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Lyapunov's type inequalities for hybrid fractional differential equations

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Abstract

We investigate new results about Lyapunov-type inequalities by considering hybrid fractional boundary value problems. We give necessary conditions for the existence of nontrivial solutions for a class of hybrid boundary value problems involving Riemann-Liouville fractional derivative of order $2 < \alpha \leq 3$. The investigation is based on a construction of Green's functions and on finding its corresponding maximum value. In order to illustrate the results, we provide numerical examples.

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1 Introduction and preliminaries

It is well known that various type integral inequalities play a dominant role in the study of quantitative properties of solutions of differential and integral equations. One of them is Lyapunov-type inequality which has been proved to be very useful in studying the zeros of solutions of differential equations. The well-known Lyapunov result [1] states that if the boundary value problem

$$\begin{cases} y''(t) + q(t)y(t) = 0, \quad a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{1.1}$$

has a nontrivial solution, where q is a real and continuous function, then

$$\int_a^b |q(s)|\,ds > \frac{4}{b-a}. \tag{1.2}$$

This result found many practical applications in differential equations (oscillation theory, disconjugacy, eigenvalue problems, *etc.*); see, for instance, [2–7] and references therein.

The search for Lyapunov-type inequalities in which the starting differential equation is constructed via fractional differential operators has begun very recently. The first work in this direction is due to Ferreira [8], where he derived a Lyapunov-type inequality for

Riemann-Liouville fractional boundary value problem

$$\begin{cases} D^{\alpha}y(t) + q(t)y(t) = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{1.3}$$

where D^{α} is the Riemann-Liouville fractional derivative of order $1 < \alpha \leq 2$ and $q : [a,b] \to \mathbb{R}$ is a continuous function. It has been proved that if (1.3) has a nontrivial solution then

$$\int_a^b \left|q(s)\right| ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{1.4}$$

Clearly, if we let $\alpha = 2$ in the above inequality, one obtains Lyapunov's standard inequality. Ferreira also in [9], was obtained a Lyapunov-type inequality for the Caputo fractional boundary value problem

$$\begin{cases} {}^{\mathrm{C}}D^{\alpha}y(t) + q(t)y(t) = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{1.5}$$

where ${}^{\mathrm{C}}D^{\alpha}$ is the Caputo fractional derivative of order $1 < \alpha \leq 2$. It has been proved in [9] that if (1.5) has a nontrivial solution then

$$\int_a^b \left|q(s)\right| ds > \frac{\Gamma(\alpha)\alpha^{\alpha}}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{1.6}$$

Similarly if we let $\alpha = 2$ in (1.6), one obtains Lyapunov's classical inequality (1.2).

In [10], Jleli and Samet considered the fractional differential equation

$${}^{\mathrm{C}}D^{\alpha}y(t) + q(t)y(t) = 0, \quad a < t < b, \tag{1.7}$$

with the mixed boundary conditions

$$y(a) = 0 = y'(b) \tag{1.8}$$

or

$$y'(a) = 0 = y(b). \tag{1.9}$$

For boundary conditions (1.8) and (1.9), two Lyapunov-type inequalities were established, respectively, as follows:

$$\int_a^b (b-s)^{\alpha-2}\left|q(s)\right| ds > \frac{\Gamma(\alpha)}{\max\{\alpha-1, 2-\alpha\}(b-a)} \tag{1.10}$$

and

$$\int_a^b (b-s)^{\alpha-1}\left|q(s)\right| ds > \Gamma(\alpha). \tag{1.11}$$

Recently Rong and Bai [11] considered (1.7) under boundary condition

$$y(a) = 0, \qquad {}^{C}D^{\beta} y(t) = 0, \quad 0 < \beta \leq 1, \tag{1.12}$$

and established the following Lyapunov-type inequality:

$$\int_a^b (b-s)^{\alpha-\beta-1} \left|q(s)\right| ds > \frac{(b-a)^{-\beta}}{\max\{\frac{1}{\Gamma(\alpha)} - \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{2-\alpha}{\alpha-1} \cdot \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\}}. \tag{1.13}$$

For other work on Lyapunov-type inequalities for fractional boundary value problems we refer the reader to [12–14].

