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An inexact generalized PRSM with LQP regularization for structured variational inequalities and its applications to traffic equilibrium problems

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Abstract

As one of the operator splitting methods, the Peaceman-Rachford splitting method (PRSM) has attracted considerable attention recently. This paper proposes a generalized PRSM for structured variational inequalities with positive orthants. In fact, we apply the well-developed LQP regularization to regularize the subproblems of the recently proposed strictly contractive PRSM, thus the resulting subproblems reduce to two nonlinear equation systems, which are much easier to solve than the subproblems of PRSM. Furthermore, these two nonlinear equations are allowed to be solved inexactly. For the new method, we prove its global convergence and establish its worst-case convergence rate in the ergodic sense. Numerical experiments show that the proposed method is quite efficient for the traffic equilibrium problems with link capacity bound.

MSC: 90C25; 90C30

Keywords: Peaceman-Rachford splitting method; structured variational inequalities; traffic equilibrium problems

1 Introduction

In this paper, we focus on the variational inequalities with separable structures and positive orthants. That is, find $u^* \in \Omega$ such that

$$(u - u^*)^\top T(u^*) \geq 0, \quad u \in \Omega, \tag{1}$$

with

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad T(u) = \begin{pmatrix} f(x) \\ g(y) \end{pmatrix}, \qquad \Omega = \{(x,y) | Ax + By = b, x \in \mathcal{R}^m_+, y \in \mathcal{R}^n_+\},$$

where $f : \mathcal{X} \to \mathcal{R}^m$ and $g : \mathcal{Y} \to \mathcal{R}^n$ are continuous and monotone operators; $A \in \mathcal{R}^{l\times m}$ and $B \in \mathcal{R}^{l\times n}$ are given matrices; $b \in \mathcal{R}^l$ is a given vector. Problem (1) is a standard mathematical model arising from several scientific fields and admits a large number of applications in network economics, traffic assignment, game theoretic problems, *etc.*; see [1–3] and the references therein. Throughout, we assume that the solution of Problem (1) (denoted by Ω^*) is nonempty.

By attaching a Lagrange multiplier vector $\lambda \in \mathcal{R}^l$ to the linear constraints $Ax + By = b$, Problem (1) can be equivalently transformed into the following compact form, denoted by $\mathrm{VI}(\mathcal{W}, Q)$: Find $w^* \in \mathcal{W}$, such that

$$\left(w - w^*\right)^\top Q\left(w^*\right) \geq 0, \quad \forall w \in \mathcal{W}, \tag{2}$$

where

$$w = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix}, \qquad Q(w) = \begin{pmatrix} f(x) - A^\top \lambda \\ g(y) - B^\top \lambda \\ Ax + By - b \end{pmatrix}, \qquad \mathcal{W} = \mathcal{R}^m_+ \times \mathcal{R}^n_+ \times \mathcal{R}^l.$$

We denote by $\mathcal{W}^*$ the solution of $\mathrm{VI}(\mathcal{W}, Q)$. Obviously, $\mathcal{W}^*$ is nonempty under the assumption that Ω^* is nonempty. In addition, due to the monotonicity of $f(\cdot)$ and $g(\cdot)$, the mapping $Q(\cdot)$ of $\mathrm{VI}(\mathcal{W}, Q)$ is also monotone.

A simple but powerful operator splitting algorithm in the literature is the alternating direction method of multipliers (ADMM) proposed in [4–6]. For the developments of ADMM on structured variational inequalities (2), we refer to [7–9]. Similar to ADMM, the Peaceman-Rachford splitting method (PRSM) is also a simple algorithm for Problem (2); see [10–12]. For solving (2), the iterative scheme of PRSM is

$$\begin{cases} 0 \leq x^{k+1} \perp \{f(x^{k+1}) - A^\top[\lambda^k - \beta(Ax^{k+1} + By^k - b)]\} \geq 0, \\ \lambda^{k+\frac{1}{2}} = \lambda^k - \beta(Ax^{k+1} + By^k - b), \\ 0 \leq y^{k+1} \perp \{g(y^{k+1}) - B^\top[\lambda^{k+\frac{1}{2}} - \beta(Ax^{k+1} + By^{k+1} - b)]\} \geq 0, \\ \lambda^{k+1} = \lambda^{k+\frac{1}{2}} - \beta(Ax^{k+1} + By^{k+1} - b), \end{cases} \tag{3}$$

where $\beta > 0$ is a penalty parameter. Different from the ADMM, the PRSM updates the Lagrange multiplier twice at each iteration. However, the global convergence of PRSM cannot be guaranteed without any further assumptions on the model (2). To solve this issue, He *et al.* [13] developed the following strictly contractive PRSM (SC-PRSM):

$$\begin{cases} 0 \leq x^{k+1} \perp \{f(x^{k+1}) - A^\top[\lambda^k - \beta(Ax^{k+1} + By^k - b)]\} \geq 0, \\ \lambda^{k+\frac{1}{2}} = \lambda^k - r\beta(Ax^{k+1} + By^k - b), \\ 0 \leq y^{k+1} \perp \{g(y^{k+1}) - B^\top[\lambda^{k+\frac{1}{2}} - \beta(Ax^{k+1} + By^{k+1} - b)]\} \geq 0, \\ \lambda^{k+1} = \lambda^{k+\frac{1}{2}} - r\beta(Ax^{k+1} + By^{k+1} - b), \end{cases} \tag{4}$$

where $r \in (0,1)$ is an underdetermined relaxation factor. The global convergence of SC-PRSM is proved via the analytic framework of contractive type methods in [13].

Note that the computational load of SC-PRSM (4) relies on the resulting two complementarity problems, which are computationally expensive, especially for large-scale problems. Therefore, how to alleviate the difficulty of these subproblems deserves intensive research. In this paper, motivated by well-developed logarithmic-quadratic proximal (LQP) regularization proposed in [14], we regularize the two complementarity problems in (4) by LQP, which forces the solutions of the two complementarity problems to be interior points of $\mathcal{R}^m_+$ and $\mathcal{R}^n_+$, respectively, thus the two complementarity problems reduce to two easier nonlinear equation systems. On the other hand, it is well known that the generalized ADMM [15, 16] includes the classical ADMM as a special case, and it can numerically accelerate the original ADMM with some values of the relaxation factor. Therefore, inspired by the above analysis, we get the following iterative scheme:

$$\begin{cases} 0 \le x^{k+1} \perp \{f(x^{k+1}) - A^\top[\lambda^k - \beta(Ax^{k+1} + By^k - b)] \\ \qquad + R[(x^{k+1} - x^k) + \mu(x^k - P_k^2(x^{k+1})^{-1})]\} \ge 0, \\ \lambda^{k+\frac{1}{2}} = \lambda^k - r\beta(Ax^{k+1} + By^k - b), \\ 0 \le y^{k+1} \perp \{g(y^{k+1}) - B^\top[\lambda^{k+\frac{1}{2}} - \beta(\alpha Ax^{k+1} - (1-\alpha)(By^{k+1} - b) + By^{k+1} - b)] \\ \qquad + S[(y^{k+1} - y^k) + \mu(y^k - Q_k^2(y^{k+1})^{-1})]\} \ge 0, \\ \lambda^{k+1} = \lambda^{k+\frac{1}{2}} - \beta[\alpha Ax^{k+1} - (1-\alpha)(By^{k+1} - b) + By^{k+1} - b], \end{cases} \tag{5}$$

