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Two weighted inequalities for B-fractional integrals

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Abstract

In this paper we prove a two weighted inequality for Riesz potentials $I_{\alpha,\gamma}f$ (B-fractional integrals) associated with the Laplace-Bessel differential operator $\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{j=1}^{k} \frac{\gamma_j}{x_j} \frac{\partial}{\partial x_j}$. This result is an analog of Heinig's result (Indiana Univ. Math. J. 33(4):573-582, 1984) for the B-fractional integral. Further, the Stein-Weiss inequality for B-fractional integrals is proved as an application of this result.

MSC: 42B20; 42B25; 42B35; 47G10; 47B37

Keywords: Laplace-Bessel differential operator; B-fractional integrals; Stein-Weiss inequality

1 Introduction

Let $\mathbb{R}^n_{k,+} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 > 0, x_2 > 0, \dots, x_k > 0\}$, $1 \le k \le n$, and w be a weight function on $\mathbb{R}^n_{k,+}$, *i.e.*, w is a non-negative and measurable function on $\mathbb{R}^n_{k,+}$. The weighted Lebesgue space $L_{p,w,\gamma} \equiv L_{p,w,\gamma}(\mathbb{R}^n_{k,+})$, $1 \le p < \infty$, is the set of all classes of measurable functions f with finite norm

$$\|f\|_{p,w,\gamma} = \left(\int_{\mathbb{R}^n_{k,+}} \bigl|f(x)\bigr|^p w(x) \bigl(x'\bigr)^\gamma \, dx \right)^{\frac{1}{p}},$$

where $(x')^\gamma = x_1^{\gamma_1} \cdot \ldots \cdot x_k^{\gamma_k}$ and $\gamma = (\gamma_1, \dots, \gamma_k)$ is a multi-index consisting of fixed positive numbers such that $|\gamma| = \gamma_1 + \cdots + \gamma_k$.

If $p = \infty$, we assume

$$L_{\infty,w,\gamma}\bigl(\mathbb{R}^n_{k,+}\bigr) = L_\infty\bigl(\mathbb{R}^n_{k,+}\bigr) = \Bigl\{f : \|f\|_{L_{\infty,w}} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n_{k,+}} \bigl|w(x)f(x)\bigr| < \infty\Bigr\}.$$

The fractional integral operators play an important role in the theory of harmonic analysis, differentiation theory and PDE's. Many mathematicians have dealt with the fractional integrals and related topics associated with the Laplace-Bessel differential operator $\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{j=1}^{k} \frac{\gamma_j}{x_j} \frac{\partial}{\partial x_j}$ such as Aliev and Gadjiev [2], Guliyev [3], Gadjiev and Hajibayov [4], Guliyev *et al.* [5] and others. In this paper we consider fractional (B-fractional) integrals in the weighted Lebesgue space $L_{p,w,\gamma}(\mathbb{R}^n_{k,+})$ associated with the generalized shift operator defined by (see, for example [6, 7])

$$T^y f(x) = C_{k,\gamma} \int_0^\pi \cdots \int_0^\pi f\big((x',y')_\alpha, x''-y''\big)\, d\nu(\alpha),$$

where

$$C_{k,\gamma} = \pi^{-\frac{k}{2}} \prod_{i=1}^{k} \frac{\Gamma(\frac{\gamma_i+1}{2})}{\Gamma(\frac{\gamma_i}{2})}, \qquad (x',y')_\alpha = \big((x_1,y_1)_{\alpha_1},\dots,(x_k,y_k)_{\alpha_k}\big),$$

$$(x_i,y_i)_{\alpha_i} = \sqrt{x_i^2 - 2x_i y_i \cos\alpha_i + y_i^2}, \qquad (x',x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}, \quad \text{and}$$

$$d\nu(\alpha) = \prod_{i=1}^{k} \sin^{\gamma_i-1}\alpha_i\, d\alpha_i, \quad 1 \le i \le k, 1 \le k \le n.$$

It is well known that the generalized shift operator T^y is closely related to the Laplace-Bessel differential operator Δ_B. Furthermore, T^y generates the corresponding B-convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y)\big(T^y g(x)\big)\big(y'\big)^\gamma\, dy.$$

The B-fractional integral (or B-Riesz potential) is defined by

$$I_{\alpha,\gamma} f(x) = \int_{\mathbb{R}^n_{k,+}} T^y\big(|x|^{\alpha-n-|\gamma|}\big) f(y)\big(y'\big)^\gamma\, dy, \quad 0 < \alpha < n + |\gamma|.$$

The properties of the B-fractional integral has been examined extensively. We refer to [2–5, 8–10] and for more general case to [11, 12].