The aim of this manuscript is to establish some Lyapunov's type inequalities for hybrid fractional boundary value problem

$$\begin{cases} D_a^{\alpha}[\frac{y(t)}{f(t,y(t))} - \sum_{i=1}^{n} I_a^{\beta} h_i(t,y(t))] + g(t)y(t) = 0, \quad t \in (a,b), \\ y(a) = y'(a) = y(b) = 0, \end{cases} \tag{1.14}$$

where D_a^{α} denotes the Riemann-Liouville fractional derivative of order $\alpha \in (2,3]$ starting from a point a, the functions $y \in C([a,b],\mathbb{R})$, $g \in L^1((a,b),\mathbb{R})$, $f \in C^1([a,b] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $h_i \in C([a,b] \times \mathbb{R}, \mathbb{R})$, $\forall i = 1,2,\ldots,n$, and I_a^{β} is the Riemann-Liouville fractional integral of order $\beta \geq \alpha$ with the lower limit at a point a.

We recall the basic definitions, [15–17].

Definition 1.1 The fractional integral of order q with the lower limit a for a function f is defined as

$$I_a^q f(t) = \frac{1}{\Gamma(q)} \int_a^t \frac{f(s)}{(t-s)^{1-q}} \, ds, \quad t > a, q > 0,$$

provided the right-hand side is point-wise defined on $[a,\infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(q) = \int_0^{\infty} t^{q-1} e^{-t} \, dt$.

Definition 1.2 The Riemann-Liouville fractional derivative with the lower limit a of order $q > 0$, $n-1 < q < n$, $n \in N$, is defined as

$$D_a^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-q-1} f(s) \, ds,$$

where the function f has absolutely continuous derivative up to order $(n-1)$.

2 Main results

We consider two cases: (I) $h_i = 0$, $i = 1,2,\ldots,n$, and (II) $h_i \neq 0$, $i = 1,2,\ldots,n$.

2.1 Case I: $h_i = 0$, $i = 1,2,\ldots,n$

We consider problem (1.14) with $h_i(t,\cdot) = 0$ for all $t \in [a,b]$. For $\alpha \in (2,3]$, we first construct a Green's function for the following boundary value problem:

$$\begin{cases} D_a^{\alpha}[\frac{y(t)}{f(t,y(t))}] + g(t)y(t) = 0, \quad t \in (a,b), \\ y(a) = y'(a) = y(b) = 0, \end{cases} \tag{2.1}$$

with the assumption that f is continuously differentiable and $f(t,y(t)) \neq 0$ for all $t \in [a,b]$.

Lemma 2.1 *Let $y \in AC([a,b],\mathbb{R})$ be a solution of problem* (2.1). *Then the function y satisfies the following integral equation*:

$$y = f(t,y)\int_a^b G(t,s)g(s)y(s)\,ds, \tag{2.2}$$

where $G(t,s)$ is the Green's function defined by

$$G(t,s) = \begin{cases} \frac{(b-s)^{\alpha-1}(t-a)^{\alpha-1}}{\Gamma(\alpha)(b-a)^{\alpha-1}}, & a \le t \le s \le b, \\ \frac{(b-s)^{\alpha-1}(t-a)^{\alpha-1}}{\Gamma(\alpha)(b-a)^{\alpha-1}} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & a \le s \le t \le b. \end{cases} \tag{2.3}$$

Proof Taking the Riemann-Liouville fractional integral of order α from a to t of both sides of (2.1), we obtain

$$\frac{y(t)}{f(t,y(t))} = c_0(t-a)^{\alpha-1} + c_1(t-a)^{\alpha-2} + c_2(t-a)^{\alpha-3} - I_a^\alpha g(t)y(t). \tag{2.4}$$

Putting $t = a$ in (2.4), we get a constant $c_2 = 0$. Differentiating both sides of equation (2.4) with respect to t, we have

$$\frac{f(t,y(t))y'(t) - y(t)f_t(t,y(t))}{f^2(t,y(t))} = c_0(\alpha-1)(t-a)^{\alpha-2} + c_1(\alpha-2)(t-a)^{\alpha-3} - I_a^{\alpha-1}g(t)y(t).$$