where $\alpha \in (0,2)$, $r \in (0, 2-\alpha)$, and $\mu \in (0,1)$ are three constants, and $R = \operatorname{diag}(r_1, r_2, \dots, r_m) \in \mathcal{R}^{m\times m}$ and $S = \operatorname{diag}(s_1, s_2, \dots, s_n) \in \mathcal{R}^{n\times n}$ are symmetric positive definite matrices, $P_k = \operatorname{diag}(1/x_1^k, 1/x_2^k, \dots, 1/x_m^k)$, $Q_k = \operatorname{diag}(1/y_1^k, 1/y_2^k, \dots, 1/y_n^k)$, and $(x^{k+1})^{-1}$ (or $(y^{k+1})^{-1}$) is a vector whose jth element is $1/x_j^{k+1}$ (or $1/y_j^{k+1}$). By Lemma 2.2 (see Section 2), the new iterate (x^{k+1}, y^{k+1}) generated by (5) lies in the interior of $\mathcal{R}^{m+n}$, provided that the previous iterate (x^k, y^k) does. Therefore, the two complementarity problems in (5) can reduce to the nonlinear equation systems, and we get the following iterative scheme:

$$\begin{cases} f(x^{k+1}) - A^\top[\lambda^k - \beta(Ax^{k+1} + By^k - b)] + R[(x^{k+1} - x^k) + \mu(x^k - P_k^2(x^{k+1})^{-1})] = 0, \\ \lambda^{k+\frac{1}{2}} = \lambda^k - r\beta(Ax^{k+1} + By^k - b), \\ g(y^{k+1}) - B^\top[\lambda^{k+\frac{1}{2}} - \beta(\alpha Ax^{k+1} - (1-\alpha)(By^{k+1} - b) + By^{k+1} - b)] \\ \quad + S[(y^{k+1} - y^k) + \mu(y^k - Q_k^2(y^{k+1})^{-1})] = 0, \\ \lambda^{k+1} = \lambda^{k+\frac{1}{2}} - \beta[\alpha Ax^{k+1} - (1-\alpha)(By^{k+1} - b) + By^{k+1} - b]. \end{cases}$$

Obviously, the above iterative scheme includes two nonlinear equations, which are not easy to solve exactly in many applications. This motivates us to propose the following inexact version:

$$\begin{cases} \text{Find } x^{k+1} \in R_{++}^m, \text{such that } \|x^{k+1} - x_*^{k+1}\| \le \nu_k, \\ \lambda^{k+\frac{1}{2}} = \lambda^k - r\beta(Ax^{k+1} + By^k - b), \\ \text{Find } y^{k+1} \in R_{++}^n, \text{such that } \|y^{k+1} - y_*^{k+1}\| \le \nu_k, \\ \lambda^{k+1} = \lambda^{k+\frac{1}{2}} - \beta[\alpha Ax^{k+1} - (1-\alpha)(By^{k+1} - b) + By^{k+1} - b], \end{cases} \tag{6}$$

where $\{\nu_k\}$ is a nonnegative sequence satisfying $\sum_{k=0}^{\infty} \nu_k < +\infty$, and x_*^{k+1}, y_*^{k+1} satisfy

$$f(x_*^{k+1}) - A^\top\bigl[\lambda^k - \beta\bigl(Ax_*^{k+1} + By^k - b\bigr)\bigr] + R\bigl[\bigl(x_*^{k+1} - x^k\bigr) + \mu\bigl(x^k - P_k^2\bigl(x_*^{k+1}\bigr)^{-1}\bigr)\bigr] = 0,$$

$$g\bigl(y_*^{k+1}\bigr) - B^\top\bigl[\lambda_*^{k+\frac{1}{2}} - \beta\bigl(\alpha Ax_*^{k+1} - (1-\alpha)\bigl(By_*^{k+1} - b\bigr) + By_*^{k+1} - b\bigr)\bigr]$$
$$\quad + S\bigl[\bigl(y_*^{k+1} - y^k\bigr) + \mu\bigl(y^k - Q_k^2\bigl(y_*^{k+1}\bigr)^{-1}\bigr)\bigr] = 0.$$

Here $\lambda_*^{k+\frac{1}{2}} = \lambda^k - r\beta(Ax_*^{k+1} + By^k - b)$.

The rest of this paper is organized as follows. In Section 2, we summarize preliminaries which are useful for further discussion, and we present the new method. In Section 3, the global convergence and the worst-case convergence rate in the ergodic sense of the new method are proved. In Section 4, we apply the proposed method to solve the traffic equilibrium problems with link capacity bound. Finally, some concluding remarks are made in Section 5.

2 Preliminaries

In this section, we first of all summarize some notations and lemmas which are used frequently in the sequent analysis, and then present our proposed method in detail.

First, we define four matrices as follows:

$$\begin{aligned} M &= \begin{pmatrix} I_m & 0 & 0 \\ 0 & I_n & 0 \\ 0 & -\beta B & (r+\alpha)I_l \end{pmatrix}, \\ P &= \begin{pmatrix} (1+\mu)R & 0 & 0 \\ 0 & (1+\mu)S + \beta B^\top B & (1-r-\alpha)B^\top \\ 0 & -B & \frac{1}{\beta}I_l \end{pmatrix}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} N &= \begin{pmatrix} \mu R & 0 & 0 \\ 0 & \mu S & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ H &= \begin{pmatrix} (1+\mu)R & 0 & 0 \\ 0 & (1+\mu)S + \frac{\beta}{r+\alpha}B^\top B & \frac{1-r-\alpha}{r+\alpha}B^\top \\ 0 & \frac{1-r-\alpha}{r+\alpha}B & \frac{1}{\beta(r+\alpha)}I_l \end{pmatrix}. \end{aligned} \tag{8}$$

The four matrices M, P, N, H just defined satisfy the following assertions.

Lemma 2.1 *If $\mu \in (0,1)$, $\alpha \in (0,2)$, $r \in (0, 2-\alpha)$, and R, S are symmetric positive definite, then we have*:

(1) *The matrices M, P, H defined, respectively, in* (7), (8) *have the following relationship*:

$$HM = P. \tag{9}$$

(2) *The two matrices H and $\tilde{H} := P^\top + P - M^\top HM - 2N$ are symmetric positive definite.*

Proof Item (1) holds evidently. As for item (2), it is obvious that H and $\tilde{H}$ are symmetric. Now, we prove that they are positive definite. Note that $\mu \in (0,1)$, $\alpha \in (0,2)$, and $r \in (0, 2-\alpha)$. Then for any $w = (x, y, \lambda) \neq 0$, we get

$$\begin{aligned} w^\top Hw &= (1+\mu)\|x\|_R^2 + (1+\mu)\|y\|_S^2 + \frac{1}{r+\alpha}\left(\beta\|By\|^2 + 2(1-r-\alpha)\lambda^\top By + \frac{1}{\beta}\|\lambda\|^2\right) \\ &\geq (1+\mu)\|x\|_R^2 + (1+\mu)\|y\|_S^2 + \frac{2}{r+\alpha}\min\{2-r-\alpha, r+\alpha\}\|By\| \cdot \|\lambda\| \\ &\geq (1+\mu)\|x\|_R^2 + (1+\mu)\|y\|_S^2, \end{aligned} \tag{10}$$

where the inequality follows from the Cauchy-Schwartz inequality. If $x \neq 0$ or $y = 0$, then from (10), we have $w^\top Hw > 0$. Otherwise $x = 0$, $y = 0$, and $\lambda \neq 0$, then we have $w^\top Hw =$

$\frac{\|\lambda\|^2}{\beta(r+\alpha)} > 0$. Thus, H is positive definite. As for $\tilde{H}$, using (9), we have

$$\begin{aligned}\tilde{H} &= P^\top + P - M^\top HM - 2N \\ &= P^\top + P - M^\top P - 2N \\ &= \begin{pmatrix} (1-\mu)R & 0 & 0 \\ 0 & (1-\mu)S & 0 \\ 0 & 0 & \frac{2-(r+\alpha)}{\beta}I_l \end{pmatrix}.\end{aligned}$$

Therefore $\tilde{H}$ is positive definite. The proof is complete. □

The following lemma lists a fundamental assertion with respect to the LQP regularization, which was proved in [17].

