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RESEARCH

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On the Cîrtoaje's conjecture

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Abstract

In this paper, we prove the Cîrtoaje conjecture under other conditions.

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1 Introduction and preliminaries

Within the past years, the power exponential functions have been the subject of very intensive research. Many problems concerning inequalities for the power exponential functions look so simple, but their solutions are not as simple as it seems. A lot of interesting results for inequalities with the power exponential functions have been obtained. The history and a literature review of inequalities with power exponential functions can be found for example in [1]. Some other interesting problems concerning stronger inequalities of power exponential functions can be found in [2]. In the paper, we study one inequality conjectured by Cîrtoaje [3]. Cîrtoaje, in [3], has posted the following conjecture on the inequalities with power exponential functions.

Conjecture 1.1 *If $a, b \in (0;1]$ and $r \in [0;e]$, then*

$$2\sqrt{a^{ra}b^{rb}} \geq a^{rb} + b^{ra}. \tag{1.1}$$

The conjecture was proved by Matejíčka [4]. In [1], Coronel and Huancas posted several conjectures for inequalities with the power exponential functions. Some of them are not valid as was shown by the unknown referee of this paper. We note that Theorems 1.2, 1.3, Lemma 3.1, Conjectures 3.1 and 3.2 in [1] are not valid. With the referee's kind permission we present his counterexample: $n = 3$, $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{9}$, $x_3 = \frac{2}{3}$, $r = \frac{5}{2}$, when

$$x_1^{rx_1} + x_2^{rx_2} + x_3^{rx_3} < x_1^{rx_2} + x_2^{rx_3} + x_3^{rx_1}.$$

In this paper, we show that the conjecture (1.1) is also valid under the conditions: $1 \leq \min^2\{a, b\} \leq \max\{a, b\}$ or $0 < \min\{a, b\} \leq 2/e$ and $\max\{a, b\} \geq 1$, or $0 < \min\{a, b\} \leq e$, $\max\{a, b\} \geq e$, and $r \in [0;e]$.

2 Main results

Theorem 2.1 *Let a, b be positive numbers. The inequality*

$$2\sqrt{a^{ra}b^{rb}} \geq a^{rb} + b^{ra} \tag{2.1}$$

holds for any $r \in \langle 0, e\rangle$ if one of the following three conditions is satisfied:

(a) $a \geq b^2 \geq 1$;

(b) $a \geq 1 \geq \frac{e}{2}b$;

(c) $a \geq e \geq b$.

Proof According to the power mean inequality, it suffices to consider the case where r has the maximum value, that is $r = e$. Without loss of generality, suppose $a \geq b$ and denote

$$H(x) = 2\sqrt{x^{ex}b^{eb}} - x^{eb} - b^{ex}.$$

An easy calculation gives

$$H'(x) = e\bigl(x^{\frac{ex}{2}}b^{\frac{eb}{2}}(\ln x + 1) - bx^{eb-1} - b^{ex}\ln b\bigr),$$
$$H''(x) = e\left(x^{\frac{ex}{2}}b^{\frac{eb}{2}}\left(\frac{e(\ln x + 1)^2}{2} + \frac{1}{x}\right) - b(eb-1)x^{eb-2} - eb^{ex}\ln^2 b\right).$$

Solution for (a): $a \geq b^2 \geq 1$.

Suppose $x \geq b^2 \geq 1$. We show $H''(x) \geq 0$, $H(b^2) \geq 0$, and $H'(b^2) \geq 0$. It implies $H(a) \geq 0$ for $a \geq b^2 \geq 1$.

It is easy to see that $\frac{1}{e}H''(x) \geq U + V$, where

$$U = x^{\frac{ex}{2}}b^{\frac{eb}{2}}e\ln b - eb^{ex}\ln^2 b,$$
$$V = x^{\frac{ex}{2}}b^{\frac{eb}{2}}\left(\frac{e\ln^2 b}{2} + \frac{e}{2}\right) + x^{\frac{ex}{2}-1}b^{\frac{eb}{2}} - eb^2x^{eb-2} + bx^{eb-2}.$$

Similarly, we estimate

$$U \geq eb^{ex}\ln b\bigl(b^{\frac{eb}{2}} - \ln b\bigr) \geq eb^{ex}\ln b.$$

It follows from

$$f(b) = \frac{eb}{2}\ln b - \ln(1 + \ln b) \geq 0,$$

because of

$$f'(b) = \frac{eb(1+\ln b)^2 - 2}{2b(1+\ln b)} \geq 0, \quad f(1) = 0.$$

Next we estimate

$$\begin{aligned} V &= x^{eb-2}\left(x^{\frac{ex}{2}-eb+2}b^{\frac{eb}{2}}\left(\frac{e\ln^2 b}{2}+\frac{e}{2}\right)+x^{\frac{ex}{2}+1-eb}b^{\frac{eb}{2}}-eb^2+b\right) \\ &\geq x^{eb-2}\left(b^{eb^2-2eb+4+\frac{eb}{2}}\left(\frac{e\ln^2 b}{2}+\frac{e}{2}\right)+b^{eb^2+2-2eb+\frac{eb}{2}}-eb^2+b\right). \end{aligned}$$

