} y_{i_2}\cdots y_{i_m}\right| \quad \text{for all } i \in [n]. \tag{6}$$

Then, applying the Hölder inequality to the right-hand side of equality (6), we obtain

$$\begin{aligned} |a_{i\cdots i}||y_i|^{m-1} &\le \left(\sum_{\substack{i_2,\ldots,i_m \in [n], \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|^p\right)^{\frac{1}{p}} \left(\sum_{\substack{i_2,\ldots,i_m \in [n], \\ \delta_{ii_2\cdots i_m}=0}} |y_{i_2}\cdots y_{i_m}|^q\right)^{\frac{1}{q}} \\ &= r_i^p(\mathcal{A})\left[\left(\sum_{j=1}^n |y_j|^q\right)^{m-1} - |y_i|^{(m-1)q}\right]^{\frac{1}{q}} \\ &= r_i^p(\mathcal{A})\left(\|y\|_q^{(m-1)q} - |y_i|^{(m-1)q}\right)^{\frac{1}{q}} \\ &\le r_i^p(\mathcal{A})\|y\|_q^{m-1} \\ &= r_i^p(\mathcal{A}) \quad \text{for all } i \in [n]. \end{aligned} \tag{7}$$

Since $\mathcal{A}$ is a p-norm SDD tenor, there exists an entrywise positive vector $x > 0$ such that $\|x\|_q \le 1$ and inequality (4) holds. Combining inequality (4) with (7), we obtain

$$|a_{i\cdots i}||y_i|^{m-1} \le r_i^p(\mathcal{A}) < x_i^{m-1}|a_{i\cdots i}| \quad \text{for all } i \in [n],$$

which means that

$$|y_i| < x_i \quad \text{for all } i \in [n].$$

Thus, $\|x\|_q > \|y\|_q = 1$, which contradicts $\|x\|_q \le 1$. The proof is completed. □

3 Relationships between p-norm SDD tensors and strong $\mathcal{H}$-tensors

The following lemma shows that the strong $\mathcal{H}$-tensors play an important role in identifying the positive definiteness of even-order real symmetric tensors.

Lemma 1 [21] *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an even-order real symmetric tensor with $a_{i\cdots i} > 0$ for all $i \in [n]$. If $\mathcal{A}$ is a strong $\mathcal{H}$-tensor, then $\mathcal{A}$ is positive definite.*

It is known that the strictly diagonally dominant tensors are a subclass of strong $\mathcal{H}$-tensors. An interesting problem arises: whether the class of p-norm SDD tensors is a subclass of strong $\mathcal{H}$-tensors for an arbitrary $p \in [1, \infty]$. In this section, we discuss this problem. We first recall the definition of strong $\mathcal{H}$-tensors.

Definition 5 [16] A tenor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called an $\mathcal{M}$-tensor if there exist a nonnegative tensor $\mathcal{B}$ and a positive real number $\eta \geq \rho(\mathcal{B})$ such that $\mathcal{A} = \eta\mathcal{I}\text{-}\mathcal{B}$. If $\eta > \rho(\mathcal{B})$, then $\mathcal{A}$ is called a strong $\mathcal{M}$-tensor.

Definition 6 [9] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. We call another tensor $\mathcal{M}(\mathcal{A}) = (m_{i_1 i_2 \cdots i_m})$ the comparison tensor of $\mathcal{A}$ if

$$m_{i_1 i_2 \cdots i_m} = \begin{cases} +|a_{i_1 i_2 \cdots i_m}| & \text{if } (i_2, i_3, \ldots, i_m) = (i_1, i_1, \ldots, i_1), \\ -|a_{i_1 i_2 \cdots i_m}| & \text{if } (i_2, i_3, \ldots, i_m) \neq (i_1, i_1, \ldots, i_1). \end{cases}$$

Definition 7 [9] We call a tensor an $\mathcal{H}$-tensor if its comparison tensor is an $\mathcal{M}$-tensor. We call it a strong $\mathcal{H}$-tensor if its comparison tensor is a strong $\mathcal{M}$-tensor.

Note that Li *et al.* [21] also provided an equivalent definition of strong $\mathcal{H}$-tensors; for details, see [21].

In [22], the multiplication of matrices has been extended to tensors. In the following, we state these results for reference.