Lemma 2.2 *Let* $\bar{P} = \operatorname{diag}(p_1, p_2, \dots, p_t) \in \mathcal{R}^{t\times t}$ *be a positive definite diagonal matrix,* $q(u) \in \mathcal{R}^t$ *be a monotone mapping of* u *with respect to* $\mathcal{R}^t_{++}$*, and* $\mu \in (0,1)$*. For a given* $\bar{u} \in \mathcal{R}^t_{++}$*, we define* $\bar{U} := \operatorname{diag}(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_t)$*. Then the equation*

$$q(u) + \bar{P}\bigl[(u - \bar{u}) + \mu\bigl(\bar{u} - \bar{U}u^{-1}\bigr)\bigr] = 0$$

has the unique positive solution u*. In addition, for this positive solution* $u \in \mathcal{R}^t_{++}$ *and any* $v \in \mathcal{R}^t_{+}$*, we have*

$$(v-u)^\top q(u) \geq \frac{1+\mu}{2}\bigl(\|u-v\|^2_{\bar{P}} - \|\bar{u}-v\|^2_{\bar{P}}\bigr) + \frac{1-\mu}{2}\|\bar{u}-u\|^2_{\bar{P}}. \tag{11}$$

Now we present the generalized PRSM with LQP regularization for solving the Problem (2).

Remark 2.1 Note that Algorithm 1 includes many LQP-type methods as special cases, such as:

- If $r = 0$ and $\nu_k = 0$ $(\forall k)$, we obtain the generalized alternating direction method with LQP regularization proposed in [18].
- If $\alpha = 1$, we obtain a method similar to the method proposed in [19], and their difference only lies in the latter is designed for the separable convex programming.

Algorithm 1 A generalized PRSM with LQP regularization for VI($\mathcal{W}, Q$)

Input $\mu \in (0,1)$, $\alpha \in (0,2)$, $\beta > 0$, $r \in (0, 2-\alpha)$, $R = \operatorname{diag}(r_1, r_2, \dots, r_m)$, and $S = \operatorname{diag}(s_1, s_2, \dots, s_n)$ are positive definite matrices. Let $\{\nu_k\}$ be a nonnegative sequence satisfying $\sum_{k=0}^{\infty} \nu_k < +\infty$. Initialize $(x, y, \lambda) = (x^0, y^0, \lambda^0)$ with $x^0 > 0$, $y^0 > 0$. Set $k = 0$.
while ‘not converged’, **do**
  (1) Compute $w^{k+1} = (x^{k+1}, y^{k+1}, \lambda^{k+1})$ according to (6).
  (2) $k = k + 1$.
end while
Output x^{k+1}, y^{k+1}.

Remark 2.2 Obviously, by the relationship of the PRSM and the generalized ADMM presented in [20], the iterative scheme (6) is equivalent to

$$\begin{cases} \text{Find } x^{k+1} \in R^m_{++}, \text{such that } \|x^{k+1} - x^{k+1}_*\| \le \nu_k, \\ \tilde{\lambda}^{k+\frac{1}{2}} = \lambda^k - \mu\beta(Ax^{k+1} + By^k - b), \\ \text{Find } y^{k+1} \in R^n_{++}, \text{such that } \|y^{k+1} - y^{k+1}_{**}\| \le \nu_k, \\ \lambda^{k+1} = \tilde{\lambda}^{k+\frac{1}{2}} - \beta(Ax^{k+1} + By^{k+1} - b), \end{cases}$$

where $\mu = \alpha - 1 + r$ and y^{k+1}_{**} satisfy

$$\begin{aligned} &g\bigl(y^{k+1}_{**}\bigr) - B^\top\bigl[\lambda^{k+\frac{1}{2}}_* - \beta\bigl(Ax^{k+1}_* + By^{k+1}_{**} - b\bigr)\bigr] \\ &\quad + S\bigl[\bigl(y^{k+1}_{**} - y^k\bigr) + \mu\bigl(y^k - Q^2_k\bigl(y^{k+1}_{**}\bigr)^{-1}\bigr)\bigr] = 0. \end{aligned}$$

Obviously, when $\alpha = 1$, $r = 0$, that is, $\tilde{\lambda}^{k+\frac{1}{2}} = \lambda^{k+\frac{1}{2}}_* = \lambda^k$, the above iterative scheme reduces to the first inexact ADMM with LQP in [21].

3 Global convergence and convergence rate

In this section, we aim to prove the global convergence of Algorithm 1, and establish its worst-case convergence rate in a nonergodic sense.

To prove the global convergence, we need to define some auxiliary sequences as follows:

$$\lambda^{k+1}_* = \lambda^{k+\frac{1}{2}}_* - \beta\bigl(Ax^{k+1}_* + By^{k+1}_* - b\bigr),$$

and

$$w^{k+1}_* = \begin{pmatrix} x^{k+1}_* \\ y^{k+1}_* \\ \lambda^{k+1}_* \end{pmatrix} \quad \text{and} \quad \hat{w}^k = \begin{pmatrix} \hat{x}^k \\ \hat{y}^k \\ \hat{\lambda}^k \end{pmatrix} = \begin{pmatrix} x^{k+1}_* \\ y^{k+1}_* \\ \lambda^k - \beta(Ax^{k+1}_* + By^k - b) \end{pmatrix}. \tag{12}$$

Thus, based on (6) and (12), we immediately have

$$\begin{aligned} &x^{k+1}_* = \hat{x}^k, \qquad y^{k+1}_* = \hat{y}^k, \qquad \lambda^{k+\frac{1}{2}}_* = \lambda^k - r\bigl(\lambda^k - \hat{\lambda}^k\bigr), \\ &\lambda^{k+1}_* = \lambda^k - \bigl[(r+\alpha)\bigl(\lambda^k - \hat{\lambda}^k\bigr) - \beta B\bigl(y^k - \hat{y}^k\bigr)\bigr]. \end{aligned}$$

This and (7), (11) show that

$$w^{k+1}_* = w^k - M\bigl(w^k - \hat{w}^k\bigr). \tag{13}$$

Lemma 3.1 *The sequence* $\{w^k_*\}$ *defined by* (12) *and the sequence* $\{w^k\}$ *generated by Algorithm* 1 *satisfy the following inequality*:

$$\bigl\|w^{k+1}_* - w^{k+1}\bigr\|_H \le \rho\nu_k, \quad \forall k \ge 0, \tag{14}$$

where $\rho > 0$ *and* H *is defined by* (8).