In the case $w = 1$, for the classical Riesz potential $I_\alpha f$, the classical Hardy-Littlewood-Sobolev theorem [13] states that, if $1 < p < \infty$ and $\alpha p < n$, then $I_\alpha f$ is an operator of strong type (p,q), where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and if $p = 1$, then $I_\alpha f$ is an operator of weak type $(1,q)$, where $\frac{1}{q} = 1 - \frac{\alpha}{n}$.

In the following we give the Heinig's result [1] for the boundedness of the classical Riesz potential $I_\alpha f$ in weighted Lebesgue spaces which is a generalization of the Hardy-Littlewood-Sobolev theorem for $I_\alpha f$.

Theorem A ([1]) *Suppose u and v are defined on R^n and $U = u^*$, $\frac{1}{V} = (\frac{1}{v})^*$. If $1 \le p \le q \le \infty$, $p < \infty$, and, for some r, $1 < r < \frac{n}{\alpha}$,*

$$\sup_{s>0} \left(\int_s^\infty \big[U(t) t^{\alpha-n}\big]^q t^{n-1}\, dt \right)^{\frac{1}{q}} \left(\int_0^s V(t)^{-p'} t^{n-1}\, dt \right)^{\frac{1}{p'}} < \infty$$

and

$$\sup_{s>0} \left(\int_0^s \big[U(t) t^{\alpha-\frac{n}{r}}\big]^q t^{n-1}\, dt \right)^{\frac{1}{q}} \left(\int_s^\infty \big[V(t) t^{\frac{n}{r'}}\big]^{-p'} t^{n-1}\, dt \right)^{\frac{1}{p'}} < \infty,$$

then $I_\alpha : L_{p,v}(\mathbb{R}^n) \to L_{q,u}(\mathbb{R}^n)$ is bounded.

Our purpose in this paper is to give an analog of Heinig's result for the B-fractional integral $I_{\alpha,\gamma}f$. Further, the Stein-Weiss inequality for B-Riesz potential is proved as an application of this result. Note that the Stein-Weiss inequality for the classical Riesz potentials was given in [14]. For the B-fractional integrals, this inequality was proved in [9] and [10].

2 Preliminaries

Let $1 \leq p \leq \infty$. In the case $w = 1$, if f is in $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and φ is in $L_{1,\gamma}(\mathbb{R}^n_{k,+})$, then the function $f \otimes \varphi$ belongs to $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|f \otimes \varphi\|_{p,\gamma} \leq \|f\|_{p,\gamma} \|\varphi\|_{1,\gamma}.$$

Suppose f is a measurable function defined on $\mathbb{R}^n_{k,+}$. For any measurable set $E \subset \mathbb{R}^n_{k,+}$, let $|E|_\gamma = \int_E (x')^\gamma \, dx$. The distribution function $f_{*,\gamma}$ of the function f is given by

$$f_{*,\gamma}(s) = \left| \left\{ x : x \in \mathbb{R}^n_{k,+}, \left| f(x) \right| > s \right\} \right|_\gamma, \quad \text{for } s \geq 0.$$

The distribution function $f_{*,\gamma}$ is non-negative, non-increasing, and continuous from the right (see [15]). With the distribution function we associate the non-increasing rearrangement of f on $[0,\infty)$ defined by

$$f^{*,\gamma}(t) = \inf\left\{ s > 0 : f_{*,\gamma}(s) \leq t \right\}.$$

If $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$, $1 \leq p < \infty$, then

$$\left(\int_{\mathbb{R}^n_{k,+}} \left| f(x) \right|^p \left(x'\right)^\gamma \, dx \right)^{\frac{1}{p}} = \left(p \int_0^\infty s^{p-1} f_{*,\gamma}(s) \, ds \right)^{\frac{1}{p}} = \left(\int_0^\infty \left(f^{*,\gamma}(t) \right)^p dt \right)^{\frac{1}{p}}.$$

In the following we give several inequalities which we will need in the proof of our main results.