Definition 8 [22] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ and $\mathcal{B} = (b_{i_1 i_2 \cdots i_k})$ be n-dimensional tensors of orders $m \geq 2$ and $k \geq 1$, respectively. The product $\mathcal{AB}$ is the following n-dimensional tensor $\mathcal{C}$ of order $(m-1)(k-1)+1$ with entries

$$c_{i\alpha_1\alpha_2\cdots\alpha_{m-1}} = \sum_{i_2,\ldots,i_m \in [n]} a_{ii_2\cdots i_m} b_{i_2\alpha_1} \cdots b_{i_m\alpha_{m-1}},$$

where $i \in [n]$ and $\alpha_1, \ldots, \alpha_{m-1} \in \{j_2 j_3 \cdots j_k : j_l \in [n], l = 2, 3, \ldots, k\}$.

Remark 4 When $m = 2$ and $\mathcal{A} = (a_{ij})$ is a matrix of dimension n, then $\mathcal{AB}$ is an mth-order n-dimensional tensor, and we have

$$(\mathcal{AB})_{i_1 i_2 \cdots i_m} = \sum_{l_2 \in [n]} a_{i_1 l_2} b_{l_2 i_2 \cdots i_m}, \quad i_j \in [n], j \in [m].$$

In particular, the product of a diagonal matrix $X = \operatorname{diag}(x_1, x_2, \ldots, x_n)$ and the tensor $\mathcal{A}$ is given by

$$(X\mathcal{A})_{i_1 i_2 \cdots i_m} = x_{i_1} a_{i_1 i_2 \cdots i_m}, \quad i_j \in [n], j \in [m].$$

Remark 5 Given an n-by-n matrix X and two mth order n-dimensional tensors $\mathcal{A}$, $\mathcal{B}$, we have the right distributive law for tensors [22], that is,

$$X \cdot \mathcal{A} + X \cdot \mathcal{B} = X \cdot (\mathcal{A} + \mathcal{B}).$$

Based on this multiplication of tensors, Kannan, Shaked-Monderer, and Berman [23] established a necessary and sufficient condition for a tensor to be a strong $\mathcal{H}$-tensor.

Lemma 2 [23] *Let $\mathcal{A} \in \mathbb{C}^{[m,n]}$. Then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor if and only if $a_{i\cdots i} \neq 0$ for all $i \in [n]$ and*

$$\rho\big(\mathcal{I} - D_{\mathcal{M}(\mathcal{A})}^{-1}\mathcal{M}(\mathcal{A})\big) < 1,$$

where $D_{\mathcal{M}(\mathcal{A})}$ is the diagonal matrix with the same diagonal entries as $\mathcal{M}(\mathcal{A})$.

The following lemma is given by Qi [3].

Lemma 3 [3] *Let $\mathcal{A} \in \mathbb{C}^{[m,n]}$ and $\mathcal{B} = a(\mathcal{A} + b\mathcal{I})$, where a and b are two complex numbers. Then μ is an eigenvalue of $\mathcal{B}$ if and only if $\mu = a(\lambda + b)$ and λ is an eigenvalue of $\mathcal{A}$. In this case, they have the same eigenvectors.*

Next, we present an equivalence condition for singular tensors.

Lemma 4 *Let $\mathcal{A} \in \mathbb{C}^{[m,n]}$. Then $\mathcal{A}$ is singular if and only if $D\mathcal{A}$ is singular, where $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ is a positive diagonal matrix.*

Proof Suppose that $D\mathcal{A}$ is singular, that is, $0 \in \sigma(D\mathcal{A})$. Then there exists a vector $x = (x_1, x_2, \ldots, x_n)^T \neq 0$ such that

$$\sum_{i_2,\ldots,i_m \in [n]} d_i a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} = 0 \quad \text{for all } i \in [n],$$

which is equivalent to

$$\sum_{i_2,\ldots,i_m \in [n]} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} = 0 \quad \text{for all } i \in [n],$$

which implies $0 \in \sigma(\mathcal{A})$, that is, $\mathcal{A}$ is singular. The proof is completed. □

By applying Lemmas 2, 3, and 4, we can now reveal the relationship of p-norm SDD tensors and strong $\mathcal{H}$-tensors.

