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Monotonicity of the ratio for the complete elliptic integral and Stolarsky mean

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Abstract

In the article, we prove that the function $r\mapsto\mathcal{E}(r)/S_{9/2-p,p}(1,r')$ is strictly increasing on $(0,1)$ for $p\leq 7/4$ and strictly decreasing on $(0,1)$ for $p\in[2,9/4]$, where $r'=\sqrt{1-r^2}$, $\mathcal{E}(r)=\int_0^{\pi/2}\sqrt{1-r^2\sin^2(t)}\,dt$ is the complete elliptic integral of the second kind, and $S_{p,q}(a,b)=[q(a^p-b^p)/(p(a^q-b^q))]^{1/(p-q)}$ is the Stolarsky mean of a and b. As applications, we present several new bounds for $\mathcal{E}(r)$, the Toader mean $T(a,b)=(2/\pi)\int_0^{\pi/2}\sqrt{a^2\cos^2 t+b^2\sin^2 t}\,dt$, and the Toader-Qi mean $\mathrm{TQ}(a,b)=(2/\pi)\int_0^{\pi/2}a^{\cos^2\theta}b^{\sin^2\theta}\,d\theta$.

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Keywords: complete elliptic integral; Stolarsky mean; Toader mean; Toader-Qi mean

1 Introduction

For $r\in(0,1)$, $p,q\in\mathbb{R}$, and $a,b>0$, the Stolarsky mean $S_{p,q}(a,b)$ [1] and the complete elliptic integral $\mathcal{E}(r)$ [2] of the second kind are defined by

$$S_{p,q}(a,b)=\begin{cases}[\frac{q(a^p-b^p)}{p(a^q-b^q)}]^{1/(p-q)}, & a\neq b, p\neq q, pq\neq 0,\\ [\frac{a^p-b^p}{p(\log a-\log b)}]^{1/p}, & a\neq b, p\neq 0, q=0,\\ [\frac{a^q-b^q}{q(\log a-\log b)}]^{1/q}, & a\neq b, p=0, q\neq 0,\\ \exp(\frac{a^p\log a-b^p\log b}{a^p-b^p}-\frac{1}{p}), & a\neq b, p=q\neq 0,\\ \sqrt{ab}, & p=q=0,\\ a, & a=b,\end{cases}\tag{1.1}$$

$$\mathcal{E}(r)=\int_0^{\pi/2}\sqrt{1-r^2\sin^2(t)}\,dt,\tag{1.2}$$

respectively.

It is well known that the Stolarsky mean $S_{p,q}(a,b)$ is continuous on the domain $\{(p,q;a,b)|p,q\in\mathbb{R};a,b>0\}$, symmetric with respect to its parameters p and q or variables a and b, and strictly increasing with respect its parameter p or q and variable a or b. Many classical bivariate means are special cases of the Stolarsky mean $S_{p,q}(a,b)$, for example,

$S_{1,0}(a,b)=(a-b)/(\log a-\log b)=L(a,b)$ is the logarithmic mean;
$S_{1,1}(a,b)=(1/e)(a^a/b^b)^{1/(a-b)}=I(a,b)$ is the identic mean;
$S_{2,1}(a,b)=(a+b)/2=A(a,b)$ is the arithmetic mean;

$S_{3/2,1/2}(a,b) = (a+\sqrt{ab}+b)/3 = \mathrm{He}(a,b)$ is the Heronian mean;
$S_{2p,p}(a,b) = [A(a^p,b^p)]^{1/p} = A_p(a,b)$ is the p-order arithmetic mean;
$S_{3p/2,p/2}(a,b) = [\mathrm{He}(a^p,b^p)]^{1/p} = \mathrm{He}_p(a,b)$ is the p-order Heronian mean;
$S_{p,0}(a,b) = [L(a^p,b^p)]^{1/p} = L_p(a,b)$ is the p-order logarithmic mean;
$S_{p,p}(a,b) = [I(a^p,b^p)]^{1/p} = I_p(a,b)$ is the p-order identric mean.