Lemma 1 (Hardy inequalities [16–18]) *Suppose ξ and θ are non-negative locally integrable functions defined on $(0,\infty)$ and $1 < p \leq q < \infty$. Then there exists a constant $C > 0$ such that for all non-negative Lebesgue measurable function ψ on $(0,\infty)$, the inequality*

$$\left[\int_0^\infty \left(\int_0^t \psi(s) \, ds \right)^q \xi(t) \, dt \right]^{\frac{1}{q}} \leq C \left(\int_0^\infty \left(\psi(t) \right)^p \theta(t) \, dt \right)^{\frac{1}{p}} \tag{1}$$

is satisfied if and only if

$$\sup_{s>0} \left(\int_s^\infty \xi(t) \, dt \right)^{\frac{1}{q}} \left(\int_0^s \left(\theta(t) \right)^{1-p'} dt \right)^{\frac{1}{p'}} < \infty. \tag{2}$$

Similarly for the dual operator,

$$\left[\int_0^\infty \left(\int_t^\infty \psi(s) \, ds \right)^q \xi(t) \, dt \right]^{\frac{1}{q}} \leq C \left(\int_0^\infty \left(\psi(t) \right)^p \theta(t) \, dt \right)^{\frac{1}{p}} \tag{3}$$

is satisfied if and only if

$$\sup_{s>0}\left(\int_0^s \xi(t)\,dt\right)^{\frac{1}{q}}\left(\int_s^\infty \bigl(\theta(t)\bigr)^{1-p'}\,dt\right)^{\frac{1}{p'}}<\infty. \tag{4}$$

Lemma 2 ([1, 15, 17]) *Let f and g be non-negative measurable functions on $\mathbb{R}^n_{k,+}$. Then*

$$\int_{\mathbb{R}^n_{k,+}} f(x)g(x)\bigl(x'\bigr)^{\gamma}\,dx \le \int_0^\infty f^{*,\gamma}(t)g^{*,\gamma}(t)\,dt \tag{5}$$

and

$$\int_0^\infty f^{*,\gamma}(t)\frac{1}{(\frac{1}{g})^{*,\gamma}(t)}\,dt \le \int_{\mathbb{R}^n_{k,+}} f(x)g(x)\bigl(x'\bigr)^{\gamma}\,dx. \tag{6}$$

Lemma 3 ([19]) *Let $1\le p_1<p_2<\infty$ and $1\le q_1<q_2<\infty$. A sublinear operator T satisfies weak-type hypotheses (p_1,q_1) and (p_2,q_2) if and only if*

$$(Tf)^{*,\gamma}(t)\le C\left(t^{-\frac{1}{q_1}}\int_0^{t^{\frac{\sigma_1}{\sigma_2}}} s^{\frac{1}{p_1}-1}f^{*,\gamma}(s)\,ds + t^{-\frac{1}{q_2}}\int_{t^{\frac{\sigma_1}{\sigma_2}}}^\infty s^{\frac{1}{p_2}-1}f^{*,\gamma}(s)\,ds\right), \tag{7}$$

where $\sigma_1=\frac{1}{q_1}-\frac{1}{q_2}$ and $\sigma_2=\frac{1}{p_1}-\frac{1}{p_2}$.

3 Two weighted inequalities for *B*-fractional integrals

In this section we prove an analog of Heinig's result for the B-fractional integral $I_{\alpha,\gamma}f$. Further, the Stein-Weiss inequality for B-Riesz potential is proved as an application of this result. In the following theorem we formulate analog of the Heinig's result for the B-fractional integral $I_{\alpha,\gamma}f$.

Theorem 1 *Let $0<\alpha<n+|\gamma|$, $1<r<\frac{n+|\gamma|}{\alpha}$, $1<p\le q<\infty$. Suppose that u and v are non-negative locally integrable functions on $\mathbb{R}^n_{k,+}$ with conditions*

$$\sup_{s>0}\left(\int_s^\infty u^{*,\gamma}(t)t^{-q(1-\frac{\alpha}{n+|\gamma|})}\,dt\right)^{\frac{1}{q}}\left(\int_0^s \left(\left(\frac{1}{v}\right)^{*,\gamma}(t)\right)^{p'-1}dt\right)^{\frac{1}{p'}}<\infty \tag{8}$$

and

$$\sup_{s>0}\left(\int_0^s u^{*,\gamma}(t)t^{-q(\frac{1}{r}-\frac{\alpha}{n+|\gamma|})}\,dt\right)^{\frac{1}{q}}\left(\int_s^\infty \left(\left(\frac{1}{v}\right)^{*,\gamma}(t)\right)^{p'-1}t^{p'(\frac{1}{r}-1)}\,dt\right)^{\frac{1}{p'}}<\infty. \tag{9}$$