Theorem 4 *Let $\mathcal{A} \in \mathbb{C}^{[m,n]}$. If $\mathcal{A}$ is a p-norm SDD tensor, then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor.*

Proof By Lemma 2 the theorem will be proved if we can show that $\rho(\mathcal{I} - D_{\mathcal{M}(\mathcal{A})}^{-1}\mathcal{M}(\mathcal{A})) < 1$. Assume, on the contrary, that there exists $\lambda \in \sigma(\mathcal{I} - D_{\mathcal{M}(\mathcal{A})}^{-1}\mathcal{M}(\mathcal{A}))$ such that $|\lambda| \geq 1$. Then, by Lemma 3,

$$0 \in \sigma\big(\lambda\mathcal{I} - \mathcal{I} + D_{\mathcal{M}(\mathcal{A})}^{-1}\mathcal{M}(\mathcal{A})\big).$$

According to the right distributive law for tensors, we have

$$\lambda\mathcal{I} - \mathcal{I} + D_{\mathcal{M}(\mathcal{A})}^{-1}\mathcal{M}(\mathcal{A}) = D_{\mathcal{M}(\mathcal{A})}^{-1}\big((\lambda - 1)D_{\mathcal{M}(\mathcal{A})}\mathcal{I} + \mathcal{M}(\mathcal{A})\big),$$

which, together with Lemma 4, yields

$$0 \in \sigma\big((\lambda - 1)D_{\mathcal{M}(\mathcal{A})}\mathcal{I} + \mathcal{M}(\mathcal{A})\big). \tag{8}$$

However, there exists an entrywise positive vector $x > 0$ such that $\|x\|_q \le 1$, and inequality (4) holds because $\mathcal{A}$ is a p-norm SDD tensor. By inequality (4) we have

$$\begin{aligned} x_i^{m-1}\big|(\lambda-1)|a_{i\cdots i}| + |a_{i\cdots i}|\big| &= x_i^{m-1}|\lambda||a_{i\cdots i}| \\ &\ge x_i^{m-1}|a_{i\cdots i}| \\ &> r_i^p(\mathcal{A}) \\ &= r_i^p\big((\lambda-1)D_{\mathcal{M}(\mathcal{A})}\mathcal{I} + \mathcal{M}(\mathcal{A})\big), \end{aligned}$$

which implies that $(\lambda-1)D_{\mathcal{M}(\mathcal{A})}\mathcal{I} + \mathcal{M}(\mathcal{A})$ is a p-norm SDD tensor. By Theorem 3 we have

$$0 \notin \sigma\big((\lambda - 1)D_{\mathcal{M}(\mathcal{A})}\mathcal{I} + \mathcal{M}(\mathcal{A})\big),$$

which contradicts (8). The proof is completed. □

4 Eigenvalue localization

Similarly to matrices, a nonsingular class of tensors can lead to an eigenvalue localization result. In this section, we illustrate this fact with the class of p-norm SDD tensors.

Theorem 5 *Let* $\mathcal{A} \in \mathbb{C}^{[m,n]}$ *and* $p \in [1,\infty]$. *Then*

$$\sigma(\mathcal{A}) \subseteq \Phi^p(\mathcal{A}).$$

When $p = 1$, $\Phi^1(\mathcal{A}) = \Gamma(\mathcal{A})$. *When* $p > 1$,

$$\Phi^p(\mathcal{A}) = \left\{ z \in \mathbb{C} : \sum_{i\in[n]} \left[\frac{r_i^p(\mathcal{A})}{|z - a_{i\cdots i}|}\right]^{\frac{p}{(m-1)(p-1)}} \ge 1 \right\}.$$

Proof Clearly, if $p = 1$, $\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$ can be easily obtained from Theorem 1. If $p > 1$, suppose that there exists $\lambda \in \sigma(\mathcal{A})$ such that $\lambda \notin \Phi^p(\mathcal{A})$, that is,

$$\sum_{i\in[n]} \left[\frac{r_i^p(\mathcal{A})}{|\lambda - a_{i\cdots i}|}\right]^{\frac{p}{(m-1)(p-1)}} < 1. \tag{9}$$