The complete elliptic integral $\mathcal{E}(r)$ of the second kind can be expressed as

$$\mathcal{E}(r) = \frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right) = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-1/2)_n(1/2)_n}{(n!)^2}r^{2n},$$

where

$$F(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} \quad (-1 < x < 1)$$

is the Gaussian hypergeometric function, $(a)_n = \Gamma(a+n)/\Gamma(a)$ and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ $(x > 0)$ is the gamma function. It is well known that $\mathcal{E}(r)$ is strictly decreasing on $(0,1)$ and satisfies

$$\mathcal{E}(0^+) = \frac{\pi}{2}, \qquad \mathcal{E}(1^-) = 1.$$

Recently, the bounds for the complete elliptic integral $\mathcal{E}(r)$ of the second kind have attracted the interest of many researchers. In particular, many remarkable inequalities for $\mathcal{E}(r)$ can be found in the literature [3–11].

Let $r \in (0,1)$ and $r' = \sqrt{1-r^2}$. Then making use of (1.1) and (1.2) together with the power series formulas we have

$$\begin{aligned}
&\frac{2}{\pi}\mathcal{E}(r) - S_{p,q}(1,r') \\
&\quad = -\frac{p+q-9/2}{96}r^4 - \frac{p+q-9/2}{128}r^6 \\
&\qquad + \frac{8(p+q)(2p^2+2q^2-5p-5q-550)+19{,}845}{45\times 2^{14}}r^8 + o(r^8).
\end{aligned} \tag{1.3}$$

Let $p+q = 9/2$, then (1.3) becomes

$$\frac{2}{\pi}\mathcal{E}(r) - S_{9/2-p,p}(1,r') = \frac{(4p-7)(4p-11)}{5\times 2^{14}}r^8 + o(r^8). \tag{1.4}$$

Motivated by equation (1.4), we discuss the monotonicity of the function $r \mapsto \mathcal{E}(r)/S_{9/2-p,p}(1,r')$ for certain $p \in \mathbb{R}$, and present several new bounds for the complete elliptic integral of the second kind $\mathcal{E}(r)$ and the Toader mean

$$T(a,b) = \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2 t + b^2\sin^2 t}\,dt.$$

2 Lemmas

In order to prove our main results we need several lemmas, which we present in this section. Out of the next eight lemmas five were taken from other papers and the last three are due to the authors.

Lemma 2.1 (See [12], Theorem 2) *Let* $r \in (0,1)$ *and* $r' = \sqrt{1-r^2}$. *Then the function*

$$F(r) = \frac{1 - 2\mathcal{E}(r)/\pi}{1 - S_{5/2,2}(1, r')}$$

is strictly increasing from $(0,1)$ *onto* $(1, 25(\pi - 2)/(9\pi))$.

Lemma 2.2 (See [13], Theorem 3.1) *Let* $r \in (0,1)$ *and* $r' = \sqrt{1-r^2}$. *Then the function*

$$G(r) = \frac{1 - 2\mathcal{E}(r)/\pi}{1 - S_{11/4,7/4}(1, r')}$$

is strictly decreasing from $(0,1)$ *onto* $(11(\pi - 2)/(4\pi), 1)$.

Lemma 2.3 (See [14], equation (3.14), [15], Corollary 1.1) *Let* $c > 0$ *and* $a, b > 0$ *with* $a \neq b$. *Then the function* $p \mapsto S_{2c-p,p}(a,b)$ *is strictly increasing on* $(-\infty, c]$ *and strictly decreasing on* $[c, \infty)$.

Lemma 2.4 (See [15], Corollary 1.2) *Let* $c > 0$, $p \in (0, 2c)$, $a, b > 0$ *with* $a \neq b$ *and* $\theta(p,c)$ *be defined by*

$$\theta(p,c) = \lim_{r \to 0^+} S_{2c-p,p}(1,r) = \begin{cases} (\frac{2c-p}{p})^{1/(2p-2c)}, & p \neq c, \\ e^{-1/c}, & p = c. \end{cases} \tag{2.1}$$

Then the function $p \mapsto S_{2c-p,p}(a,b)/\theta(p,c)$ *is strictly decreasing on* $(0, c]$ *and strictly increasing on* $[c, 2c)$.