Then $I_{\alpha,\gamma}$ is a bounded operator from $L_{p,v,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,u,\gamma}(\mathbb{R}^n_{k,+})$, that is, there exists a constant $C>0$ such that for any $f\in L_{p,v,\gamma}(\mathbb{R}^n_{k,+})$,

$$\|I_{\alpha,\gamma}f\|_{q,u,\gamma}\le C\|f\|_{p,v,\gamma}.$$

Proof It is known that $I_{\alpha,\gamma}f$ is an operator of weak type $(1,\frac{1}{1-\frac{\alpha}{n+|\gamma|}})$ and is an operator of strong type $(r,\frac{1}{\frac{1}{r}-\frac{\alpha}{n+|\gamma|}})$, where $1<r<\infty$. Refer to $I_{\alpha,\gamma}f$, Lemma 3, taking

$$p_1=1,\qquad q_1=\frac{1}{1-\frac{\alpha}{n+|\gamma|}},\qquad p_2=r,\qquad q_2=\frac{1}{\frac{1}{r}-\frac{\alpha}{n+|\gamma|}}.$$

Then

$$\sigma_1 = \frac{1}{q_1} - \frac{1}{q_2} = 1 - \frac{1}{r}, \qquad \sigma_2 = \frac{1}{p_1} - \frac{1}{p_2} = 1 - \frac{1}{r}, \qquad \frac{\sigma_1}{\sigma_2} = 1,$$

and

$$\begin{aligned}
&\left[\int_0^\infty \bigl((I_{\alpha,\gamma}f)^{*,\gamma}(t)\bigr)^q u^{*,\gamma}(t)\,dt\right]^{\frac{1}{q}} \\
&\quad\le C\left[\int_0^\infty \left(t^{\frac{\alpha}{n+|\gamma|}-1}\int_0^t f^{*,\gamma}(s)\,ds + t^{\frac{\alpha}{n+|\gamma|}-\frac{1}{r}}\int_t^\infty s^{\frac{1}{r}-1} f^{*,\gamma}(s)\,ds\right)^q u^{*,\gamma}(t)\,dt\right]^{\frac{1}{q}}.
\end{aligned}$$

Applying the Minkowski inequality we obtain

$$\begin{aligned}
&\left[\int_0^\infty \bigl((I_{\alpha,\gamma}f)^{*,\gamma}(t)\bigr)^q u^{*,\gamma}(t)\,dt\right]^{\frac{1}{q}} \\
&\quad\le C\left[\int_0^\infty u^{*,\gamma}(t)t^{(\frac{\alpha}{n+|\gamma|}-1)q}\left(\int_0^t f^{*,\gamma}(s)\,ds\right)^q dt\right]^{\frac{1}{q}} \\
&\qquad + C\left[\int_0^\infty u^{*,\gamma}(t)t^{(\frac{\alpha}{n+|\gamma|}-\frac{1}{r})q}\left(\int_t^\infty s^{\frac{1}{r}-1} f^{*,\gamma}(s)\,ds\right)^q dt\right]^{\frac{1}{q}}.
\end{aligned} \tag{10}$$

If we take the notation

$$\xi(t) = u^{*,\gamma}(t)t^{(\frac{\alpha}{n+|\gamma|}-1)q}, \qquad \psi(t) = f^{*,\gamma}(t), \qquad \theta(t) = \frac{1}{(\frac{1}{v})^{*,\gamma}(t)},$$

then we have (2) from (8) and applying (1)

$$\begin{aligned}
&\left[\int_0^\infty u^{*,\gamma}(t)t^{(\frac{\alpha}{n+|\gamma|}-1)q}\left(\int_0^t f^{*,\gamma}(s)\,ds\right)^q dt\right]^{\frac{1}{q}} \\
&\quad\le C\left(\int_0^\infty \frac{1}{(\frac{1}{v})^{*,\gamma}(t)}\bigl(f^{*,\gamma}(t)\bigr)^p\,dt\right)^{\frac{1}{p}}.
\end{aligned} \tag{11}$$