Applying the conditions of problem (2.1), the constant c_1 is vanished. Replacing t by b with $c_1 = c_2 = 0$ in (2.4) and using the last condition of (2.1), the constant c_0 is obtained as follows:

$$c_0 = \frac{I_a^\alpha g(b)y(b)}{(b-a)^{\alpha-1}}.$$

Hence a solution y of problem (2.1) satisfies the following integral equation:

$$y(t) = f\bigl(t,y(t)\bigr)\left[\int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}g(s)y(s)\,ds - \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)y(s)\,ds\right]. \tag{2.5}$$

By the definition of the Green's function as in (2.3), equation (2.5) can be written in the form of (2.2). The proof is completed. □

Lemma 2.2 *The Green's function defined in* (2.3) *satisfies*:

(i) $G(t,s) \ge 0$, $\forall t,s \in [a,b]$.
(ii) $G(t,s) \le H(s) := \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha-1)}$.
(iii) $\max_{s\in[a,b]} H(s) = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha-1)}$.

Proof First of all, we define the following two functions:

$$g_1(t,s) = \frac{(b-s)^{\alpha-1}(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}, \quad a \le t \le s \le b,$$

$$g_2(t,s) = \frac{(b-s)^{\alpha-1}(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-1}, \quad a \leq s \leq t \leq b.$$

(i) It is obvious that $g_1(t,s) \geq 0$. To show that $g_2(t,s) \geq 0$, we use the following observation of Ferreira in [8]:

$$a + \frac{(s-a)(b-a)}{t-a} \geq s \quad \text{is equivalent to } s \geq a.$$

Then we have

$$\begin{aligned}
(t-s)^{\alpha-1} &= (t-a+a-s)^{\alpha-1} \\
&= \left[(t-a)\left(1+\frac{a-s}{t-a}\right)\right]^{\alpha-1} \\
&= \left[(b-a)\left(1+\frac{a-s}{t-a}\right)\right]^{\alpha-1} \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} \\
&= \left[b-\left(a+\frac{(b-a)(s-a)}{t-a}\right)\right]^{\alpha-1} \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}},
\end{aligned} \tag{2.6}$$

which leads to

$$\begin{aligned}
g_2(t,s) &= \frac{(b-s)^{\alpha-1}(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-1} \\
&= \frac{(b-s)^{\alpha-1}(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} - \left[b-\left(a+\frac{(b-a)(s-a)}{t-a}\right)\right]^{\alpha-1} \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} \\
&\geq \frac{(b-s)^{\alpha-1}(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} - (b-s)^{\alpha-1}\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} \\
&= 0.
\end{aligned}$$

Therefore, part (i) is proved.

(ii) For $a \leq s \leq t \leq b$, we have

$$\begin{aligned}
g_2(t,s) &= \left[\frac{(b-s)(t-a)}{b-a}\right]^{\alpha-1} - (t-s)^{\alpha-1} \\
&= (\alpha-1)\int_{t-s}^{[(b-s)(t-a)]/(b-a)} x^{\alpha-2}\,dx \\
&\leq (\alpha-1)\left[\frac{(b-s)(t-a)}{b-a}\right]^{\alpha-2}\left[\frac{(b-s)(t-a)}{b-a} - (t-s)\right] \\
&\leq (\alpha-1)\frac{(b-s)^{\alpha-1}(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} \\
&\leq (\alpha-1)(b-s)^{\alpha-1},
\end{aligned}$$

and consequently

$$\frac{g_2(t,s)}{\Gamma(\alpha)} \leq \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha-1)}.$$

For $a \leq t \leq s \leq b$, we have

$$\begin{aligned} g_1(t,s) &= \left[\frac{(b-s)(t-a)}{b-a}\right]^{\alpha-1} \\ &\leq \frac{(b-s)^{\alpha-1}(s-a)^{\alpha-1}}{(b-a)^{\alpha-1}} \\ &= (\alpha-1)\int_0^{[(b-s)(s-a)]/(b-a)} x^{\alpha-2}\,dx \\ &\leq (\alpha-1)\frac{(b-s)^{\alpha-2}(s-a)^{\alpha-2}}{(b-a)^{\alpha-2}}\left[\frac{(b-s)(s-a)}{b-a}\right] \\ &= (\alpha-1)\frac{(b-s)^{\alpha-1}(s-a)^{\alpha-1}}{(b-a)^{\alpha-1}} \\ &\leq (\alpha-1)(b-s)^{\alpha-1}, \end{aligned}$$

which yields

$$\frac{g_1(t,s)}{\Gamma(\alpha)} \leq \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha-1)}.$$