Let $\mathcal{B} := \lambda\mathcal{I} - \mathcal{A} = (b_{i_1 i_2 \cdots i_m})$. Since $\lambda \in \sigma(\mathcal{A})$, this, together with Lemma 3, yields that $\mathcal{B}$ is surely singular. On the other hand, by the definition of $\mathcal{B}$ we obtain $r_i^p(\mathcal{B}) = r_i^p(\mathcal{A})$ and $|b_{i\cdots i}| = |\lambda - a_{i\cdots i}|$ for all $i \in [n]$, so that (9) becomes

$$\sum_{i\in[n]} \left[\frac{r_i^p(\mathcal{B})}{|b_{i\cdots i}|}\right]^{\frac{p}{(m-1)(p-1)}} < 1,$$

which implies

$$\left\|\delta_p(\mathcal{B})\right\|_q < 1,$$

where q is Hölder's complement of p, which means that $\mathcal{B}$ is a p-norm SDD tensor. By Theorem 3, $\mathcal{B}$ is nonsingular. This leads to a contradiction. □

Remark 6 When $m = 2$, Theorem 5 reduces to the result of [20].

Remark 7 In particular, taking $p = \infty$, we have

$$\Phi^{\infty}(\mathcal{A}) = \left\{ z \in \mathbb{C} : \sum_{i \in [n]} \left[\frac{\max_{\substack{i_2,\ldots,i_m \in [n], \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|}{|z - a_{i\cdots i}|} \right]^{\frac{1}{(m-1)}} \geq 1 \right\}.$$

It follows from Theorem 5 that

$$\sigma(\mathcal{A}) \subseteq \Phi^{p}(\mathcal{A}),$$

but since this conclusion holds for any $p \in [1, \infty]$, we immediately have the following theorem.

Theorem 6 *Let $\mathcal{A} \in \mathbb{C}^{[m,n]}$. Then*

$$\sigma(\mathcal{A}) \subseteq \bigcap_{p \in [1,\infty]} \Phi^{p}(\mathcal{A}),$$

where $\Phi^{p}(\mathcal{A})$ is defined as in Theorem 5.

The following example shows that $\Phi^{\infty}(\mathcal{A})$ is tighter than $\Omega(\mathcal{A})$ in some case.

Example 2 Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{[3,2]}$ and $\mathcal{B} = (b_{ijk}) \in \mathbb{R}^{[3,2]}$ with elements defined as follows:

$$a_{111} = 1.96, \qquad a_{222} = 16, \quad \text{and the remaining} \quad a_{ijk} = 1,$$
$$b_{111} = 18, \qquad b_{222} = 16, \quad \text{and the remaining} \quad b_{ijk} = 1,$$

respectively. The eigenvalue inclusion regions $\Omega(\mathcal{A})$ ($\Omega(\mathcal{B})$), $\Phi^{\infty}(\mathcal{A})$ ($\Phi^{\infty}(\mathcal{B})$) and the exact eigenvalues of $\mathcal{A}$ ($\mathcal{B}$) are drawn in Figure 1A (Figure 1B), where $\Omega(\mathcal{A})$ ($\Omega(\mathcal{B})$), $\Phi^{\infty}(\mathcal{A})$ ($\Phi^{\infty}(\mathcal{B})$) and the exact eigenvalues of $\mathcal{A}$ ($\mathcal{B}$) are respectively denoted by the blue area, the green area, and red asterisks. In addition, by Corollary 7.8 in [10] we have

$$\sigma(\mathcal{A}) = \{15.6288, 16.7844, 1.7534 + 0.4687i, 1.7534 - 0.4687i\}$$

and

$$\sigma(\mathcal{B}) = \{20.1811, 17.3673, 15.2258 + 0.5245i, 15.2258 - 0.5245i\}.$$

It is easy to see that $\Phi^{\infty}(\mathcal{A}) \subseteq \Omega(\mathcal{A})$, but $\Omega(\mathcal{B}) \subseteq \Phi^{\infty}(\mathcal{B})$.

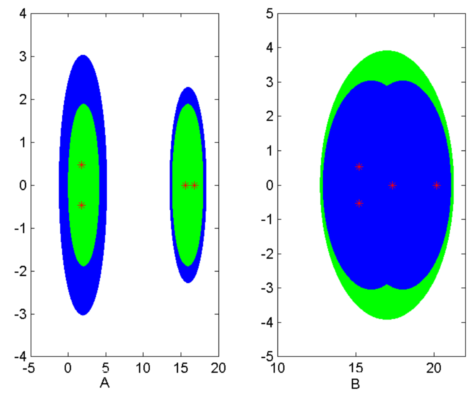

Figure 1 The eigenvalue inclusion sets of tensors $\mathcal{A}$ and $\mathcal{B}$ for Example 2.

