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More results on generalized singular number inequalities of τ-measurable operators

Yazhou Han* and Jingjing Shao

*Correspondence:
hyz0080@aliyun.com
College of Mathematics and Systems Science, Xinjiang University, Urumqi, 830046, China

Abstract
In this article we give some generalized singular number inequalities for products and sums of τ-measurable operators. Some related arithmetic-geometric mean and Heinz mean inequalities for a generalized singular number of τ-measurable operators are proved.

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1 Introduction

Let $\mathbb{M}_n$ be the space of $n \times n$ complex matrices. Given $A \in \mathbb{M}_n$, we define $|A| = (A^*A)^{\frac{1}{2}}$. The singular values of A, *i.e.*, the eigenvalues of the operator $|A|$, enumerated in decreasing order, will be denoted by $S_j(A)$, $j = 1, 2, \dots, n$. The arithmetic-geometric mean inequality for singular values due to Bhatia and Kittaneh [1] says that

$$2S_j\bigl(AB^*\bigr) \le S_j\bigl(A^*A + B^*B\bigr), \quad j = 1, 2, \dots, n, \tag{1.1}$$

holds for any $A, B \in \mathbb{M}_n$. In 2000, Zhan [2] proved that

$$2S_j(A - B) \le S_j(A \oplus B), \quad j = 1, 2, \dots, n, \tag{1.2}$$

for positive semidefinite matrices $A, B \in \mathbb{M}_n$. On the other hand, Tao [3] observed that if $A, B, K \in \mathbb{M}_n$ with $\begin{pmatrix} A & K \\ K^* & B \end{pmatrix} \ge 0$, then

$$2S_j(K) \le S_j\left(\begin{pmatrix} A & K \\ K^* & B \end{pmatrix}\right), \quad j = 1, 2, \dots, n. \tag{1.3}$$

It was pointed out in [3] that inequalities (1.1), (1.2), and (1.3) are equivalent. According to inequality (1.3), Audenaert [4] (see also [5]) gave a Heinz mean inequality for singular values, that is, if $A, B \in \mathbb{M}_n$ are positive semidefinite matrices and $0 \le r \le 1$, then

$$S_j\bigl(A^rB^{1-r} + A^{1-r}B^r\bigr) \le S_j(A + B), \quad j = 1, 2, \dots, n. \tag{1.4}$$

Among other things, in 2012, Albadawi [6] showed that if $A_i, B_i, X_i \in B(\mathcal{H})$ $(i = 1, 2, \ldots, n)$ with $X_i \geq 0$, then

$$2S_j\left(\sum_{i=1}^{n} A_i X_i B_i^*\right) \leq \Big(\max_{i=1,2,\ldots,n} \|X_i\|\Big) S_j^2\left(\begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ B_1 & B_2 & \cdots & B_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\right) \tag{1.5}$$

holds for $j = 1, 2, \ldots$. Inequality (1.5) yields the well-known arithmetic-geometric mean inequality for singular values as special cases.

Using the notion of the generalized singular number studied by Fack and Kosaki [7], we generalize inequalities (1.1)-(1.5) for τ-measurable operators associated with a semifinite von Neumann algebra $\mathcal{M}$.

2 Preliminaries

Unless stated otherwise, $\mathcal{M}$ will always denote a semifinite von Neumann algebra acting on a Hilbert space $\mathcal{H}$, with a normal faithful semifinite trace τ. We refer to [7, 8] for noncommutative integration. We denote the identity of $\mathcal{M}$ by 1 and let $\mathcal{P}$ denote the projection lattice of $\mathcal{M}$. A closed densely defined linear operator x in $\mathcal{H}$ with domain $D(x) \subseteq \mathcal{H}$ is said to be affiliated with $\mathcal{M}$ if $u^*xu = x$ for all unitary operators u which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If x is affiliated with $\mathcal{M}$, we define its distribution function by $\lambda_s(x) = \tau(e_s^{\perp}(|x|))$ and x will be called τ-measurable if and only if $\lambda_s(x) < \infty$ for some $s > 0$, where $e_s^{\perp}(|x|) = e_{(s,\infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (s, ∞). The set of all τ-measurable operators will be denoted by $\overline{\mathcal{M}}$. The set $\overline{\mathcal{M}}$ is a $*$-algebra with sum and product being the respective closures of the algebraic sum and product.