Lemma 2.5 (See [16], Theorem 5) *Let* $c > 0$ *and* $0 < x < y < z$. *Then the function* $p \mapsto S_{2c-p,p}(x,y)/S_{2c-p,p}(x,z)$ *is strictly decreasing on* $(-\infty, c]$ *and strictly increasing on* $[c, \infty)$.

Lemma 2.6 *Let* $r \in (0,1)$ *and* $r' = \sqrt{1-r^2}$. *Then the function*

$$r \mapsto \frac{2\mathcal{E}(r)/\pi}{S_{11/4,7/4}(1, r')}$$

is strictly increasing from $(0,1)$ *onto* $(1, 22/7\pi)$.

Proof Let $G(r)$ be defined by Lemma 2.2. Then we clearly see that

$$\frac{2\mathcal{E}(r)/\pi}{S_{11/4,7/4}(1,r')} = 1 + \frac{[1 - G(r)][1 - S_{11/4,7/4}(1,r')]}{S_{11/4,7/4}(1,r')}, \tag{2.2}$$

and both the functions $r \mapsto 1 - S_{11/4,7/4}(1,r')$ and $r \mapsto 1/S_{11/4,7/4}(1,r')$ are positive and strictly increasing on $(0,1)$. It follows from (1.1) and (1.2) together with Lemma 2.2 that

$$\lim_{r \to 0^+} \frac{2\mathcal{E}(r)/\pi}{S_{11/4,7/4}(1,r')} = 1, \qquad \lim_{r \to 1^-} \frac{2\mathcal{E}(r)/\pi}{S_{11/4,7/4}(1,r')} = \frac{22}{7\pi}, \tag{2.3}$$

and the function $r \mapsto 1 - G(r)$ is also positive and strictly increasing on $(0,1)$.

Therefore, Lemma 2.6 follows from (2.2) and (2.3) together with the monotonicity and positivity of the functions $r \mapsto 1 - S_{11/4,7/4}(1,r')$, $r \mapsto 1/S_{11/4,7/4}(1,r')$, and $r \mapsto 1 - G(r)$ on $(0,1)$. □

Lemma 2.7 *Let $r \in (0,1)$ and $r' = \sqrt{1-r^2}$. Then the function*

$$r \mapsto \frac{2\mathcal{E}(r)/\pi}{S_{5/2,2}(1,r')}$$

is strictly decreasing from $(0,1)$ onto $(25/8\pi,1)$.

Proof Let $F(r)$ be defined by Lemma 2.1. Then from (1.1) and (1.2) together with Lemma 2.1 we clearly see that

$$\lim_{r\to 0^+}\frac{2\mathcal{E}(r)/\pi}{S_{5/2,2}(1,r')} = 1, \qquad \lim_{r\to 1^-}\frac{2\mathcal{E}(r)/\pi}{S_{5/2,2}(1,r')} = \frac{25}{8\pi}, \tag{2.4}$$

$$\frac{2\mathcal{E}(r)/\pi}{S_{5/2,2}(1,r')} = 1 - \frac{[F(r)-1][1-S_{5/2,2}(1,r')]}{S_{5/2,2}(1,r')}, \tag{2.5}$$

and all the functions $r \mapsto F(r) - 1$, $r \mapsto 1 - S_{5/2,2}(1,r')$, and $r \mapsto 1/S_{5/2,2}(1,r')$ are positive and strictly increasing on $(0,1)$.

Therefore, Lemma 2.7 follows easily from (2.4) and (2.5) together with the monotonicity and positivity of the functions $r \mapsto F(r) - 1$, $r \mapsto 1 - S_{5/2,2}(1,r')$, and $r \mapsto 1/S_{5/2,2}(1,r')$ on $(0,1)$. □

Lemma 2.8 *Let $c > 0$, $r \in (0,1)$, and $r' = \sqrt{1-r^2}$. Then the function*

$$r \mapsto \frac{S_{2c-p_0,p_0}(1,r')}{S_{2c-p,p}(1,r')}$$

is strictly increasing (decreasing) on $(0,1)$ if $p < p_0 \le c$ ($p_0 < p \le c$).