It follows that

$$G(t,s) \leq H(s), \quad \forall s,t \in [a,b],$$

which is the proof of part (ii).

(iii) This is obvious, since $H'(s) < 0$. The proof is complete. □

Theorem 2.1 *The necessary condition for the existence of a nontrivial solution for the boundary value problem* (2.1) *is*

$$\frac{\Gamma(\alpha-1)}{\|f\|} \leq \int_a^b (b-s)^{\alpha-1}\big|g(s)\big|\,ds, \tag{2.7}$$

where $\|f\| = \sup_{t\in[a,b],y\in\mathbb{R}} |f(t,y)|$.

Proof From Lemma 2.1, the solution of (2.1) satisfies the following integral equation:

$$y(t) = f\big(t,y(t)\big)\left(\int_a^b G(t,s)g(s)y(s)\,ds\right).$$

The continuity of functions y and f on their compact domains yield

$$\|y\| \leq \|f\|\left(\int_a^b \big|G(t,s)\big|\big|g(s)\big|\big|y(s)\big|\,ds\right).$$

Simplifying above inequality, we get

$$1 \leq \|f\|\left(\int_a^b H(s)\big|g(s)\big|\,ds\right). \tag{2.8}$$

Applying the result in Lemma 2.2, the desired inequality in (2.7) is obtained. □

Corollary 2.1 *The necessary condition for the existence of a nontrivial solution for the boundary value problem* (2.1) *is*

$$\frac{\Gamma(\alpha-1)}{\|f\|}(b-a)^{1-\alpha} \leq \|g\|_{L^1}. \tag{2.9}$$

Corollary 2.2 *Consider the fractional Sturm-Liouville problem given by*

$$\begin{cases} D_0^\alpha[\frac{y(t)}{f(t,y(t))}] + \lambda y(t) = 0, & \alpha \in (2,3], t \in (0,1), \\ y(0) = y'(0) = y(1) = 0, \end{cases} \tag{2.10}$$

where $f(t,y(t)) \neq 0$ *for all* $t \in [0,1]$ *and* $\lambda \in \mathbb{R}$. *The necessary condition for the existence of a nontrivial solution for the boundary value problem* (2.10) *is*

$$|\lambda| \geq \frac{\alpha\Gamma(\alpha-1)}{\|f\|}. \tag{2.11}$$

Proof From the Lyapunov-type inequality in Theorem 2.1 and replacing the values $a = 0$, $b = 1$, and $g(t) \equiv \lambda$ for $t \in [0,1]$, the inequality (2.7) becomes

$$\frac{\Gamma(\alpha-1)}{\|f\|} \leq \int_0^1 (1-s)^{\alpha-1}|\lambda|\,ds = \frac{1}{\alpha}|\lambda|.$$

This completes the proof. □

Corollary 2.3 *Consider the fractional Sturm-Liouville problem of the form*

$$\begin{cases} D_a^\alpha[\frac{y(t)}{f(t,y(t))}] + \lambda y(t) = 0, & \alpha \in (2,3], t \in (a,b), \\ y(a) = y'(a) = y(b) = 0, \end{cases} \tag{2.12}$$

where $f(t,y(t)) \neq 0$ *for all* $t \in [a,b]$ *and* $\lambda \in \mathbb{R}$. *The necessary condition for the existence of a nontrivial solution for the boundary value problem* (2.12) *is*

$$|\lambda| \geq \frac{\Gamma(\alpha-1)}{(b-a)^\alpha\|f\|}. \tag{2.13}$$

Proof From Corollary 2.1, we get

$$\frac{\Gamma(\alpha-1)}{\|f\|}(b-a)^{1-\alpha} \leq \int_a^b |\lambda|\,ds = (b-a)|\lambda|,$$

which is the inequality in (2.13). This completes the proof. □

Example 2.1 Consider the following boundary value problem of the hybrid fractional differential equation:

$$\begin{cases} D_1^{5/2}[\frac{y(t)}{(t+1)+\frac{|y(t)|+3}{|y(t)|+5}}] + \lambda y(t) = 0, & t \in (1,3), \\ y(1) = y'(1) = y(3) = 0. \end{cases} \tag{2.14}$$