Proof By the definitions of λ^{k+1} and λ_*^{k+1}, we have

$$\lambda_*^{k+1} - \lambda^{k+1} = (1+r)\beta A\big(x^{k+1} - x_*^{k+1}\big) + \beta B\big(y^{k+1} - y_*^{k+1}\big).$$

This and (6), (12) imply (14) immediately. The lemma is proved. □

Lemma 3.2 *If* $w^k = w^{k+1}$, *then* $w^{k+1} = (x^{k+1}, y^{k+1}, \lambda^{k+1})$ *produced by Algorithm* 1 *is a solution of* $\mathrm{VI}(\mathcal{W}, Q)$.

Proof For any $x \in \mathcal{R}_+^m$, applying Lemma 2.2 to the x-subproblem of (6) by setting $\bar{u} = x^k$, $u = \hat{x}^k$, $v = x$, and

$$q(u) = f\big(\hat{x}^k\big) - A^\top\big[\lambda^k - \beta\big(A\hat{x}^k + By^k - b\big)\big]$$

in (11), we have

$$\begin{aligned}
&\big(x - \hat{x}^k\big)^\top\big\{f\big(\hat{x}^k\big) - A^\top\big[\lambda^k - \beta\big(A\hat{x}^k + By^k - b\big)\big]\big\} \\
&\quad\geq \frac{1+\mu}{2}\big(\big\|\hat{x}^k - x\big\|_R^2 - \big\|x^k - x\big\|_R^2\big) + \frac{1-\mu}{2}\big\|x^k - \hat{x}^k\big\|_R^2 \\
&\quad= \frac{1+\mu}{2}\big(\big\|\hat{x}^k - x\big\|_R^2 - \big\|x^k - \hat{x}^k + \hat{x}^k - x\big\|_R^2\big) + \frac{1-\mu}{2}\big\|x^k - \hat{x}^k\big\|_R^2 \\
&\quad= (1+\mu)\bigg[\big(\hat{x}^k - x\big)^\top R\big(\hat{x}^k - x^k\big) - \frac{1}{2}\big\|x^k - \hat{x}^k\big\|_R^2\bigg] + \frac{1-\mu}{2}\big\|x^k - \hat{x}^k\big\|_R^2 \\
&\quad= (1+\mu)\big(\hat{x}^k - x\big)^\top R\big(\hat{x}^k - x^k\big) - \mu\big\|x^k - \hat{x}^k\big\|_R^2,
\end{aligned}$$

from which we get

$$\big(x - \hat{x}^k\big)^\top\big[(1+\mu)R\big(x^k - \hat{x}^k\big) - f\big(\hat{x}^k\big) + A^\top\hat{\lambda}^k\big] \leq \mu\big\|x^k - \hat{x}^k\big\|_R^2. \tag{15}$$

For any $y \in \mathcal{R}_+^n$, applying Lemma 2.2 to the y-subproblem of (6) by setting $\bar{u} = y^k$, $u = \hat{y}^k$, $v = y$, and

$$q(u) = g\big(\hat{y}^k\big) - B^\top\big[\lambda_*^{k+\frac{1}{2}} - \beta\big(\alpha A\hat{x}^k - (1-\alpha)\big(B\hat{y}^k - b\big) + B\hat{y}^k - b\big)\big]$$

in (11), we have

$$\begin{aligned}
&\big(y - \hat{y}^k\big)^\top\big\{g\big(\hat{y}^k\big) - B^\top\big[\lambda_*^{k+\frac{1}{2}} - \beta\big(\alpha A\hat{x}^k - (1-\alpha)\big(B\hat{y}^k - b\big) + B\hat{y}^k - b\big)\big]\big\} \\
&\quad\geq \frac{1+\mu}{2}\big(\big\|\hat{y}^k - y\big\|_S^2 - \big\|y^k - y\big\|_S^2\big) + \frac{1-\mu}{2}\big\|y^k - \hat{y}^k\big\|_S^2 \\
&\quad= (1+\mu)\big(\hat{y}^k - y\big)^\top S\big(\hat{y}^k - y^k\big) - \mu\big\|y^k - \hat{y}^k\big\|_S^2,
\end{aligned}$$

from the above inequality and $\lambda_*^{k+\frac{1}{2}} - \beta(\alpha A\hat{x}^k - (1-\alpha)(B\hat{y}^k - b) + B\hat{y}^k - b) = \hat{\lambda}^k + (1-r-\alpha)(\lambda^k - \hat{\lambda}^k) - \beta B(\hat{y}^k - y^k)$, we get

$$\begin{aligned}
&\big(y - \hat{y}^k\big)^\top\big\{\big[(1+\mu)S + \beta B^\top B\big]\big(y^k - \hat{y}^k\big) - g\big(\hat{y}^k\big) + B^\top\hat{\lambda}^k - (1-r-\alpha)B^\top\big(\hat{\lambda}^k - \lambda^k\big)\big\} \\
&\quad\leq \mu\big\|y^k - \hat{y}^k\big\|_S^2.
\end{aligned} \tag{16}$$

In addition, from (12) again, we have

$$\big(A\hat{x}^k + B\hat{y}^k - b\big) - B\big(\hat{y}^k - y^k\big) + \frac{1}{\beta}\big(\hat{\lambda}^k - \lambda^k\big) = 0. \tag{17}$$

Then, combining (15), (16), (17), for any $w = (x, y, \lambda) \in \mathcal{W}$, we have

$$\begin{aligned}
&\big(\hat{w}^k - w\big)^\top \left\{ \begin{pmatrix} f(\hat{x}^k) - A^\top \hat{\lambda}^k \\ g(\hat{y}^k) - B^\top \hat{\lambda}^k \\ A_1\hat{x}_1^k + A_2\hat{x}_2^k - b \end{pmatrix} \right. \\
&\quad \left. + \begin{pmatrix} (1+\mu)R(\hat{x}_1^k - x_1^k) \\ (1 - r - \alpha)B^\top(\hat{\lambda}^k - \lambda^k) + [(1+\mu)S + \beta B^\top B](\hat{y}^k - y^k) \\ -B(\hat{y}^k - y^k) + (\hat{\lambda}^k - \lambda^k)/\beta \end{pmatrix} \right\} \\
&\le \mu\big\|x^k - \hat{x}^k\big\|_R^2 + \mu\big\|y^k - \hat{y}^k\big\|_S^2.
\end{aligned}$$

Then, recalling the definitions of P in (7) and N in (8), the above inequality can be written as

$$\big(\hat{w}^k - w\big)^\top Q\big(\hat{w}^k\big) \le \big\|w^k - \hat{w}^k\big\|_N^2 - \big(w - \hat{w}^k\big)^\top P\big(w^k - \hat{w}^k\big), \quad \forall w \in \mathcal{W}. \tag{18}$$

In addition, if $w^k = w^{k+1}$, then we have $w^k = \hat{w}^k$, which together with (18) indicates that

$$\big(w - \hat{w}^k\big)^\top Q\big(\hat{w}^k\big) \ge 0, \quad \forall w \in \mathcal{W}.$$

This implies that $\hat{w}^k = (\hat{x}_1^k, \hat{x}_2^k, \hat{\lambda}^k)$ is a solution of $\mathrm{VI}(\mathcal{W}, Q)$. Since $\hat{w}^k = w^{k+1}$, therefore w^{k+1} is also a solution of $\mathrm{VI}(\mathcal{W}, Q)$. This completes the proof. □

The next lemma further refines the right term of (17) and express it in terms of some quadratic terms, and its proof is motivated by Lemma 3.3 in [13].