Proof We clearly see that it suffices to prove that the function

$$r' \mapsto \frac{S_{2c-p_0,p_0}(1,r')}{S_{2c-p,p}(1,r')}$$

is strictly decreasing (increasing) on $(0,1)$ if $p < p_0 \le c$ ($p_0 < p \le c$).

Let $r'_1, r'_2 \in (0,1)$ with $r'_1 < r'_2$. Then $1 < 1/r'_2 < 1/r'_1$ and Lemma 2.5 leads to

$$\frac{S_{2c-p_0,p_0}(1,1/r'_2)}{S_{2c-p_0,p_0}(1,1/r'_1)} < (>) \frac{S_{2c-p,p}(1,1/r'_2)}{S_{2c-p,p}(1,1/r'_1)}$$

or

$$\frac{S_{2c-p_0,p_0}(1,1/r'_2)}{S_{2c-p,p}(1,1/r'_2)} < (>) \frac{S_{2c-p_0,p_0}(1,1/r'_1)}{S_{2c-p,p}(1,1/r'_1)} \tag{2.6}$$

if $p < p_0 \le c$ ($p_0 < p \le c$). The homogeneity of degree 1 for the Stolarsky mean and (2.6) give the desired result. □

3 Main results

Theorem 3.1 *Let $p \in (-\infty, 9/4]$, $r \in (0,1)$, $r' = \sqrt{1-r^2}$, and*

$$R_p(r) = \frac{2\mathcal{E}(r)/\pi}{S_{9/2-p,p}(1,r')}. \tag{3.1}$$

Then we have

(1) *the function $r \mapsto R_p(r)$ is strictly increasing on $(0,1)$ if and only if $p \in (-\infty, 7/4]$;*

(2) *the function $r \mapsto R_p(r)$ is strictly decreasing on $(0,1)$ if $p \in [2, 9/4]$.*

Proof (1) If the function $r \mapsto R_p(r)$ is strictly increasing on $(0,1)$, then we clearly see that

$$\lim_{r\to 0^+} \frac{d(\log R_p(r))/dr}{8r^7} \geq 0. \tag{3.2}$$

From (1.1), (1.2), (1.4), and (3.1) one has

$$\lim_{r\to 0^+} S_{9/2-p,p}\big(1,r'\big) = \lim_{r\to 0^+} \frac{2}{\pi}\mathcal{E}(r) = \lim_{r\to 0^+} R_p(r) = 1, \tag{3.3}$$

$$\frac{2}{\pi}\mathcal{E}(r) - S_{9/2-p,p}\big(1,r'\big) = \frac{(p-7/4)(p-11/4)}{5\times 2^{10}} r^8 + o\big(r^8\big). \tag{3.4}$$

It follows from (3.1), (3.3), (3.4), and L'Hôspital's rule that

$$\begin{aligned}
&\lim_{r\to 0^+} \frac{d(\log R_p(r))/dr}{d(2\mathcal{E}(r)/\pi - S_{9/2-p,p}(1,r'))/dr} \\
&\quad = \lim_{r\to 0^+} \frac{\log R_p(r)}{2\mathcal{E}(r)/\pi - S_{9/2-p,p}(1,r')} \\
&\quad = \lim_{r\to 0^+} \frac{\log R_p(r)}{(R_p(r)-1)S_{9/2-p,p}(1,r')} = 1,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\lim_{r\to 0^+} \frac{d(2\mathcal{E}(r)/\pi - S_{9/2-p,p}(1,r'))/dr}{8r^7} \\
&\quad = \lim_{r\to 0^+} \frac{2\mathcal{E}(r)/\pi - S_{9/2-p,p}(1,r')}{r^8} \\
&\quad = \frac{(p-7/4)(p-11/4)}{5\times 2^{10}}.
\end{aligned} \tag{3.6}$$

Note that

$$\begin{aligned}
\frac{d(\log R_p(r))/dr}{8r^7} = {}& \frac{d(\log R_p(r))/dr}{d(2\mathcal{E}(r)/\pi - S_{9/2-p,p}(1,r'))/dr} \\
& \times \frac{d(2\mathcal{E}(r)/\pi - S_{9/2-p,p}(1,r'))/dr}{8r^7}.
\end{aligned} \tag{3.7}$$

It follows from (3.2), (3.5)-(3.7) that

$$\left(p - \frac{7}{4}\right)\left(p - \frac{11}{4}\right) \geq 0. \tag{3.8}$$

Therefore, $p \in (-\infty, 7/4]$ follows from $p \in (-\infty, 9/4]$ and (3.8).