Here $\alpha = 5/2$, $a = 1$, $b = 3$, $f(t,y) = (t+1) + (|y|+3)/(|y|+5)$. We find that $\|f\| = 5$. Applying Corollary 2.3, we see that the necessary condition for the existence of a nontrivial solution for the boundary value problem (2.14) is

$$|\lambda| \geq 0.03133285342.$$

2.2 Case II: $h_i \not\equiv 0$, $i = 1, 2, \ldots, n$

In this section we will construct Lyapunov-type inequalities for the boundary value problem (1.14). We recall that f is continuously differentiable.

Lemma 2.3 *Let $y \in AC[a,b]$ be a solution of problem* (1.14). *Then the function y can be written as*

$$y(t) = f\big(t, y(t)\big)\left[\int_a^b G(t,s)g(s)y(s)\,ds - \sum_{i=1}^n \int_a^b G^*(t,s)h_i\big(s, y(s)\big)\,ds\right], \tag{2.15}$$

where $G(t,s)$ is defined as in (2.3) *and $G^*(t,s)$ is defined by*

$$G^*(t,s) = \begin{cases} \frac{(b-s)^{\beta-1}(t-a)^{\alpha-1}}{\Gamma(\beta)(b-a)^{\alpha-1}} - \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}, & a \leq s \leq t \leq b, \\ \frac{(b-s)^{\beta-1}(t-a)^{\alpha-1}}{\Gamma(\beta)(b-a)^{\alpha-1}}, & a \leq t \leq s \leq b. \end{cases} \tag{2.16}$$

Proof The general solution of problem (1.14) is given by

$$\begin{aligned} &\frac{y(t)}{f(t,y(t))} - \sum_{i=1}^n I_a^\beta h_i\big(t, y(t)\big) \\ &\quad = -I_a^\alpha g(t)y(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + c_3(t-a)^{\alpha-3}. \end{aligned} \tag{2.17}$$

By condition $y(a) = 0$, the constant $c_3 = 0$. Differentiating equation (2.17), we get

$$\begin{aligned} &\frac{f(t,y(t))y'(t) - y(t)f_t(t,y(t))}{f^2(t,y(t))} - \sum_{i=1}^n I_a^{\beta-1} h_i\big(t, y(t)\big) \\ &\quad = -I_a^{\alpha-1} g(t)y(t) + c_1(\alpha-1)(t-a)^{\alpha-2} + c_2(\alpha-2)(t-a)^{\alpha-3}. \end{aligned}$$

Replacing t by a to the above equation, we have $c_2 = 0$. Equation (2.17) becomes

$$y(t) = f\big(t, y(t)\big)\left[-I_a^\alpha g(t)y(t) + c_1(t-a)^{\alpha-1} + \sum_{i=1}^n I_a^\beta h_i\big(t, y(t)\big)\right]. \tag{2.18}$$

Since $y(b) = 0$, we have

$$c_1 = (b-a)^{1-\alpha}\left(I_a^\alpha g(b)y(b) - \sum_{i=1}^n I_a^\beta h_i\big(b, y(b)\big)\right).$$