Definition 2.1 Let $x \in \overline{\mathcal{M}}$ and $t > 0$. The 'tth singular number (or generalized singular number) of x' $\mu_t(x)$ is defined by

$$\mu_t(x) = \inf\bigl\{\|xe\| : e \text{ is a projection in } \mathcal{M} \text{ with } \tau\bigl(e^{\perp}\bigr) \leq t\bigr\}.$$

From Lemma 2.5 in [7] we see that the generalized singular number function $t \to \mu_t(x)$ is decreasing right-continuous and

$$\mu_t(uxv) \leq \|v\|\|u\|\mu_t(x), \quad t > 0, \tag{2.1}$$

for all $u, v \in \mathcal{M}$ and $x \in \overline{\mathcal{M}}$. Moreover,

$$\mu_t\bigl(f(x)\bigr) = f\bigl(\mu_t(x)\bigr), \quad t > 0, \tag{2.2}$$

whenever $0 \leq x \in \overline{\mathcal{M}}$ and f is an increasing continuous function on $[0, \infty)$ satisfying $f(0) = 0$. Proposition 2.2 in [7] implies that

$$\mu_t(x) = \inf\bigl\{s \geq 0; \lambda_s(x) \leq t\bigr\} = \inf\bigl\{s \geq 0; \tau\bigl(e_{(s,\infty)}\bigl(|x|\bigr)\bigr) \leq t\bigr\}, \quad t > 0, \tag{2.3}$$

and

$$\lambda_{\mu_t(x)}(x) \le t, \quad t > 0. \tag{2.4}$$

The space $\overline{\mathcal{M}}$ is a partially ordered vector space under the ordering $x \ge 0$ defined by $(x\xi, \xi) \ge 0$, $\xi \in D(x)$. The trace τ on $\mathcal{M}^+$ (the positive part of $\mathcal{M}$) extends uniquely to an additive, positively homogeneous, unitarily invariant, and normal functional $\widetilde{\tau} : \overline{\mathcal{M}} \to [0, \infty]$, which is given by $\widetilde{\tau}(x) = \int_0^\infty \mu_t(x)\,dt$, $x \in \mathcal{M}^+$. This extension is also denoted by τ. Further,

$$\tau\big(f(x)\big) = \int_0^\infty f\big(\mu_t(x)\big)\,dt$$

whenever $0 \le x \in \overline{\mathcal{M}}$ and f is non-negative Borel function which is bounded on a neighborhood of 0 and satisfies $f(0) = 0$. See [7, 9] for basic properties and detailed information on the generalized singular number. For $0 < p < \infty$, $L^p(\mathcal{M})$ is defined as the set of all densely defined closed operators x affiliated with $\mathcal{M}$ such that

$$\|x\|_p = \tau\big(|x|^p\big)^{\frac{1}{p}} = \left(\int_0^\infty \mu_t(x)^p\,dt\right)^{\frac{1}{p}} < \infty.$$

As usual, we put $L^\infty(\mathcal{M}; \tau) = \mathcal{M}$ and denote by $\|\cdot\|_\infty$ $(= \|\cdot\|)$ the usual operator norm. It is well known that $L^p(\mathcal{M})$ is a Banach space under $\|\cdot\|_p$ $(1 \le p \le \infty)$ (*cf.* [8]).

Let $\mathbb{M}_n(\mathcal{M})$ denote the linear space of $n \times n$ matrices

$$x = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

with entries $x_{ij} \in \mathcal{M}$, $i, j = 1, 2, \ldots, n$. Let $\mathcal{H}^n = \bigoplus_{i=1}^n \mathcal{H}$. Then $\mathbb{M}_n(\mathcal{M})$ is a von Neumann algebra in the Hilbert space $\mathcal{H}^n$. For $x \in \mathbb{M}_n(\mathcal{M})$, define $\tau_n(x) = \sum_{i=1}^n \tau(x_{ii})$, then τ_n is a normal faithful semifinite trace on $\mathbb{M}_n(\mathcal{M})$. The direct sum of operators $x_1, x_2, \ldots, x_n \in \overline{\mathcal{M}}$, denoted by $\bigoplus_{i=1}^n x_i$, is the block-diagonal operator matrix defined on $\mathcal{H}^n$ by

$$\bigoplus_{i=1}^n x_i = \begin{pmatrix} x_{11} & 0 & \cdots & 0 \\ 0 & x_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & x_{nn} \end{pmatrix}.$$