Next, we prove that the function $r \mapsto R_p(r)$ is strictly increasing on $(0,1)$ if $p \in (-\infty, 7/4]$. We divide the proof into two cases.

Case 1: $p = 7/4$. Then the desired result follows directly from Lemma 2.6.

Case 2: $p < 7/4$. Then $R_p(r)$ can be rewritten as

$$\begin{aligned} R_p(r) &= \frac{2\mathcal{E}(r)/\pi}{S_{9/2-7/4,7/4}(1,r')} \times \frac{S_{9/2-7/4,7/4}(1,r')}{S_{9/2-p,p}(1,r')} \\ &= \frac{2\mathcal{E}(r)/\pi}{S_{11/4,7/4}(1,r')} \times \frac{S_{9/2-7/4,7/4}(1,r')}{S_{9/2-p,p}(1,r')}. \end{aligned} \tag{3.9}$$

Therefore, the function $r \mapsto R_p(r)$ is strictly increasing on $(0,1)$ follows from Lemmas 2.6 and 2.8 together with (3.9).

(2) If $p \in [2, 9/4]$, then we divide the proof into two cases.

Case 1: $p = 2$. Then the desired result follows directly from Lemma 2.7.

Case 2: $p \in (2, 9/4]$. Then $R_p(r)$ can be expressed as

$$\begin{aligned} R_p(r) &= \frac{2\mathcal{E}(r)/\pi}{S_{9/2-2,2}(1,r')} \times \frac{S_{9/2-2,2}(1,r')}{S_{9/2-p,p}(1,r')} \\ &= \frac{2\mathcal{E}(r)/\pi}{S_{5/2,2}(1,r')} \times \frac{S_{9/2-2,2}(1,r')}{S_{9/2-p,p}(1,r')}. \end{aligned} \tag{3.10}$$

Therefore, the function $r \mapsto R_p(r)$ is strictly decreasing on $(0,1)$ follows from Lemmas 2.7 and 2.8 together with (3.10). □

From (1.1), (1.2), and Theorem 3.1 we get Corollary 3.2 immediately.

Corollary 3.2 *Let* $r \in (0,1)$, $r' = \sqrt{1-r^2}$, *and* $\theta(p,c)$ *be defined by* (2.1). *Then the double inequality*

$$S_{9/2-p,p}(1,r') < \frac{2}{\pi}\mathcal{E}(r) < \frac{2}{\pi\theta(p,9/4)} S_{9/2-p,p}(1,r')$$

holds for all $r \in (0,1)$ *and* $p \in (0, 7/4]$, *and the double inequality*

$$\frac{2}{\pi\theta(p,9/4)} S_{9/2-p,p}(1,r') < \frac{2}{\pi}\mathcal{E}(r) < S_{9/2-p,p}(1,r')$$

holds for all $r \in (0,1)$ *and* $p \in [2, 9/4]$.

Letting $p = 9/8, 3/2, 7/4; 2, 9/4$ and making use of (2.1), Lemmas 2.3 and 2.4, then Corollary 3.2 leads to Corollary 3.3.

Corollary 3.3 *Let* $r \in (0,1)$ *and* $r' = \sqrt{1-r^2}$. *Then the inequalities*

$$\begin{aligned} &\mathrm{He}_{9/4}(1,r') < A_{3/2}(1,r') < S_{11/4,7/4}(1,r') < \frac{2}{\pi}\mathcal{E}(r) \\ &\quad < \frac{22}{7\pi} S_{11/4,7/4}(1,r') < \frac{2^{5/3}}{\pi} A_{3/2}(1,r') < \frac{2 \times 3^{4/9}}{\pi}\mathrm{He}_{9/4}(1,r') \end{aligned}$$

and

$$\frac{2e^{4/9}}{\pi}I_{9/4}\big(1,r'\big) < \frac{25}{8\pi}S_{5/2,2}\big(1,r'\big) < \frac{2}{\pi}\mathcal{E}(r) < S_{5/2,2}\big(1,r'\big) < I_{9/4}\big(1,r'\big)$$

hold for all $r \in (0,1)$.