Substituting the constant c_1 into equation (2.18), the solution of problem (1.14) is in the form

$$\begin{aligned}
y(t) &= f\big(t,y(t)\big)\left[-I_a^\alpha g(t)y(t) + \left(\frac{t-a}{b-a}\right)^{\alpha-1}\left(I_a^\alpha g(b)y(b) - \sum_{i=1}^n I_a^\beta h_i\big(b,y(b)\big)\right)\right.\\
&\quad \left. + \sum_{i=1}^n I_a^\beta h_i\big(t,y(t)\big)\right]\\
&= f\big(t,y(t)\big)\left[-\int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)y(s)\,ds\right.\\
&\quad + \left(\frac{t-a}{b-a}\right)^{\alpha-1}\left(\int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)y(s)\,ds\right.\\
&\quad \left.\left. - \sum_{i=1}^n \int_a^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} h_i\big(s,y(s)\big)\,ds\right) + \sum_{i=1}^n \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h_i\big(s,y(s)\big)\,ds\right]\\
&= f\big(t,y(t)\big)\left[\int_a^b G(t,s)g(s)y(s)\,ds - \sum_{i=1}^n \int_a^b G^*(t,s)h_i\big(s,y(s)\big)\,ds\right],
\end{aligned}$$

where the Green's functions $G(t,s)$ and $G^*(t,s)$ are defined by (2.3) and (2.16), respectively. The proof is completed. □

Lemma 2.4 *The Green's function* $G^*(t,s)$, *which is given by* (2.16), *satisfies the following inequalities*:

(i) $G^*(t,s) \geq 0$, $\forall t,s \in [a,b]$;

(ii) $G^*(t,s) \leq J(s) := \frac{(\alpha-1)(b-s)^{\beta-1}}{\Gamma(\beta)}$.

Also we have

(iii) $\max_{s\in[a,b]} J(s) = \frac{(\alpha-1)(b-a)^{\beta-1}}{\Gamma(\beta)}$.

Proof From Lemma 2.3, we define

$$\begin{aligned}
g_3(t,s) &= \frac{(b-s)^{\beta-1}(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\beta-1}, \quad a \leq s \leq t \leq b,\\
g_4(t,s) &= \frac{(b-s)^{\beta-1}(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}, \quad a \leq t \leq s \leq b.
\end{aligned}$$

It is obvious that $g_4(t,s) \geq 0$. By using (2.6) with replacing α by β, we have

$$\begin{aligned}
g_3(t,s) &= \frac{(b-s)^{\beta-1}(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\beta-1}\\
&\geq \frac{(b-s)^{\beta-1}(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} - \frac{(b-s)^{\beta-1}(t-a)^{\beta-1}}{(b-a)^{\beta-1}}\\
&= (b-s)^{\beta-1}\left[\left(\frac{t-a}{b-a}\right)^{\alpha-1} - \left(\frac{t-a}{b-a}\right)^{\beta-1}\right].
\end{aligned}$$

As $\beta \geq \alpha$, we deduce that $g_3(t,s) \geq 0$. Therefore, we have $G^*(t,s) \geq 0$ for all $t,s \in [a,b]$.

We omit the proofs of (ii) and (iii), since these are similar to that for the Green's function $G(t,s)$ in Lemma 2.2. □

In the following results will be used the following condition:
(H) $|h_i(t,y(t))| \leq |x_i(t)||y(t)|$ where $x_i \in C([a,b],\mathbb{R})$, $i = 1,2,\ldots,n$.

Theorem 2.2 *Assume that the condition* (H) *holds with* $[a,b] = [0,1]$. *The necessary condition for the existence of a nontrivial solution for problem* (1.14) *on* $[0,1]$ *is*

$$\Gamma(\alpha-1)\left(\frac{1}{\|f\|} - \frac{(\alpha-1)}{\Gamma(\beta+1)}\sum_{i=1}^{n}\|x_i\|\right) \leq \int_0^1 (1-s)^{\alpha-1}\bigl|g(s)\bigr|\,ds. \tag{2.19}$$

Proof From Lemma 2.3, the solution of problem (1.14) on $[0,1]$ is given by

$$y(t) = f\bigl(t,y(t)\bigr)\left[\int_0^1 G(t,s)g(s)y(s)\,ds - \sum_{i=1}^{n}\int_0^1 G^* h_i\bigl(s,y(s)\bigr)\,ds\right].$$