Lemma 3.3 *Let the sequence $\{w^k\}$ be generated by Algorithm 1. Then, for any $w \in \mathcal{W}$, we have*

$$\begin{aligned}
&\big\|w^k - \hat{w}^k\big\|_N^2 - \big(w - \hat{w}^k\big)^\top P\big(w^k - \hat{w}^k\big) \\
&\quad = \frac{1}{2}\big(\big\|w - w^k\big\|_H^2 - \big\|w - w_*^{k+1}\big\|_H^2\big) - \frac{1}{2}\big\|w^k - \hat{w}^k\big\|_{\tilde{H}}^2.
\end{aligned} \tag{19}$$

Proof Taking $a = w$, $b = \hat{w}^k$, $c = w^k$, $d = w_*^{k+1}$ in the identity

$$(a-b)^\top H(c-d) = \frac{1}{2}\big(\|a-d\|_H^2 - \|a-c\|_H^2\big) + \frac{1}{2}\big(\|c-b\|_H^2 - \|d-b\|_H^2\big),$$

we get

$$\begin{aligned}
&\big(w - \hat{w}^k\big)^\top H\big(w^k - w_*^{k+1}\big) \\
&\quad = \frac{1}{2}\big(\big\|w - w_*^{k+1}\big\|_H^2 - \big\|w - w^k\big\|_H^2\big) + \frac{1}{2}\big(\big\|w^k - \hat{w}^k\big\|_H^2 - \big\|w_*^{k+1} - \hat{w}^k\big\|_H^2\big),
\end{aligned}$$

which combined with (9) and (13) yields

$$\begin{aligned}
&(w-\hat{w}^k)^\top P(w^k-\hat{w}^k)\\
&\quad=\frac{1}{2}\bigl(\|w-w_*^{k+1}\|_H^2-\|w-w^k\|_H^2\bigr)+\frac{1}{2}\bigl(\|w^k-\hat{w}^k\|_H^2-\|w_*^{k+1}-\hat{w}^k\|_H^2\bigr).
\end{aligned}\tag{20}$$

For the last term of (20), we have

$$\begin{aligned}
&\|w^k-\hat{w}^k\|_H^2-\|w_*^{k+1}-\hat{w}^k\|_H^2\\
&\quad=\|w^k-\hat{w}^k\|_H^2-\|(w^k-\hat{w}^k)-(w^k-w_*^{k+1})\|_H^2\\
&\quad=\|w^k-\hat{w}^k\|_H^2-\|(w^k-\hat{w}^k)-M(w^k-\hat{w}^k)\|_H^2\\
&\quad=2(w^k-\hat{w}^k)^\top HM(w^k-\hat{w}^k)-(w^k-\hat{w}^k)^\top M^\top HM(w^k-\hat{w}^k)\\
&\quad=(w^k-\hat{w}^k)(P^\top+P-M^\top HM)(w^k-\hat{w}^k).
\end{aligned}$$

Substituting it in (20), we obtain (19). The proof is complete. □

The following theorem indicates the sequence generated by Algorithm 1 is Fejèr monotone with respect to $\mathcal{W}^*$.

Theorem 3.1 *Let* $\{w^k\}$ *be the sequence generated by Algorithm* 1. *Then, for any* $w^*\in\mathcal{W}^*$, *we have*

$$\|w_*^{k+1}-w^*\|_H^2\le\|w^k-w^*\|_H^2-\|w^k-\hat{w}^k\|_{\tilde{H}}^2.\tag{21}$$

Proof From (18), (19), and the monotonicity of Q, we obtain

$$\begin{aligned}
&(\hat{w}^k-w)^\top Q(w)\\
&\quad\le(\hat{w}^k-w)^\top Q(\hat{w}^k)\\
&\quad\le\|w^k-\hat{w}^k\|_N^2-(w-\hat{w}^k)^\top P(w^k-\hat{w}^k)\\
&\quad=\frac{1}{2}\bigl(\|w-w^k\|_H^2-\|w-w_*^{k+1}\|_H^2\bigr)-\frac{1}{2}\|w^k-\hat{w}^k\|_{\tilde{H}}^2.
\end{aligned}\tag{22}$$

The assertion (21) follows immediately by setting $w=w^*\in\mathcal{W}^*$ in (22). The theorem is proved. □

Now, we are ready to prove the global convergence of Algorithm 1.

Theorem 3.2 *The sequence* $\{w^k\}$ *generated by Algorithm* 1 *converges to some* w^∞, *which belongs to* $\mathcal{W}^*$.

Proof First, by (21), for any given $w^*\in\mathcal{W}^*$, we have

$$\|w_*^{k+1}-w^*\|_H\le\|w^k-w^*\|_H,$$

which together with (14) implies that

$$\|w^{k+1}-w^*\|_H \le \|w^{k+1}-w_*^{k+1}\|_H + \|w_*^{k+1}-w^*\|_H \le \rho\nu_k + \|w^k-w^*\|_H.$$

Therefore, for any $l \le k$, we have

$$\|w^{k+1}-w^*\|_H \le \|w^l-w^*\|_H + \rho\sum_{i=l}^{k}\nu_i.$$

Since $\sum_{k=0}^{\infty}\nu_k < +\infty$, there is a constant $C_{w^*} > 0$, such that

$$\|w^{k+1}-w^*\|_H \le C_{w^*} < +\infty, \quad \forall k \ge 0. \tag{23}$$

Therefore the sequence $\{w^k\}$ generated by Algorithm 1 is bounded. Furthermore, it follows from (14), (21), (23) that

$$\begin{aligned}
&\|w^{k+1}-w^*\|_H^2 \\
&\quad = \|(w^{k+1}-w_*^{k+1}) + (w_*^{k+1}-w^*)\|_H^2 \\
&\quad \le \|w^{k+1}-w_*^{k+1}\|_H^2 + 2\|w^{k+1}-w_*^{k+1}\|_H \times \|w_*^{k+1}-w^*\|_H + \|w_*^{k+1}-w^*\|_H^2 \\
&\quad \le \|w^k-w^*\|_H^2 - \|w^k-\hat{w}^k\|_{\tilde{H}}^2 + 2\rho\nu_k\|w^k-w^*\|_H + \rho^2\nu_k^2 \\
&\quad \le \|w^k-w^*\|_H^2 - \|w^k-\hat{w}^k\|_{\tilde{H}}^2 + 2\rho\nu_k C_{w^*} + \rho^2\nu_k^2.
\end{aligned} \tag{24}$$