3 Arithmetic-geometric mean and Heinz mean inequalities for generalized singular number of τ-measurable operators

Let $x \in \overline{\mathcal{M}}$ and $d_{\mu(x)}(t)$ be the classical distribution function of $s \to \mu_s(x)$. By Proposition 1.2 of [10], we deduce

$$\lambda_t(x) = d_{\mu(x)}(t) = m\big(\{s \in (0, \infty) : \mu_s(x) > t\}\big), \quad t > 0,$$

where m is the Lebesgue measure on $(0,\infty)$. Since $s \to \mu_s(x)$ is non-increasing and continuous from the right (see, Lemma 2.5 of [7]), we have

$$\lambda_t(x) = \inf\{s > 0 : \mu_s(x) \leq t\}, \quad t > 0.$$

Moreover,

$$\mu_{\lambda_s(x)}(x) \leq s, \quad s > 0. \tag{3.1}$$

The following lemma, which includes a basic property of generalized singular number, plays a central role in our investigation.

Lemma 3.1 *Let $x_i \in \overline{\mathcal{M}}$, $i = 1, 2, \ldots, n$. Then*

$$\mu_t\left(\bigoplus_{i=1}^n x_i\right) = \inf\left\{\max\{\mu_{s_1}(x_1), \mu_{s_2}(x_2), \ldots, \mu_{s_n}(x_n)\} : s_i \geq 0, \sum_{i=1}^n s_i \leq t\right\}.$$

Moreover,

$$\mu_t\left(\bigoplus_{i=1}^n x_i\right) = \inf\left\{\max\{\mu_{s_1}(x_1), \mu_{s_2}(x_2), \ldots, \mu_{s_n}(x_n)\} : s_i \geq 0, \sum_{i=1}^n s_i = t\right\}.$$

Proof Let $s_i \geq 0$ with $\sum_{i=1}^n s_i \leq t$. By (2.4), we get

$$\tau\left(\bigoplus_{i=1}^n e_{(\mu_{s_i}(x_i),\infty)}(|x_i|)\right) \leq \sum_{i=1}^n s_i \leq t.$$

Therefore, according to the definition of generalized singular number, we obtain

$$\mu_t\left(\bigoplus_{i=1}^n x_i\right) \leq \left\|\bigoplus_{i=1}^n x_i e_{[0,\mu_{s_i}(x_i)]}(|x_i|)\right\| \leq \max_{i=1,2,\ldots,n}\{\mu_{s_i}(x_i)\}.$$

For the reverse inclusion, from (2.3), we get

$$\mu_t\left(\bigoplus_{i=1}^n x_i\right) = \inf\left\{s \geq 0 : \tau\left(e_{(s,\infty)}\left(\left|\bigoplus_{i=1}^n x_i\right|\right)\right) \leq t\right\}.$$

Since

$$e_{(s,\infty)}\left(\left|\bigoplus_{i=1}^n x_i\right|\right) = e_{(s,\infty)}\left(\bigoplus_{i=1}^n |x_i|\right) = \bigoplus_{i=1}^n e_{(s,\infty)}(|x_i|),$$

we have

$$\mu_t\left(\bigoplus_{i=1}^n x_i\right) = \inf\left\{s \geq 0 : \sum_{i=1}^n \tau\left(e_{(s,\infty)}(|x_i|)\right) \leq t\right\}.$$

Let $s_i = \tau(e_{(s,\infty)}(|x_i|))$. It follows from inequality (3.1) that $\mu_{s_i}(x_i) \le s$. Hence

$$\max_{i=1,2,\ldots,n}\{\mu_{s_i}(x_i)\} \le \mu_t\left(\bigoplus_{i=1}^{n} x_i\right). \qquad \square$$

Remark 3.2

(1) Let $x \in \overline{\mathcal{M}}$. If $x_i = x$, $i = 1,2,\ldots,n$, it follows from Lemma 3.1 that $\mu_t(\bigoplus_{i=1}^{n} x_i) = \mu_{\frac{t}{n}}(x)$, $t > 0$.