Now if we take

$$\xi(t) = u^{*,\gamma}(t)t^{(\frac{\alpha}{n+|\gamma|}-\frac{1}{r})q}, \qquad \psi(t) = t^{\frac{1}{r}-1}f^{*,\gamma}(t), \qquad \theta(t) = \frac{1}{(\frac{1}{v})^{*,\gamma}(t)}t^{p(\frac{1}{r}-1)},$$

then we have (4) from (9) and applying (3) we can assert that

$$\begin{aligned}
&\left[\int_0^\infty u^{*,\gamma}(t)t^{(\frac{\alpha}{n+|\gamma|}-\frac{1}{r})q}\left(\int_t^\infty s^{\frac{1}{r}-1} f^{*,\gamma}(s)\,ds\right)^q dt\right]^{\frac{1}{q}} \\
&\quad\le \left(\int_0^\infty \bigl(t^{\frac{1}{r}-1}f^{*,\gamma}(t)\bigr)^p \frac{1}{(\frac{1}{v})^{*,\gamma}(t)} t^{p(\frac{1}{r}-1)}\,dt\right)^{\frac{1}{p}} \\
&\quad= \left(\int_0^\infty \frac{1}{(\frac{1}{v})^{*,\gamma}(t)}\bigl(f^{*,\gamma}(t)\bigr)^p\,dt\right)^{\frac{1}{p}}.
\end{aligned} \tag{12}$$

Combining (10), (11), (12) yields

$$\left[\int_0^{\infty}\big((I_{\alpha,\gamma}f)^{*,\gamma}(t)\big)^q u^{*,\gamma}(t)\,dt\right]^{\frac{1}{q}} \le C\left(\int_0^{\infty}\frac{1}{(\frac{1}{\nu})^{*,\gamma}(t)}\big(f^{*,\gamma}(t)\big)^p\,dt\right)^{\frac{1}{p}}. \tag{13}$$

Applying (5), (13), and (6) we have

$$\begin{aligned}
&\left[\int_{\mathbb{R}^n_{k,+}}\big((I_{\alpha,\gamma}f(x))\big)^q u(x)\big(x'\big)^{\gamma}\,dx\right]^{\frac{1}{q}} \\
&\quad\le \left[\int_0^{\infty}\big((I_{\alpha,\gamma}f)^{*,\gamma}(t)\big)^q u^{*,\gamma}(t)\,dt\right]^{\frac{1}{q}} \\
&\quad\le C\left(\int_0^{\infty}\frac{1}{(\frac{1}{\nu})^{*,\gamma}(t)}\big(f^{*,\gamma}(t)\big)^p\,dt\right)^{\frac{1}{p}} \\
&\quad\le C\left[\int_{\mathbb{R}^n_{k,+}}\big(f(x)\big)^p \nu(x)\big(x'\big)^{\gamma}\,dx\right]^{\frac{1}{p}}.
\end{aligned}$$

Thus the proof the theorem is completed. □

In the following theorem we prove the Stein-Weiss inequality for *B*-fractional integrals by using Theorem 1. Note that the Stein-Weiss inequality for classical Riesz potentials was given in [14]. For *B*-fractional integrals, this inequality was proved in [9] and [10].

Theorem 2 *Let* $0 < \alpha < n + |\gamma|$, $1 < p < \frac{n+|\gamma|}{\alpha}$, $\beta < 0$, $0 < \beta + \alpha p < (n + |\gamma|)(p - 1)$, $u(x) = |x|^{\beta}$, *and* $\nu(x) = |x|^{\beta+\alpha p}$, *for* $x \in \mathbb{R}^n_{k,+}$. *Then* $I_{\alpha,\gamma}$ *is a bounded operator from* $L_{p,\nu,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{q,u,\gamma}(\mathbb{R}^n_{k,+})$, *that is, there exists a constant* $C > 0$ *such that, for any* $f \in L_{p,\nu,\gamma}(\mathbb{R}^n_{k,+})$,

$$\left(\int_{\mathbb{R}^n_{k,+}}\big|I_{\alpha,\gamma}f(x)\big|^p|x|^{\beta}\big(x'\big)^{\gamma}\,dx\right)^{\frac{1}{p}} \le C\left(\int_{\mathbb{R}^n_{k,+}}\big|f(x)\big|^p|x|^{\beta+\alpha p}\big(x'\big)^{\gamma}\,dx\right)^{\frac{1}{p}}.$$