5 Determining the positive (semi)definiteness for an even-order real symmetric tensor

By applying the results obtained in Sections 3 and 4 we give a sufficient condition for the positive (semi)definiteness of an even-order real symmetric tensor.

Theorem 7 *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an even-order symmetric tensor with $a_{i\cdots i} > 0$ for all $i \in [n]$. If $\mathcal{A}$ is a p-norm SDD tensor, then $\mathcal{A}$ is positive definite.*

Proof The theorem follows immediately from Lemma 1 and Theorem 4. □

Theorem 8 *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an even-order symmetric tensor with $a_{i\cdots i} > 0$ for all $i \in [n]$, and $p \in [1, \infty]$. If*

$$\left\| \delta_p(\mathcal{A}) \right\|_q \le 1,$$

where q is Hölder's complement of p. Then $\mathcal{A}$ is positive semidefinite.

Proof If $p = 1$, then

$$\left\| \delta_1(\mathcal{A}) \right\|_\infty = \max_{i \in [n]} \left(\frac{r_i(\mathcal{A})}{|a_{ii\cdots i}|} \right)^{\frac{1}{m-1}} \le 1,$$

which implies that $\mathcal{A}$ is diagonally dominant. By Theorem 3 of [24] it follows that $\mathcal{A}$ is positive semidefinite.

If $p > 1$, then let λ be an H-eigenvalue of $\mathcal{A}$, and $\lambda < 0$. By Theorem 5 we have $\lambda \in \Phi^p(\mathcal{A})$, which implies that

$$\sum_{i \in [n]} \left[\frac{r_i^p(\mathcal{A})}{|\lambda - a_{i\cdots i}|} \right]^{\frac{p}{(m-1)(p-1)}} \ge 1.$$

However, it follows from $a_{ii\cdots i} > 0$ for all $i \in [n]$ that

$$\sum_{i \in [n]} \left[\frac{r_i^p(\mathcal{A})}{|a_{i\cdots i}|} \right]^{\frac{p}{(m-1)(p-1)}} > \sum_{i \in [n]} \left[\frac{r_i^p(\mathcal{A})}{|\lambda - a_{i\cdots i}|} \right]^{\frac{p}{(m-1)(p-1)}} \ge 1,$$

which implies

$$\left\|\delta_p(\mathcal{A})\right\|_q > 1,$$

which contradicts $\|\delta_p(\mathcal{A})\|_q \le 1$. This completes the proof. □

Example 3 Let $\mathcal{A} \in \mathbb{R}^{[4,2]}$ be a symmetric tensor with elements defined as follows:

$$a_{1111} = 21, \qquad a_{1222} = a_{2122} = a_{2212} = a_{2221} = -3, \qquad a_{2222} = 8,$$

and the remaining $a_{i_1 i_2 i_3 i_4} = 0$. By computation,

$$|a_{2222}| = 8 < 9 = r_2^{\Delta_2}(\mathcal{A}) = \sum_{\substack{(i_2,i_3,i_4)\in\Delta_2,\\ \delta_{2i_2i_3i_4}=0}} |a_{2i_2i_3i_4}|,$$

which means that the statement (I) of Proposition 2.4 in [1] does not hold, and hence we cannot use Proposition 2.4 in [1] to determine the positive definiteness of $\mathcal{A}$. However, it is easy to verify that $\mathcal{A}$ is a ∞-norm SDD tensor. By Theorem 7, $\mathcal{A}$ is positive definite.

6 Conclusions

In this paper, we proposed a new class of nonsingular tensors (p-norm SDD tensors) and proved that the class of p-norm SDD tensors is a subclass of strong $\mathcal{H}$-tensors. Furthermore, we presented a new eigenvalue inclusion set, which is tighter than those provided by Li *et al.* [1] in some case. Based on this set, we presented a checkable sufficient condition for the positive (semi)definiteness of an even-order symmetric tensor.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to this work. Both authors read and approved the final manuscript.

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