(2) Let $x \in \overline{\mathcal{M}}$. If $x_1 = x$ and $x_i = 0$, $i = 2,3,\ldots,n$, it follows from Lemma 3.1 that $\mu_t(\bigoplus_{i=1}^{n} x_i) = \mu_t(x)$, $t > 0$.

(3) Let $x_1, x_2, y_1, y_2 \in \overline{\mathcal{M}}$ such that $\mu_t(x_i) \le \mu_t(y_i)$, $t > 0$, $i = 1,2$. From Lemma 3.1, we deduce $\mu_t(x_1 \oplus x_2) \le \mu_t(y_1 \oplus y_2)$, $t > 0$.

As an application of Lemma 3.1 we now obtain the desired generalized singular number inequality (1.3) for τ-measurable operators.

Lemma 3.3 *Let $x, y, z \in \overline{\mathcal{M}}$. If $\left(\begin{smallmatrix} x & z \\ z^* & y \end{smallmatrix}\right) \ge 0$, then*

$$2\mu_t(z) \le \mu_t\left(\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}\right), \quad t > 0.$$

Proof Let $N = \left(\begin{smallmatrix} 0 & z \\ z^* & 0 \end{smallmatrix}\right)$, $M = \left(\begin{smallmatrix} x & z \\ z^* & y \end{smallmatrix}\right)$, and $U = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then

$$\begin{aligned}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &= \begin{pmatrix} x & -z \\ -z^* & y \end{pmatrix} \\ &= \begin{pmatrix} x & z \\ z^* & y \end{pmatrix} - 2N.\end{aligned}$$

Hence $N = \frac{1}{2}(M - UMU^*)$. Let $N = N^+ - N^-$ be the Jordan decomposition of N. It follows from Lemma 6 of [11] that $\mu_t(N^+) \le \mu_t(\frac{1}{2}M)$, $t > 0$, and

$$\mu_t(N^-) \le \mu_t\left(\frac{1}{2}UMU^*\right) \le \|U\|\left\|U^*\right\|\mu_t\left(\frac{1}{2}M\right) \le \mu_t\left(\frac{1}{2}M\right), \quad t > 0.$$

By Theorem 6 of [12], we have

$$\mu_t(N) = \mu_t\left(N^+ - N^-\right) \le \mu_t\left(N^+ \oplus N^-\right), \quad t > 0.$$

Therefore, from Lemma 3.1 we obtain

$$\mu_{2t}(N) \le \mu_{2t}\left(N^+ \oplus N^-\right) \le \mu_{2t}\left(\frac{1}{2}M \oplus \frac{1}{2}M\right) = \mu_t\left(\frac{1}{2}M\right), \quad t > 0,$$

i.e.,

$$2\mu_{2t}\left(\begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}\right) = 2\mu_{2t}(N) \le \mu_t\left(\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}\right), \quad t > 0.$$

It is clear that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & z \\ z^* & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} z & 0 \\ 0 & z^* \end{smallmatrix}\right)^*$ and $\|\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\| = 1$. Then Lemma 2.5 of [7] and Lemma 3.1 imply that

$$2\mu_t(z) = 2\mu_{2t}\left(\begin{pmatrix} z & 0 \\ 0 & z^* \end{pmatrix}\right) \le 2\mu_{2t}(N) \le \mu_t\left(\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}\right), \quad t > 0. \qquad \square$$

Combing Lemma 3.3 with the following theorem we see that inequalities (1.1), (1.2), and (1.3) hold for τ-measurable operators.

Theorem 3.4 *The following statements are equivalent*:

(1) *Let* $0 \le x, y \in \overline{\mathcal{M}}$. *Then* $\mu_t(x - y) \le \mu_t(x \oplus y)$, $t > 0$.

(2) *For any* $x, y \in \overline{\mathcal{M}}$, $2\mu_t(xy^*) \le \mu_t(x^*x + y^*y)$, $t > 0$.