Proof It is known that $|B(0,r)|_{\gamma} = w(n,k,\gamma)r^{n+|\gamma|}$, where $w(n,k,\gamma) = |B(0,1)|_{\gamma}$. Since $\beta < 0$ we have

$$\begin{aligned}
u_{*,\gamma}(s) &= \big|\big\{x \in \mathbb{R}^n_{k,+} : |x|^{\beta} > s\big\}\big|_{\gamma} \\
&= \big|\big\{x : x \in \mathbb{R}^n_{k,+}, |x| < s^{\frac{1}{\beta}}\big\}\big|_{\gamma} \\
&= \big|B\big(0, s^{\frac{1}{\beta}}\big)\big|_{\gamma} \\
&= w(n,k,\gamma)r^{\frac{n+|\gamma|}{\beta}}
\end{aligned}$$

and

$$\begin{aligned}
u^{*,\gamma}(t) &= \inf\big\{s > 0 : u_{*,\gamma}(s) \le t\big\} \\
&= \left(\frac{1}{w(n,k,\gamma)}\right)^{\frac{\beta}{n+|\gamma|}} t^{\frac{\beta}{n+|\gamma|}}.
\end{aligned}$$

Since $\beta + \alpha p > 0$ we can write

$$\begin{aligned}\left(\frac{1}{\nu}\right)_{*,\gamma}(s) &= \left|\left\{x : x \in \mathbb{R}^n_{k,+} |x|^{-(\beta+\alpha p)} > s\right\}\right|_\gamma \\ &= \left|\left\{x : x \in \mathbb{R}^n_{k,+}, |x| < s^{-\frac{1}{\beta+\alpha p}}\right\}\right|_\gamma \\ &= \left|B\left(0, s^{-\frac{1}{\beta+\alpha p}}\right)\right|_\gamma \\ &= w(n,k,\gamma) r^{-\frac{n+|\gamma|}{\beta+\alpha p}}\end{aligned}$$

and

$$\left(\frac{1}{\nu}\right)^{*,\gamma}(t) = \inf\left\{s > 0 : \left(\frac{1}{\nu}\right)_{*,\gamma}(s) \le t\right\} = \left(\frac{1}{w(n,k,\gamma)}\right)^{-\frac{\beta+\alpha p}{n+|\gamma|}} t^{-\frac{\beta+\alpha p}{n+|\gamma|}}.$$

Take $p = q = r$ and examine (8) and (9). Since $\beta + \alpha p < (n + |\gamma|)(p - 1)$ we have $\frac{\beta+\alpha p}{n+|\gamma|} - p < -1$ and $-\frac{\beta+\alpha p}{n+|\gamma|}(p' - 1) > -1$. Then

$$\begin{aligned}&\sup_{s>0}\left(\int_s^\infty u^{*,\gamma}(t) t^{-q(1-\frac{\alpha}{n+|\gamma|})}\,dt\right)^{\frac{1}{q}} \left(\int_0^s \left(\left(\frac{1}{\nu}\right)^{*,\gamma}(t)\right)^{p'-1} dt\right)^{\frac{1}{p'}} \\ &\quad= \sup_{s>0}\left(\int_s^\infty \left(\frac{1}{w(n,k,\gamma)}\right)^{\frac{\beta}{n+|\gamma|}} t^{\frac{\beta}{n+|\gamma|}} t^{-p(1-\frac{\alpha}{n+|\gamma|})}\,dt\right)^{\frac{1}{p}} \\ &\qquad\times \left(\int_0^s \left(\left(\frac{1}{w(n,k,\gamma)}\right)^{-\frac{\beta+\alpha p}{n+|\gamma|}} t^{-\frac{\beta+\alpha p}{n+|\gamma|}}\right)^{p'-1} dt\right)^{\frac{1}{p'}} \\ &\quad= \left(\frac{1}{w(n,k,\gamma)}\right)^{\frac{-\alpha p}{(n+|\gamma|)p}} \sup_{s>0}\left(\frac{1}{\frac{\beta+\alpha p}{n+|\gamma|} - p + 1} t^{\frac{\beta+\alpha p}{n+|\gamma|}-p+1}\Big|_s^\infty\right)^{\frac{1}{p}} \\ &\qquad\times \left(\frac{1}{-\frac{\beta+\alpha p}{n+|\gamma|}(p'-1)+1} t^{-\frac{\beta+\alpha p}{n+|\gamma|}(p'-1)+1}\Big|_0^s\right)^{\frac{1}{p'}} \\ &\quad= \left(\frac{1}{w(n,k,\gamma)}\right)^{\frac{-\alpha p}{(n+|\gamma|)p}} \left(\frac{-1}{\frac{\beta+\alpha p}{n+|\gamma|} - p + 1}\right)^{\frac{1}{p}} \left(\frac{1}{-\frac{\beta+\alpha p}{n+|\gamma|}(p'-1)+1}\right)^{\frac{1}{p'}} \\ &\qquad\times \sup_{s>0} s^{(\frac{\beta+\alpha p}{n+|\gamma|}-p+1)\frac{1}{p} + (-\frac{\beta+\alpha p}{n+|\gamma|}(p'-1)+1)\frac{1}{p'}} \\ &\quad= \left(\frac{1}{w(n,k,\gamma)}\right)^{\frac{-\alpha p}{(n+|\gamma|)p}} \left(\frac{-1}{\frac{\beta+\alpha p}{n+|\gamma|} - p + 1}\right)^{\frac{1}{p}} \left(\frac{1}{-\frac{\beta+\alpha p}{n+|\gamma|}(p'-1)+1}\right)^{\frac{1}{p'}} < \infty.\end{aligned}$$