Corollary 3.4 *Let* $p \in (-\infty, 9/4]$, $r \in (0,1)$ *and* $r' = \sqrt{1-r^2}$. *Then the inequality*

$$\frac{2}{\pi}\mathcal{E}(r) > S_{9/2-p,p}\big(1,r'\big) \tag{3.11}$$

holds for all $r \in (0,1)$ *if and only if* $p \in (-\infty, 7/4]$, *and inequality* (3.11) *is reversed if* $p \in [2, 9/4]$.

Proof From Lemma 2.3 and Corollary 3.2 we clearly see that inequality (3.11) holds for all $r \in (0,1)$ if $p \in (-\infty, 7/4]$, and inequality (3.11) is reversed if $p \in [2, 9/4]$.

If inequality (3.11) holds for all $r \in (0,1)$, then (3.4) leads to the conclusion that $p \in (-\infty, 7/4]$. □

Let $p = -9/2, -9/4, 0, 9/8, 3/2, 7/4; 2, 9/4$. Then Lemma 2.3 and Corollary 3.4 lead to Corollary 3.5.

Corollary 3.5 *Let* $r \in (0,1)$ *and* $r' = \sqrt{1-r^2}$. *Then the inequalities*

$$\begin{aligned} A_{9/2}^{1/3}\big(1,r'\big)G^{2/3}\big(1,r'\big) &< G^{1/2}\big(1,r'\big)\,\mathrm{He}_{9/2}^{1/2}\big(1,r'\big) < L_{9/2}\big(1,r'\big) < \mathrm{He}_{9/4}\big(1,r'\big) \\ &< A_{3/2}\big(1,r'\big) < S_{11/4,7/4}\big(1,r'\big) < \frac{2}{\pi}\mathcal{E}(r) < S_{5/2,2}\big(1,r'\big) < I_{9/4}\big(1,r'\big) \end{aligned}$$

hold for all $r \in (0,1)$, *where* $G(a,b) = \sqrt{ab}$ *is the geometric mean of* a *and* b.

The Toader mean $T(a,b)$ [17] of two positive real numbers a and b is defined by

$$T(a,b) = \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2 t + b^2\sin^2 t}\,dt = \begin{cases} \frac{2}{\pi}\mathcal{E}(\sqrt{1-(\frac{b}{a})^2}), & a > b, \\ \frac{2}{\pi}\mathcal{E}(\sqrt{1-(\frac{a}{b})^2}), & a < b, \\ a, & a = b. \end{cases} \tag{3.12}$$

From (3.12) we clearly see that all the results given in Corollaries 3.2-3.5 can be restated by the Toader mean $T(a,b)$.

Remark 3.6 Let $\theta(p,c)$ be defined by (2.1). Then the double inequality

$$S_{9/2-p,p}(a,b) < T(a,b) < \frac{2}{\pi\theta(p,9/4)}S_{9/2-p,p}(a,b)$$

holds for all $a,b > 0$ with $a \neq b$ and $p \in (0, 7/4]$, and the double inequality

$$\frac{2}{\pi\theta(p,9/4)}S_{9/2-p,p}(a,b) < T(a,b) < S_{9/2-p,p}(a,b)$$

holds for all $a,b > 0$ with $a \neq b$ and $p \in [2, 9/4]$.