(3) *Let* $x, y, z \in \overline{\mathcal{M}}$. *If* $\left(\begin{smallmatrix} x & z \\ z^* & y \end{smallmatrix}\right) \ge 0$, *then*

$$2\mu_t(z) \le \mu_t\left(\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}\right), \quad t > 0.$$

Proof (1) $\Rightarrow$ (2): For any $x, y \in \overline{\mathcal{M}}$, we write $X = \left(\begin{smallmatrix} x & 0 \\ y & 0 \end{smallmatrix}\right)$, $Y = \left(\begin{smallmatrix} x & 0 \\ -y & 0 \end{smallmatrix}\right)$. Then $X^*X = \left(\begin{smallmatrix} x^*x+y^*y & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $Y^*Y = \left(\begin{smallmatrix} x^*x+y^*y & 0 \\ 0 & 0 \end{smallmatrix}\right)$. It follows from Lemma 3.1 and (1) that

$$\begin{aligned} 2\mu_t\left(\begin{pmatrix} yx^* & 0 \\ 0 & xy^* \end{pmatrix}\right) = 2\mu_t\left(\begin{pmatrix} 0 & xy^* \\ yx^* & 0 \end{pmatrix}\right) &= \mu_t\big(XX^* - YY^*\big) \\ &\le \mu_t\big(XX^* \oplus YY^*\big) \\ &= \inf\big\{\max\big(\mu_a\big(XX^*\big), \mu_b\big(YY^*\big)\big) : a, b \ge 0, a + b = t\big\} \\ &= \inf\big\{\max\big(\mu_a\big(X^*X\big), \mu_b\big(Y^*Y\big)\big) : a, b \ge 0, a + b = t\big\} \\ &= \inf_{a,b\ge 0, a+b=t}\big\{\max\big(\mu_a\big(x^*x + y^*y\big), \mu_b\big(x^*x + y^*y\big)\big)\big\} \\ &= \mu_t\left(\begin{pmatrix} x^*x + y^*y & 0 \\ 0 & x^*x + y^*y \end{pmatrix}\right), \quad t > 0. \end{aligned}$$

Lemma 3.1 ensures that $2\mu_t(xy^*) \le \mu_t(x^*x + y^*y)$, $t > 0$.

(2) $\Rightarrow$ (1): Let $0 \le x, y \in \overline{\mathcal{M}}$ and let

$$S = \begin{pmatrix} x^{\frac{1}{2}} & -y^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} x^{\frac{1}{2}} & y^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}.$$

From (2) we have $2\mu_t(ST^*) \le \mu_t(S^*S + T^*T)$, $t > 0$. Then the result follows from Lemma 3.1.

From Lemma 3.3 we have (1) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1): For any $0 \le x, y \in \overline{\mathcal{M}}$, we have the following unitary similarity transform:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{x+y}{2} & \frac{x-y}{2} \\ \frac{x-y}{2} & \frac{x+y}{2} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \ge 0.$$

According to (3), we obtain

$$\mu_t(x-y) \le \mu_t\left(\begin{pmatrix} \frac{x+y}{2} & \frac{x-y}{2} \\ \frac{x-y}{2} & \frac{x+y}{2} \end{pmatrix}\right) \le \mu_t \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad t > 0. \qquad \square$$

Lemma 3.5 *Let* $0 \le x, y \in \overline{\mathcal{M}}$ *and* $0 \le r \le 1$. *Then*

$$2\mu_t\left(x^{1+r} + y^{1+r}\right) \ge \mu_t\left((x+y)^{\frac{1}{2}}\left(x^r + y^r\right)(x+y)^{\frac{1}{2}}\right), \quad t > 0.$$

Proof Let $0 \le x, y \in \overline{\mathcal{M}}$ and $0 \le r \le 1$. Since $1 \le 1+r \le 2$, the function $t \to t^{1+r}$ is operator convex. Hence

$$\frac{x^{1+r} + y^{1+r}}{2} \ge \left(\frac{x+y}{2}\right)^{1+r} = \frac{1}{2}(x+y)^{\frac{1}{2}}\left(\frac{x+y}{2}\right)^r (x+y)^{\frac{1}{2}}.$$

Note that $t \to t^r$ $(0 \le r \le 1)$ is operator concave, we obtain $\frac{x^r+y^r}{2} \le (\frac{x+y}{2})^r$. Therefore,

$$x^{1+r} + y^{1+r} \ge \frac{1}{2}(x+y)^{\frac{1}{2}}\left(x^r + y^r\right)(x+y)^{\frac{1}{2}}.$$

This completes the proof. $\square$

Based on Lemma 3.5 we now obtain the desired generalized singular number inequality (1.4) for τ-measurable operators.