Now examine (9). Since $\beta + \alpha p > 0$ we have $\frac{\beta+\alpha p}{n+|\gamma|} - 1 > -1$ and $-\frac{\beta+\alpha p}{n+|\gamma|}(p' - 1) - 1 < -1$. Then

$$\begin{aligned}&\sup_{s>0}\left(\int_0^s u^{*,\gamma}(t) t^{-q(\frac{1}{r}-\frac{\alpha}{n+|\gamma|})}\,dt\right)^{\frac{1}{q}} \left(\int_s^\infty \left(\left(\frac{1}{\nu}\right)^{*,\gamma}(t)\right)^{p'-1} t^{p'(\frac{1}{r}-1)}\,dt\right)^{\frac{1}{p'}} \\ &\quad= \sup_{s>0}\left(\int_0^s \left(\frac{1}{w(n,k,\gamma)}\right)^{\frac{\beta}{n+|\gamma|}} t^{\frac{\beta}{n+|\gamma|}} t^{-p(\frac{1}{p}-\frac{\alpha}{n+|\gamma|})}\,dt\right)^{\frac{1}{p}}\end{aligned}$$

$$
\begin{aligned}
&\times\left(\int_s^\infty\left(\left(\left(\frac{1}{w(n,k,\gamma)}\right)^{-\frac{\beta+\alpha p}{n+|\gamma|}}t^{-\frac{\beta+\alpha p}{n+|\gamma|}}\right)^{p'-1}t^{p'(\frac{1}{p}-1)}\,dt\right)^{\frac{1}{p'}}\\
&=\left(\frac{1}{w(n,k,\gamma)}\right)^{\frac{\alpha}{n+|\gamma|}}\sup_{s>0}\left(\frac{n+|\gamma|}{\beta+\alpha p}t^{\frac{\beta+\alpha p}{n+|\gamma|}}\Big|_0^s\right)^{\frac{1}{p}}\\
&\quad\times\left(\frac{n+|\gamma|}{(\beta+\alpha p)(1-p')}t^{-\frac{\beta+\alpha p}{n+|\gamma|}(p'-1)}\Big|_s^\infty\right)^{\frac{1}{p'}}\\
&=\left(\frac{1}{w(n,k,\gamma)}\right)^{\frac{\alpha}{n+|\gamma|}}\frac{n+|\gamma|}{\beta+\alpha p}\left(\frac{1}{p'-1}\right)^{\frac{1}{p'}}\sup_{s>0}s^{\frac{\beta+\alpha p}{n+|\gamma|}\frac{1}{p}-\frac{\beta+\alpha p}{n+|\gamma|}\frac{p'-1}{p'}}\\
&=\left(\frac{1}{w(n,k,\gamma)}\right)^{\frac{\alpha}{n+|\gamma|}}\frac{n+|\gamma|}{\beta+\alpha p}\left(\frac{1}{p'-1}\right)^{\frac{1}{p'}}<\infty.
\end{aligned}
$$

Therefore (8) and (9) are satisfied and from Theorem 1, we have the result of the corollary. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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