Then, summing the inequality (24) over $k = 0, 1, \ldots$ and by $\sum_{k=0}^{\infty}\nu_k < +\infty$, we have

$$\sum_{k=0}^{\infty}\|w^k-\hat{w}^k\|_{\tilde{H}}^2 \le \|w^0-w^*\|_H^2 + \sum_{k=0}^{\infty}(2\rho\nu_k C_{w^*} + \rho^2\nu_k^2) < +\infty,$$

which implies that

$$\lim_{k\to\infty}\|w^k-\hat{w}^k\|_{\tilde{H}}^2 = 0. \tag{25}$$

Thus the sequence $\{\hat{w}^k\}$ is also bounded, and thus it has at least one cluster point. Let w^∞ be a cluster point of $\{\hat{w}^k\}$ and let the subsequence $\{\hat{w}^{k_j}\}$ converge to w^∞. Then, by (18) and (25), we can get

$$\lim_{k\to\infty}(w-\hat{w}^k)^\top Q(\hat{w}^k) \ge 0, \quad \forall w \in \mathcal{W}. \tag{26}$$

That is,

$$(w-w^\infty)^\top Q(w^\infty) \ge 0, \quad \forall w \in \mathcal{W},$$

which implies that $w^\infty \in \mathcal{W}^*$. By (24), we have

$$\|w^{k+1}-w^\infty\|_H^2 \le \|w^l-w^\infty\|_H^2 + \sum_{i=l}^{\infty}(2\rho\nu_i C_{w^*} + \rho^2\nu_i^2), \quad \forall k \ge 0, \forall l \le k. \tag{27}$$

From $\lim_{k\to\infty}\|w^k-\hat{w}^k\|_{\tilde{H}}=0$ and $\{\hat{w}^{k_j}\}\to w^\infty$, for any given $\epsilon>0$, there exists an integer j_0, such that

$$\|w^{k_{j_0}}-w^\infty\|_{\tilde{H}}<\frac{\epsilon}{\sqrt{2}}\quad\text{and}\quad\sum_{i=k_{j_0}}^{\infty}\bigl(2\rho\nu_iC_{w^*}+\rho^2\nu_i^2\bigr)<\frac{\epsilon^2}{2}.$$

Therefore, for any $k\geq k_{j_0}$, it follows from the above two equalities and (27) that

$$\|w^{k+1}-w^\infty\|_{\tilde{H}}\leq\sqrt{\|w^{k_{j_0}}-w^\infty\|_{\tilde{H}}^2+\sum_{i=k_{j_0}}^{\infty}\bigl(2\rho\nu_iC_{w^*}+\rho^2\nu_i^2\bigr)}<\epsilon,$$

which combining with the positive definite of $\tilde{H}$ indicates that the sequence $\{w^k\}$ converges to $w^\infty\in\mathcal{W}^*$. This completes the proof. □

Now, we are going to establish the convergence rate of Algorithm 1 in a nonergodic sense.

Theorem 3.3 *Let* $\{w^k\}$ *be the sequence generated by Algorithm* 1. *Then, for any* $w\in\mathcal{W}$, *we have*

$$(\tilde{w}_t-w)^\top Q(w)\leq\frac{1}{t+1}\left(\frac{1}{2}\|w^0-w\|_H^2+\rho\sum_{k=0}^{t}\nu_k\|w-w^{k+1}\|_H\right),\tag{28}$$

where $\tilde{w}_t=(\sum_{k=0}^t\hat{w}^k)/(t+1)$.

Proof From (22), we have

$$\bigl(w-\hat{w}^k\bigr)^\top Q(w)+\frac{1}{2}\|w-w^k\|_H^2\geq\frac{1}{2}\|w-w_*^{k+1}\|_H^2,\quad\forall w\in\mathcal{W}.$$

It follows from (14) that

$$\begin{aligned}
&\|w-w_*^{k+1}\|_H^2\\
&\quad\geq\bigl(\|w-w^{k+1}\|_H-\|w_*^{k+1}-w^{k+1}\|_H\bigr)^2\\
&\quad=\|w-w^{k+1}\|_H^2-2\|w-w^{k+1}\|_H\times\|w_*^{k+1}-w^{k+1}\|_H+\|w_*^{k+1}-w^{k+1}\|_H^2\\
&\quad\geq\|w-w^{k+1}\|_H^2-2\rho\nu_k\|w-w^{k+1}\|_H,\quad\forall w\in\mathcal{W}.
\end{aligned}$$

From the above two inequalities, we get

$$\begin{aligned}
&\bigl(w-\hat{w}^k\bigr)^\top Q(w)+\frac{1}{2}\|w-w^k\|_H^2\\
&\quad\geq\frac{1}{2}\|w-w^{k+1}\|_H^2-\rho\nu_k\|w-w^{k+1}\|_H,\quad\forall w\in\mathcal{W}.
\end{aligned}$$

Summing the above inequality over $k = 0, 1, \dots, t$, we obtain

$$\begin{aligned}
&\left[(t+1)w - \left(\sum_{k=0}^{t} \hat{w}^k\right)\right]^\top Q(w) + \frac{1}{2}\left\|w^0 - w\right\|_H^2 \\
&\quad \geq \frac{1}{2}\left\|w - w^{t+1}\right\|_H^2 - \rho \sum_{k=0}^{t} \nu_k \left\|w - w^{k+1}\right\|_H \\
&\quad \geq -\rho \sum_{k=0}^{t} \nu_k \left\|w - w^{k+1}\right\|_H, \quad \forall w \in \mathcal{W}.
\end{aligned}$$

Using the notation of $\tilde{w}_t$, we have

$$(w - \tilde{w}_t)^\top Q(w) + \frac{1}{2(t+1)}\left\|w^0 - w\right\|_H^2 \geq -\frac{\rho}{t+1}\sum_{k=0}^{t} \nu_k \left\|w - w^{k+1}\right\|_H, \quad \forall w \in \mathcal{W}.$$

The assertion (28) follows from the above inequality immediately. The proof is completed. □

Remark 3.1 From the proof of Theorem 3.2, there is a constant $D > 0$, such that

$$\left\|w^k\right\|_H \leq D \quad \text{and} \quad \left\|\hat{w}^k\right\|_H \leq D, \quad \forall k \geq 0.$$

Since $\tilde{w}_t = (\sum_{k=0}^{t} \hat{w}^k)/(t+1)$, thus, we also have $\|\tilde{w}_t\| \leq D$. Denote $E_1 = \sum_{k=0}^{\infty} \nu_k < +\infty$. For any $w \in \mathcal{B}_{\mathcal{W}}(\tilde{w}_t) = \{w \in \mathcal{W} \mid \|w - \tilde{w}_t\|_H \leq 1\}$, by (28), we get

$$\begin{aligned}
&(\tilde{w}_t - w)^\top Q(w) \\
&\quad \leq \frac{1}{t+1}\left(\frac{1}{2}\left(\left\|w^0 - \tilde{w}_t\right\|_H + \|\tilde{w}_t - w\|_H\right)^2 + \rho\sum_{k=0}^{t} \nu_k\left(\left\|\tilde{w}_t - w^{k+1}\right\|_H + \|\tilde{w}_t - w\|_H\right)\right) \\
&\quad \leq \frac{1}{t+1}\left(\frac{1}{2}(2D+1)^2 + \rho E_1(2D+1)\right).
\end{aligned}$$

Then, for any given $\epsilon > 0$, the above inequality shows that after at most $\lceil (2D+1)(2D+1+2\rho E_1)/(2\epsilon) - 1\rceil$ iterations, we can get

$$(\tilde{w}_t - w)^\top Q(w) \leq \epsilon, \quad \forall w \in \mathcal{B}_{\mathcal{W}}(\tilde{w}_t).$$

This indicates that $\tilde{w}_t$ is an approximate solution of $\mathrm{VI}(\mathcal{W}, Q)$ with an accuracy of $\mathcal{O}(1/t)$. Thus a worst-case $\mathcal{O}(1/t)$ convergence rate of Algorithm 1 in the ergodic sense is established.