Theorem 3.6 *Let* $0 \le r \le 1$ *and* $0 \le x, y \in L^1(\mathcal{M})$. *Then*

$$\mu_t\left(x^r y^{1-r} + x^{1-r} y^r\right) \le \mu_t(x+y), \quad t > 0. \tag{3.2}$$

Proof Let $0 \le \nu \le 1$. If we replace x, y by $x^{\frac{1}{1+\nu}}$, $y^{\frac{1}{1+\nu}}$, respectively, in Lemma 3.5, we deduce

$$2\mu_t(x+y) \ge \mu_t\left(\left(x^{\frac{1}{1+\nu}} + y^{\frac{1}{1+\nu}}\right)^{\frac{1}{2}}\left(x^{\frac{\nu}{1+\nu}} + y^{\frac{\nu}{1+\nu}}\right)\left(x^{\frac{1}{1+\nu}} + y^{\frac{1}{1+\nu}}\right)^{\frac{1}{2}}\right).$$

It follows from Lemma 2 of [13] and the fact $x, y \in L^1(\mathcal{M})$ that

$$2\mu_t(x+y) \ge \mu_t\left(\left(x^{\frac{1}{1+\nu}} + y^{\frac{1}{1+\nu}}\right)\left(x^{\frac{\nu}{1+\nu}} + y^{\frac{\nu}{1+\nu}}\right)\right). \tag{3.3}$$

Note that

$$\begin{aligned}
&\mu_t\left(\begin{pmatrix} (x^{\frac{1}{1+\nu}} + y^{\frac{1}{1+\nu}})(x^{\frac{\nu}{1+\nu}} + y^{\frac{\nu}{1+\nu}}) & 0 \\ 0 & 0 \end{pmatrix}\right) \\
&\quad = \mu_t\left(\begin{pmatrix} x^{\frac{1}{2+2\nu}} & y^{\frac{1}{2+2\nu}} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x^{\frac{1}{2+2\nu}} & 0 \\ y^{\frac{1}{2+2\nu}} & 0 \end{pmatrix}\begin{pmatrix} x^{\frac{\nu}{1+\nu}} + y^{\frac{\nu}{1+\nu}} & 0 \\ 0 & 0 \end{pmatrix}\right) \\
&\quad = \mu_t\left(\begin{pmatrix} x^{\frac{1}{2+2\nu}} & 0 \\ y^{\frac{1}{2+2\nu}} & 0 \end{pmatrix}\begin{pmatrix} x^{\frac{\nu}{1+\nu}} + y^{\frac{\nu}{1+\nu}} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x^{\frac{1}{2+2\nu}} & y^{\frac{1}{2+2\nu}} \\ 0 & 0 \end{pmatrix}\right) \\
&\quad = \mu_t\left(\begin{pmatrix} x^{\frac{1}{2+2\nu}}(x^{\frac{\nu}{1+\nu}} + y^{\frac{\nu}{1+\nu}})x^{\frac{1}{2+2\nu}} & x^{\frac{1}{2+2\nu}}(x^{\frac{\nu}{1+\nu}} + y^{\frac{\nu}{1+\nu}})y^{\frac{1}{2+2\nu}} \\ y^{\frac{1}{2+2\nu}}(x^{\frac{\nu}{1+\nu}} + y^{\frac{\nu}{1+\nu}})x^{\frac{1}{2+2\nu}} & y^{\frac{1}{2+2\nu}}(x^{\frac{\nu}{1+\nu}} + y^{\frac{\nu}{1+\nu}})y^{\frac{1}{2+2\nu}} \end{pmatrix}\right).
\end{aligned}$$

Combining Lemma 3.1, Lemma 3.3, and inequality (3.3) we deduce

$$\begin{aligned}\mu_t(x+y) &\geq \mu_t\bigl(x^{\frac{1}{2+2\nu}}\bigl(x^{\frac{\nu}{1+\nu}}+y^{\frac{\nu}{1+\nu}}\bigr)y^{\frac{1}{2+2\nu}}\bigr)\\ &= \mu_t\bigl(x^{\frac{2\nu+1}{2+2\nu}}y^{\frac{1}{2+2\nu}}+x^{\frac{1}{2+2\nu}}y^{\frac{2\nu+1}{2+2\nu}}\bigr), \quad 0\leq \nu\leq 1.\end{aligned}$$