Remark 3.7 The inequalities

$$\begin{aligned}
&\mathrm{He}_{9/4}(a,b) < A_{3/2}(a,b) < S_{11/4,7/4}(a,b) < T(a,b) \\
&\quad < \frac{22}{7\pi} S_{11/4,7/4}(a,b) < \frac{2^{5/3}}{\pi} A_{3/2}(a,b) < \frac{2\times 3^{4/9}}{\pi}\,\mathrm{He}_{9/4}(a,b), \\
&\frac{2e^{4/9}}{\pi} I_{9/4}(a,b) < \frac{25}{8\pi} S_{5/2,2}(a,b) < T(a,b) < S_{5/2,2}(a,b) < I_{9/4}(a,b), \\
&A_{9/2}^{1/3}(a,b) G^{2/3}(a,b) < G^{1/2}(a,b)\,\mathrm{He}_{9/2}^{1/2}(a,b) < L_{9/2}(a,b) < \mathrm{He}_{9/4}(a,b) \\
&\quad < A_{3/2}(a,b) < S_{11/4,7/4}(a,b) < T(a,b) < S_{5/2,2}(a,b) < I_{9/4}(a,b), \\
&L_{3/2}(a,b) < \mathrm{He}_{3/4}(a,b) < A_{1/2}(a,b) < S_{11/12,7/12}(a,b) \\
&\quad < T_{1/3}(a,b) < S_{5/6,2/3}(a,b) < I_{3/4}(a,b)
\end{aligned} \tag{3.13}$$

hold for all $a, b > 0$ with $a \neq b$, where $T_{1/3}(a,b) = T^3(a^{1/3}, b^{1/3})$.

Remark 3.8 Let $p \in (-\infty, 9/4]$. Then the inequality

$$T(a,b) > S_{9/2-p,p}(a,b) \tag{3.14}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \in (-\infty, 7/4]$, and inequality (3.14) is reversed if $p \in [2, 9/4]$.

The Toader-Qi mean $\mathrm{TQ}(a,b)$ [17, 18] and Gauss arithmetic-geometric mean $\mathrm{AGM}(a,b)$ [19] of two positive real numbers a and b are, respectively, given by

$$\mathrm{TQ}(a,b) = \frac{2}{\pi}\int_0^{\pi/2} a^{\cos^2\theta} b^{\sin^2\theta}\,d\theta$$

and

$$\mathrm{AGM}(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

where a_n and b_n ($n \geq 1$) are defined by

$$a_1 = a, \qquad b_1 = b, \qquad a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}.$$

For all $a, b > 0$ with $a \neq b$, Yang *et al.* [20] proved that

$$\begin{aligned}
&\mathrm{AGM}(a,b) < L^{3/4}(a,b) A^{1/4}(a,b) < L_{3/2}(a,b) < \sqrt{L(a,b) I(a,b)} \\
&\quad < \mathrm{He}_{3/4}(a,b) < A_{1/2}(a,b) < I_{3/4}(a,b),
\end{aligned} \tag{3.15}$$

and the following inequalities can be found in Remark 3.6 of [21]:

$$\begin{aligned}
&L(a,b) < \mathrm{AGM}(a,b) < L^{3/4}(a,b) A^{1/4}(a,b) < \mathrm{TQ}(a,b) \\
&\quad < A_{1/2}(a,b) < T_{1/3}(a,b) < I_{3/4}(a,b).
\end{aligned} \tag{3.16}$$

In [22, 23], the authors proved that the double inequality

$$L_{3/2}(a,b) < \mathrm{TQ}(a,b) < \sqrt{L(a,b)I(a,b)} \tag{3.17}$$

holds for all $a, b > 0$ with $a \neq b$.

Remark 3.9 It follows from (3.13) and (3.15)-(3.17) that

$$\begin{aligned} L(a,b) &< \mathrm{AGM}(a,b) < L^{3/4}(a,b)A^{1/4}(a,b) < L_{3/2}(a,b) \\ &< \mathrm{TQ}(a,b) < \sqrt{L(a,b)I(a,b)} < \mathrm{He}_{3/4}(a,b) < A_{1/2}(a,b) \\ &< S_{11/12,7/12}(a,b) < T_{1/3}(a,b) < S_{5/6,2/3}(a,b) < I_{3/4}(a,b) \end{aligned}$$

for all $a, b > 0$ with $a \neq b$.

Remark 3.10 Unfortunately, in the article we cannot present the monotonicity conclusion of the function $r \to R_p(r)$ given by (3.1) on the interval $(0, \infty)$ for $p \in (7/4, 2) \cup (9/4, \infty)$, we leave it as an open problem to the reader.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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