Since $y \in C([0,1],\mathbb{R})$ and $f \in C^1([0,1]\times\mathbb{R},\mathbb{R}\setminus\{0\})$, we get

$$\begin{aligned}
\bigl|y(s)\bigr| &\leq \|f\|\left[\int_0^1 \bigl|G(t,s)\bigr|\bigl|g(s)\bigr|\bigl|y(s)\bigr|\,ds + \sum_{i=1}^{n}\int_0^1 \bigl|G^*(t,s)\bigr|\bigl|h_i\bigl(s,y(s)\bigr)\bigr|\,ds\right] \\
&\leq \|f\|\left[\int_0^1 H(s)\bigl|g(s)\bigr|\bigl|y(s)\bigr|\,ds + \sum_{i=1}^{n}\int_0^1 J(s)\bigl|x_i(s)\bigr|\bigl|y(s)\bigr|\,ds\right] \\
&\leq \|f\|\left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-1)}\bigl|g(s)\bigr|\bigl|y(s)\bigr|\,ds\right. \\
&\quad \left. + \frac{(\alpha-1)}{\Gamma(\beta)}\sum_{i=1}^{n}\int_0^1 (1-s)^{\beta-1}\bigl|x_i(s)\bigr|\bigl|y(s)\bigr|\,ds\right],
\end{aligned}$$

which leads to

$$\|y\| \leq \|f\|\left[\frac{\|y\|}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-1}\bigl|g(s)\bigr|\,ds + \frac{(\alpha-1)\|y\|}{\Gamma(\beta+1)}\sum_{i=1}^{n}\|x_i\|\right].$$

Therefore, we deduce that the inequality in (2.19) holds. □

Corollary 2.4 *Assume that the condition* (H) *holds with* $[a,b] = [0,1]$. *Consider the problem*

$$\begin{cases} D_0^{\alpha}\left[\frac{y(t)}{f(t,y(t))} - \sum_{i=1}^{n} I_0^{\beta} h_i(t,y(t))\right] + \lambda y(t) = 0, & \alpha \in (2,3], t \in (0,1), \\ y(0) = y'(0) = y(1) = 0. \end{cases} \tag{2.20}$$

The necessary condition for the existence of a nontrivial solution for problem (2.20) *on* $[0,1]$ *is*

$$\alpha\Gamma(\alpha-1)\left(\frac{1}{\|f\|} - \frac{(\alpha-1)}{\Gamma(\beta+1)}\sum_{i=1}^{n}\|x_i\|\right) \leq |\lambda|. \tag{2.21}$$

Proof Setting the function $g(t) \equiv \lambda$ for $t \in [0,1]$ and applying Theorem 2.2, we obtain the following inequality:

$$\Gamma(\alpha-1)\left(\frac{1}{\|f\|}-\frac{(\alpha-1)}{\Gamma(\beta+1)}\sum_{i=1}^{n}\|x_i\|\right) \leq \int_0^1 (1-s)^{\alpha-1}|\lambda|\,ds = \frac{1}{\alpha}|\lambda|,$$

from which the result in (2.21) is proved. □

Theorem 2.3 *Suppose that the condition* (H) *holds. The necessary condition for the existence of a nontrivial solution for problem* (1.14) *on* $[a,b]$, *is*

$$\|g\|_{L^1} \geq \frac{\Gamma(\alpha-1)}{(b-a)^{\alpha-1}}\left(\frac{1}{\|f\|}-(\alpha-1)\frac{(b-a)^{\beta-1}}{\Gamma(\beta)}\sum_{i=1}^{n}\|x_i\|_{L^1}\right). \tag{2.22}$$

Proof From Lemmas 2.2 and 2.4, we have

$$\begin{aligned}\big|y(s)\big| &\leq \|f\|\left[\int_a^b \big|G(t,s)\big|\big|g(s)\big|\big|y(s)\big|\,ds + \sum_{i=1}^{n}\int_a^b \big|G^*(t,s)\big|\big|h_i\big(s,y(s)\big)\big|\,ds\right] \\ &\leq \|f\|\left[\int_a^b H(s)\big|g(s)\big|\big|y(s)\big|\,ds + \sum_{i=1}^{n}\int_a^b J(s)\big|x_i(s)\big|\big|y(s)\big|\,ds\right].\end{aligned}$$