4 Numerical experiments

In this section, we apply Algorithm 1 to the traffic equilibrium problem with link capacity bound [22], which has been well studied in the literature of transportation. All codes were written by Matlab R2010a and conducted on a ThinkPad notebook with a Pentium (R) Dual-Core CPU T4400@2.2 GHz, 2GB of memory.

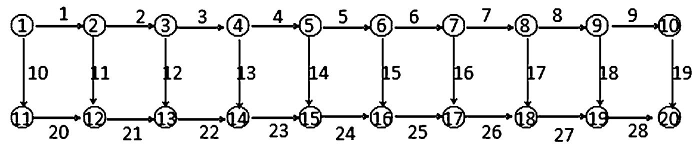

Figure 1 A directed network with 20 nodes and 28 links.

Table 1 The link traversing cost functions $t_a(\hat{f})$

$t_1(\hat{f}) = 5 \cdot 10^{-5}\hat{f}_1^4 + 5\hat{f}_1 + 2\hat{f}_2 + 500$	$t_{15}(\hat{f}) = 3 \cdot 10^{-5}\hat{f}_{15}^4 + 9\hat{f}_{15} + 2\hat{f}_{14} + 200$
$t_2(\hat{f}) = 3 \cdot 10^{-5}\hat{f}_2^4 + 4\hat{f}_2 + 4\hat{f}_1 + 200$	$t_{16}(\hat{f}) = 8\hat{f}_{16}^4 + 5\hat{f}_{12} + 300$
$t_3(\hat{f}) = 5 \cdot 10^{-5}\hat{f}_3^4 + 3\hat{f}_3 + \hat{f}_4 + 350$	$t_{17}(\hat{f}) = 3 \cdot 10^{-5}\hat{f}_{17}^4 + 7\hat{f}_{17} + 2\hat{f}_{15} + 450$
$t_4(\hat{f}) = 3 \cdot 10^{-5}\hat{f}_4^4 + 6\hat{f}_4 + 3\hat{f}_5 + 400$	$t_{18}(\hat{f}) = 5\hat{f}_{18} + \hat{f}_{16} + 300$
$t_5(\hat{f}) = 6 \cdot 10^{-5}\hat{f}_5^4 + 6\hat{f}_5 + 4\hat{f}_6 + 600$	$t_{19}(\hat{f}) = 8\hat{f}_{19} + 3\hat{f}_{17} + 600$
$t_6(\hat{f}) = 7\hat{f}_6 + 3\hat{f}_7 + 500$	$t_{20}(\hat{f}) = 3 \cdot 10^{-5}\hat{f}_{20}^4 + 6\hat{f}_{20} + \hat{f}_{21} + 300$
$t_7(\hat{f}) = 8 \cdot 10^{-5}\hat{f}_7^4 + 8\hat{f}_7 + 2\hat{f}_8 + 400$	$t_{21}(\hat{f}) = 4 \cdot 10^{-5}\hat{f}_{21}^4 + 4\hat{f}_{21} + \hat{f}_{22} + 400$
$t_8(\hat{f}) = 4 \cdot 10^{-5}\hat{f}_8^4 + 5\hat{f}_8 + 2\hat{f}_9 + 650$	$t_{22}(\hat{f}) = 2 \cdot 10^{-5}\hat{f}_{22}^4 + 6\hat{f}_{22} + \hat{f}_{23} + 500$
$t_9(\hat{f}) = 10^{-5}\hat{f}_9^4 + 6\hat{f}_9 + 2\hat{f}_{10} + 700$	$t_{23}(\hat{f}) = 3 \cdot 10^{-5}\hat{f}_{23}^4 + 9\hat{f}_{23} + 2\hat{f}_{24} + 350$
$t_{10}(\hat{f}) = 4\hat{f}_{10} + \hat{f}_{12} + 800$	$t_{24}(\hat{f}) = 2 \cdot 10^{-5}\hat{f}_{24}^4 + 8\hat{f}_{24} + \hat{f}_{25} + 400$
$t_{11}(\hat{f}) = 7 \cdot 10^{-5}\hat{f}_{11}^4 + 7\hat{f}_{11} + 4\hat{f}_{12} + 650$	$t_{25}(\hat{f}) = 3 \cdot 10^{-5}\hat{f}_{25}^4 + 9\hat{f}_{25} + 3\hat{f}_{26} + 450$
$t_{12}(\hat{f}) = 8\hat{f}_{12} + 2\hat{f}_{13} + 700$	$t_{26}(\hat{f}) = 6 \cdot 10^{-5}\hat{f}_{26}^4 + 7\hat{f}_{26} + 8\hat{f}_{27} + 300$
$t_{13}(\hat{f}) = 10^{-5}\hat{f}_{13}^4 + 7\hat{f}_{13} + 3\hat{f}_{18} + 600$	$t_{27}(\hat{f}) = 3 \cdot 10^{-5}\hat{f}_{27}^4 + 8\hat{f}_{27} + 3\hat{f}_{28} + 500$
$t_{14}(\hat{f}) = 8\hat{f}_{14} + 3\hat{f}_{15} + 500$	$t_{28}(\hat{f}) = 3 \cdot 10^{-5}\hat{f}_{28}^4 + 7\hat{f}_{28} + 650$

Consider a network $[\mathcal{N}, \mathcal{L}]$ of nodes $\mathcal{N}$ and directed links $\mathcal{L}$, which is depicted in Figure 1, and consists of 20 nodes, 28 links and 8 O/D pairs.

We use the following symbols. a: a link; p: a path; ω: an origin/destination (O/D) pair of nodes; $\mathcal{P}_\omega$: the set of all paths connecting the O/D pair ω; $\hat{A}$: the path-arc incidence matrix; E: the path-O/D pair incident matrix; x_p: the traffic flow on the path p; $\hat{f}_a$: the link load on the link a; d_ω: the traffic amount between the O/D pair ω. Thus, the link-flow vector $\hat{f}$ is given by

$$\hat{f} = \hat{A}^\top x$$

and the O/D pair-traffic amount vector d is given by

$$d = E^\top x.$$

Let $t(\hat{f}) = \{t_a, a \in \mathcal{L}\}$ be the vector of link travel costs, which is given in Table 1. For a given link travel cost vector t, the path travel cost vector θ is given by

$$\theta = \bar{A}t(\hat{f}) \quad \text{and} \quad \theta(x) = \hat{A}t(\hat{A}^\top x).$$

Associated with every O/D pair ω, there is a travel disutility $\eta_\omega(d)$, which is defined by

$$\eta_\omega(d) = -m_\omega d_\omega + q_\omega, \quad \forall \omega, \tag{29}$$

and the parameters m_ω, q_ω are given in Table 2. Now, the traffic network equilibrium prob-

Table 2 The O/D pairs and the parameters in (25)