Using

$$\begin{aligned} &l(b) = eb^2-2eb+4+\frac{eb}{2} \geq \frac{8-e}{2}, \qquad m(b) = eb^2-2eb+2+\frac{eb}{2} \geq \frac{4-e}{2}, \\ &l'(b) = 2eb-2e+\frac{e}{2} > 0, \qquad m'(b) = 2eb-2e+\frac{e}{2} > 0, \qquad l(1) = \frac{8-e}{2}, \\ &m(1) = \frac{4-e}{2}, \qquad \frac{8-e}{2} > \frac{5}{2}, \qquad \frac{4-e}{2} > \frac{1}{2}, \end{aligned}$$

we have $H''(x) \geq 0$ for $x \geq b^2 \geq 1$ if

$$p(b) = \frac{e}{2}b^{\frac{5}{2}} - eb^2 + b + b^{\frac{1}{2}} \geq 0.$$

Put $u = b^{1/2}$ then we obtain $p(b) \geq 1$ if

$$k(u) = \frac{e}{2}u^4 - eu^3 + u + 1 \geq 0.$$

It is easy to see that $k'(u) = 0$ if

$$q(u) = 2eu^3 - 3eu^2 + 1 = 0.$$

From Cardano's formula we see that the root of $q(u)$ is equal to $u = 1.4071$. From $k(1.4071) = 0.5602$ and from $k''(u) = 6eu^2 - 6eu \geq 0$ we have $k(u) \geq 0$ for $u \geq 1$. So $H''(x) \geq 0$ for $x \geq b^2 \geq 1$.

Now we show $H'(b^2) \geq 0$.

Some calculation gives

$$H'(b^2) = e\left(b^{eb^2+\frac{eb}{2}}(2\ln b+1) - b^{2eb-1} - b^{eb^2}\ln b\right).$$

From this we have $H'(b^2) \geq 0$ if

$$\left(b^{eb^2+\frac{eb}{2}} - b^{eb^2}\right)\ln b + b^{eb^2+\frac{eb}{2}}(\ln b+1) - b^{2eb-1} \geq 0.$$

Rewriting this we obtain $H'(b^2) \geq 0$ if

$$b^{2eb-1}\left(b^{eb^2+\frac{eb}{2}-2eb+1}(\ln b+1)-1\right) + b^{eb^2}\ln b\left(b^{\frac{eb}{2}}-1\right) \geq 0.$$

But this will be fulfilled if

$$b^{eb^2-3\frac{eb}{2}+1}(\ln b+1) - 1 \geq 0.$$

This is equivalent to

$$o(b) = \left(eb^2 - 3\frac{eb}{2} + 1\right)\ln b + \ln(\ln b + 1) \geq 0.$$

Using $\ln b \geq \frac{b-1}{b}$ we have $o(b) \geq 0$ if we show that

$$s(b) = \left(eb^2 - 3\frac{eb}{2}\right)\ln b + \ln(2b-1) \geq 0.$$

An easy calculation gives

$$s'(b) = \left(2eb - 3\frac{e}{2}\right)\ln b + eb - 3\frac{e}{2} + \frac{2}{2b-1}.$$

Because of $s(1) = 0$ it suffices to show that $eb - 3\frac{e}{2} + \frac{2}{2b-1} \geq 0$. But this is evident from $2 - \frac{e}{2} \geq 0$ and $eb - 3\frac{e}{2} + \frac{2}{2b-1} = \frac{1}{2b-1}(2e(b-1)^2 + 2 - \frac{e}{2})$.

Now we show that $H(b^2) \geq 0$.

An easy calculation gives

$$H\left(b^2\right) = 2b^{eb^2}b^{\frac{eb}{2}} - b^{eb^2} - b^{2eb} = b^{2eb}\left(2b^{eb^2+\frac{eb}{2}-2eb} - b^{eb^2-2eb} - 1\right) \geq 0$$

if

$$2b^{eb^2+\frac{eb}{2}-2eb} - b^{eb^2-2eb} = b^{eb^2-2eb}\left(2b^{\frac{eb}{2}} - 1\right) \geq 1.$$