Consequently the above inequality becomes

$$\|y\| \leq \|f\|\left[\frac{(b-a)^{\alpha-1}\|y\|}{\Gamma(\alpha-1)}\int_a^b |g(s)|\,ds + \frac{(\alpha-1)(b-a)^{\beta-1}\|y\|}{\Gamma(\beta)}\sum_{i=1}^{n}\int_a^b \big|x_i(s)\big|\,ds\right],$$

which leads to

$$1 \leq \|f\|\left[\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha-1)}\|g\|_{L^1} + \frac{(\alpha-1)(b-a)^{\beta-1}}{\Gamma(\beta)}\sum_{i=1}^{n}\|x_i\|_{L^1}\right].$$

Then the estimate in (2.22) holds. □

Corollary 2.5 *Let the condition* (H) *holds. Consider the fractional boundary value problem given by*

$$\begin{cases} D_a^{\alpha}\big[\frac{y(t)}{f(t,y(t))}-\sum_{i=1}^{n} I_a^{\beta}h_i(t,y(t))\big]+\lambda y(t)=0, \quad \alpha\in(2,3], t\in(a,b), \\ y(a)=y'(a)=y(b)=0. \end{cases} \tag{2.23}$$

The necessary condition for the existence of a nontrivial solution for problem (2.23) *on* $[a,b]$ *is*

$$|\lambda| \geq \frac{\Gamma(\alpha-1)}{(b-a)^{\alpha}}\left(\frac{1}{\|f\|}-\frac{(\alpha-1)(b-a)^{\beta-1}}{\Gamma(\beta)}\sum_{i=1}^{n}\|x_i\|_{L^1}\right). \tag{2.24}$$

Proof Applying the inequality in (2.22) with $g(s) = \lambda$, $s \in [a,b]$, it follows that

$$\int_a^b |\lambda|\,ds \ge \frac{\Gamma(\alpha-1)}{(b-a)^{\alpha-1}}\left(\frac{1}{\|f\|} - \frac{(\alpha-1)(b-a)^{\beta-1}}{\Gamma(\beta)}\sum_{i=1}^{n}\|x_i\|_{L^1}\right),$$

which implies the inequality in (2.24). □

Example 2.2 Consider the following boundary value problem of the hybrid fractional differential equation:

$$\begin{cases} D_0^{5/2}\left[\frac{y(t)}{(t+1)+\frac{2|y(t)|+1}{3|y(t)|+2}} - \sum_{i=1}^{3} I_0^{7/2}\frac{t^{(i+1)/(i+2)}y^2(t)}{1+|y(t)|}\right] + \lambda y(t) = 0, \quad t \in (0,\frac{1}{2}), \\ y(0) = y'(0) = y(\frac{1}{2}) = 0. \end{cases} \tag{2.25}$$

Here $\alpha = 5/2$, $\beta = 7/2$, $a = 0$, $b = 1/2$, $n = 3$, $h_i(t,y) = (t^{(i+1)/(i+2)}y^2)/(1+|y|)$, $i = 1,2,3$, $f(t,y) = (t+1) + (2|y|+1)/(3|y|+2)$. We find that $\|f\| = 13/6$, and $|h_i(t,y)| \le |t^{(i+1)/(i+2)}||y|$. Setting $x_i(t) = t^{(i+1)/(i+2)}$, $i = 1,2,3$, we have $\|x_1\|_{L^1} = 0.1889881575$, $\|x_2\|_{L^1} = 0.1698867308$, and $\|x_3\|_{L^1} = 0.1595414382$. Applying Corollary 2.5, we see that the necessary condition for the existence of a nontrivial solution for the boundary value problem (2.25) is

$$|\lambda| \ge 2.106444184.$$

Remark 2.1 The boundary value problem (1.14) can be rewritten by

$$\begin{cases} D_a^{\alpha}\left[\frac{y(t)}{f(t,y(t))}\right] + g(t)y(t) = \sum_{i=1}^{n} I_a^{\beta-\alpha} h_i(t,y(t)), \quad t \in (a,b), \\ y(a) = y'(a) = y(b) = 0, \end{cases} \tag{2.26}$$

which is a hybrid fractional integro-differential equation with boundary conditions. Therefore, all results can be applied.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this article. They read and approved the final manuscript.

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