No. of pairs	1	2	3	4	5	6	7	8
(O, D)	(1, 20)	(1, 19)	(2, 17)	(4, 20)	(6, 19)	(2, 20)	(2, 13)	(3, 14)
m_ω	5	6	1	6	10	10	5	4
q_ω	1,000	2,000	5,000	1,000	5,000	2,000	1,000	2,000
No. of paths	10	9	6	7	4	9	2	2

lem is to seek the path-flow pattern x^* [22]:

$$x^* \geq 0, \quad \left(x - x^*\right)^\top \hat{F}\left(x^*\right) \geq 0, \quad \forall x \in \mathcal{S} := \left\{x \in \mathcal{R}^n | \hat{A}^\top x \leq b, x \geq 0\right\},$$

where

$$\hat{F}_p(x) = \theta_p(x) - \eta_\omega\left(d(x)\right), \quad \forall \omega, p \in \mathcal{P}_\omega,$$

and b is the given link capacity vector. Using matrices $\hat{A}$ and E, a compact form of mapping is $\hat{F}(x) = \hat{A}t(\hat{A}^\top x) - E\eta(E^\top x)$. Introducing a slack variable $y \geq 0$ and setting $g(y) = 0$, $B = I$, the above problem can be converted into Problem (1). That is,

$$\left(u - u^*\right)^\top T\left(u^*\right) \geq 0, \quad u \in \Omega, \tag{30}$$

with

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad T(u) = \begin{pmatrix} \hat{F}(x) \\ 0 \end{pmatrix}, \qquad \Omega = \left\{(x, y) | \hat{A}^\top x + y = b, x \geq 0, y \geq 0\right\}.$$

When $v_k = 0$ ($\forall k \geq 0$), the implementation details of the two nonlinear equations of Algorithm 1 at each iteration are

$$\begin{cases} -[\lambda^k - \beta(y^{k+1} + \hat{A}^\top x^k - b)] + S[(y^{k+1} - y^k) + \mu(y^k - Q_k^2(y^{k+1})^{-1})] = 0, \\ \lambda^{k+\frac{1}{2}} = \lambda^k - r\beta(y^{k+1} + \hat{A}^\top x^k - b), \\ \hat{F}(x^{k+1}) - \hat{A}[\lambda^{k+\frac{1}{2}} - \beta[\alpha y^{k+1} - (1-\alpha)(\hat{A}^\top x^{k+1} - b) + \hat{A}^\top x^{k+1} - b] \\ \quad + R[(x^{k+1} - x^k) + \mu(x^k - P_k^2(x^{k+1})^{-1})]] = 0, \\ \lambda^{k+1} = \lambda^{k+\frac{1}{2}} - \beta[\alpha y^{k+1} - (1-\alpha)(\hat{A}^\top x^{k+1} - b) + \hat{A}^\top x^{k+1} - b]. \end{cases}$$

For the y-subproblem, it is easy to get its solution explicitly [23]:

$$y_j^{k+1} = \frac{-ss_j^k + \sqrt{(ss_j^k)^2 + 4\mu s_j(\beta + s_j)(y_j^k)^2}}{2(\beta + s_j)},$$

where

$$ss^k = -\lambda^k + \beta\left(\hat{A}^\top x^k - b\right) - (1 - \mu)Sy^k.$$

For the x-subproblem, we use the LQP-type method developed in [24] to solve it. In the test, we take $x^0 = (1, 1, \dots, 1)^\top$, $y^0 = (1, 1, \dots, 1)^\top$, and $\lambda^0 = (0, 0, \dots, 0)^\top$ as the starting point.

Table 3 Numerical results for different ϵ, α, and link capacity b

Link-flow capacity	ϵ	Algorithm 1				IADMM
		α: **0.3**	**0.6**	**0.9**	**1.2**	
$b = 30$	10^{-4}	121/0.1065	103/0.0974	97/0.0886	90/0.0884	115/0.0955
	10^{-5}	142/0.1200	127/0.1094	121/0.1091	114/0.1028	146/0.1325
	10^{-6}	163/0.1388	151/0.1284	146/0.1250	135/0.1157	169/0.1357
$b = 40$	10^{-4}	133/0.1115	121/0.1003	120/0.1026	115/0.0960	131/0.1148
	10^{-5}	152/0.1277	149/0.1290	143/0.1215	138/0.1136	156/0.1310
	10^{-6}	179/0.1373	169/0.1259	167/0.1324	165/0.1300	180/0.1760

Table 4 The optimal link flow and the toll charge on the link with $b = 40$

Link	Flow	Charge	Link	Flow	Charge
1	0	0	15	27.06	0
2	12.94	0	16	5.27	0
3	40.00	251.5	17	1.83	0
4	12.94	0	18	32.90	0
5	0	0	19	5.27	0
6	40.00	1,254.1	20	5.27	0
7	34.73	0	21	5.27	0
8	32.90	0	22	33.95	0
9	0	0	23	0	0
10	0	0	24	12.94	0
11	0	0	25	40.00	1,180.9
12	33.95	0	26	32.32	0
13	27.06	0	27	34.16	0
14	12.94	0	28	0	0

For the test problem, the stopping criterion is

$$\max\left\{\frac{\|e_x(w^k)\|_\infty}{\|e_x(w^0)\|_\infty}, \|e_y(w^k)\|_\infty, \|e_\lambda(w^k)\|_\infty\right\} \le \epsilon,$$

where

$$e(w^k) := \begin{pmatrix} e_x(w^k) \\ e_y(w^k) \\ e_\lambda(w^k) \end{pmatrix} = \begin{pmatrix} x^k - P_{\mathcal{R}^n_+}\{x^k - [\hat{F}(x^k) - \hat{A}\lambda^k]\} \\ y^k - P_{\mathcal{R}^n_+}[y^k + \lambda^k] \\ \hat{A}^\top x^k + y^k - b \end{pmatrix}.$$

In the test, we take $\mu = 0.01$, $\beta = 0.8$, $r = 0.8$, $R = 100I$, $S = 0.9I$. To illustrate the superiority of Algorithm 1, we also implement the inexact ADMM (denoted by IADMM) presented in [25] to solve this example under the same computational environment. The numerical results for different capacities ($b = 30$ and $b = 40$) and different ϵ and α are listed in Table 1, where the numbers in the tuplet '$\cdot/\cdot$' represents, respectively, the numbers of iterations (Iter.) and the CPU time in seconds. Numerical results in Table 3 indicate that Algorithm 1 is an efficient method for the traffic equilibrium problem with link capacity bound, and it is superior to the IADMM in terms of number of iteration and CPU time. Furthermore, the two criteria of Algorithm 1 decrease with respect to α, as one has pointed in [26].

In addition, for the test problem with $b = 40$, the optimal link-flow (Flow) vector $\hat{A}x^*$ and the toll charge (Charge) on the congested link $-\lambda^*$ are listed in Table 4.

5 Conclusions

In this paper, we have proposed an inexact generalized PRSM with LQP regularization for the structured variational inequalities, for which one only needs to solve two nonlinear equations approximately at each iteration. Under mild conditions, we have proved the global convergence of the new method and establish its convergence rate. Numerical results about the traffic equilibrium problem with link capacity bound indicate that the new method is quite efficient.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The first author has designed the algorithm and the second author has proved its global convergence and convergence rate. The two authors have equally contributed in the numerical results. All authors read and approved the final manuscript.

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