It suffices to show that $2b^{\frac{eb}{2}} - 1 \geq b^e$ because of

$$b^{eb^2-2eb}\left(2b^{\frac{eb}{2}} - 1\right) \geq b^{e(b-1)^2} \geq 1.$$

Denote $F(b) = 2b^{\frac{eb}{2}} - b^e$. Then

$$F'(b) = e\left(b^{\frac{eb}{2}}(\ln b + 1) - b^{e-1}\right).$$

Because of $F(1) = 1$ it suffices to show that $F'(b) \geq 0$. We will be done if we show that

$$s(b) = \left(\frac{eb}{2} - e\right)\ln b + \ln(2b-1) \geq 0.$$

We used $\ln b \geq (b-1)/b$. Because of $s(1) = 0$ it suffices to show that

$$s'(b) = \frac{e}{2}\ln b + \frac{2eb^2 + b(4-5e) + 2e}{2b(2b-1)} \geq 0.$$

But this is evident because of $j(b) = 2eb^2 + b(4-5e) + 2e \geq 0$. ($j(1) = 4 - e \geq 0$, $j'(1) = 4 - e \geq 0$, $j''(b) = 4e \geq 0$.)

Solution for (b): $a \geq 1 \geq \frac{e}{2}b$.

If $b \le 2/e$ and $x \ge 1$ then it suffices to show that

$$H'(x) = e\bigl(x^{\frac{ex}{2}} b^{\frac{eb}{2}}(\ln x + 1) - bx^{eb-1} - b^{ex}\ln b\bigr) \ge 0$$

and $H(1) \ge 0$.

It is evident that $H'(x) \ge 0$ if

$$x^{\frac{ex}{2}-eb+1} b^{\frac{eb}{2}-1}(\ln x + 1) \ge 1.$$

It follows from $\frac{ex}{2} - eb + 1 \ge 0$ and from $\frac{eb}{2} - 1 \le 0$. Finally, we show that $H(1) \ge 0$. Denote $v(b) = 2b^{\frac{eb}{2}} - 1 - b^e$.

Then we have $v(2/e) = 4/e - 1 - (2/e)^e = 0.0373$. If we show $v'(b) \le 0$ then the conjecture will be proved. An easy calculation gives

$$v'(b) = e\bigl(b^{\frac{eb}{2}}(\ln b + 1) - b^{e-1}\bigr).$$

$v'(b) \le 0$ if

$$b^{\frac{eb}{2}-e+1}(\ln b + 1) \le 1.$$

If $0 \le b \le 1/e$ then the inequality is evident. Let $1/e \le b \le 2/e$, then $v'(b) \le 0$ if

$$s(b) = \left(\frac{eb}{2} - e + 1\right)\ln b + \ln(\ln b + 1) \le 0.$$

Numerical calculation shows that $s(2/e) = -0.1461$. So it suffices to show that

$$s'(b) = \frac{e}{2}\ln b + \left(\frac{eb}{2} - e + 1\right)\frac{1}{b} + \frac{1}{\ln b + 1}\frac{1}{b} \ge 0.$$

This is equivalent to

$$\frac{1}{2}(\ln b + 1) + \frac{1}{1 + \ln b} + 1 - e \ge 0. \tag{2.2}$$

(We used $eb/2 \ge 1/2$.)

Put $t = 1 + \ln b$. Then (2.2) can be rewritten as

$$\frac{1}{t}m(t) = \frac{1}{t}\bigl(t^2 + 2(1-e)t + 2\bigr) \ge 0$$

for $0 < t \le \ln 2$. From $m'(t) = 2t + 2(1-e) \le -2.0503 < 0$ and from $m(\ln 2) = \ln^2 2 + 2 + 2(1-e)\ln 2 = 0.0984 \ge 0$ we see that the proof of the case (b) is complete.

Solution for (c) $a \ge e \ge b$.

It suffices to show that $\sqrt{a^{ra}b^{rb}} \ge a^{rb}$ and $\sqrt{a^{ra}b^{rb}} \ge b^{ra}$. The first inequality is equivalent to

$$a^{a-b} \ge \left(\frac{a}{b}\right)^b,$$

or

$$a^{x-1} \geq x, \qquad x = \frac{a}{b} \geq 1.$$

Indeed, we have

$$a^{x-1} \geq e^{x-1} \geq x.$$

The second inequality is equivalent to

$$\frac{b}{a}^{a} b^{a-b} \leq 1,$$

or

$$xb^{1-x} \leq 1, \qquad x = \frac{b}{a} \leq 1.$$

Indeed we have

$$xb^{1-x} \leq xe^{1-x} \leq 1.$$

This completes our proof. □

Note 2.1 We note that the proof of the case (c) originates from an unknown reviewer. His proof of (c) is more elegant and his formulation of (c) is more general than ours.

Note 2.2 We note that

$$2\sqrt{10^{e10}5^{e5}} - 10^{e5} - 5^{e10} = -5.6e18,$$

which implies that the inequality (1.1) is not valid for all $a, b \geq 0$.

The referee's counterexample implies that (1.1) cannot be generalized for $n = 3$.

Competing interests

The author declares that he has no competing interests.

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