Therefore,

$$\mu_t\bigl(x^r y^{1-r}+x^{1-r}y^r\bigr)\leq \mu_t(x+y), \quad \frac{1}{2}\leq r\leq \frac{3}{4}.$$

On the one hand, we have

$$\begin{aligned}&\mu_t\left(\begin{pmatrix}(x^{\frac{1}{1+\nu}}+y^{\frac{1}{1+\nu}})(x^{\frac{\nu}{1+\nu}}+y^{\frac{\nu}{1+\nu}}) & 0\\ 0 & 0\end{pmatrix}\right)\\ &\quad= \mu_t\left(\begin{pmatrix}x^{\frac{\nu}{2+2\nu}} & 0\\ y^{\frac{\nu}{2+2\nu}} & 0\end{pmatrix}\begin{pmatrix}x^{\frac{1}{1+\nu}}+y^{\frac{1}{1+\nu}} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x^{\frac{\nu}{2+2\nu}} & y^{\frac{\nu}{2+2\nu}}\\ 0 & 0\end{pmatrix}\right).\end{aligned}$$

Repeating the arguments above we get

$$\mu_t\bigl(x^r y^{1-r}+x^{1-r}y^r\bigr)\leq \mu_t(x+y), \quad \frac{3}{4}\leq r\leq 1.$$

By the symmetry property of inequality (3.2) with respect to $r=\frac{1}{2}$, we see that inequality (3.2) holds for all $0\leq r\leq 1$. □

Let $0\leq x,y\in\overline{\mathcal{M}}$. Then Lemma 3.1 and Theorem 3.4 imply that

$$\mu_t\bigl((x-y)\oplus 0\bigr)\leq \mu_t(x\oplus y), \quad t>0.$$

If $x,y\in\overline{\mathcal{M}}$ with $\mu_t(x)\leq\mu_t(y)$, $t>0$, Lemma 3.1 gives us that

$$\mu_t(x)=\mu_t(x\oplus 0)\leq \mu_t(y\oplus y), \quad t>0.$$

Some examples of such inequalities related to ones discussed above are presented below.

Lemma 3.7 *Let $x,y\in\overline{\mathcal{M}}^{sa}:=\{z\in\overline{\mathcal{M}}; z=z^*\}$ such that $\pm y\leq x$. If $x\geq 0$, then*

$$\mu_t(y)\leq\mu_t(x\oplus x)$$

and

$$\int_0^t \mu_t(y)\,ds\leq\int_0^t \mu_s(x)\,ds, \quad t>0.$$

Proof Since $\pm y\leq x$, we have $-x\leq y\leq x$. Then Theorem 1 of [14] indicates that $2|y|\leq x+uxu^*$ for some unitary $u\in\overline{\mathcal{M}}^{sa}$. From Theorem 4.4 and Lemma 2.5 of [7], we deduce

$$2\mu_t(y) \le \mu_t(x + uxu^*) \le \mu_{\frac{t}{2}}(uxu^*) + \mu_{\frac{t}{2}}(x) \le 2\mu_{\frac{t}{2}}(x) = 2\mu_t(x \oplus x), \quad t > 0,$$

and

$$2\int_0^t \mu_s(y)\,ds \le \int_0^t \mu_s(x + uxu^*)\,ds \le 2\int_0^t \mu_s(x)\,ds, \quad t > 0. \qquad \square$$

We conclude this section with a series of inequalities which are related to the Heinz mean inequality for a generalized singular number of τ-measurable operators.

Proposition 3.8 *Let $x, y \in \overline{\mathcal{M}}$. Then*

$$\mu_t(x^*y + y^*x) \le \mu_t\big((x^*x + y^*y) \oplus (x^*x + y^*y)\big), \quad t > 0, \tag{3.4}$$

and

$$\mu_t(yx^* + xy^*) \le \mu_t\big((x^*x + y^*y) \oplus (x^*x + y^*y)\big), \quad t > 0. \tag{3.5}$$

Proof Since $(x \pm y)^*(x \pm y) \ge 0$, we have $\pm(x^*y + y^*x) \le x^*x + y^*y$. Thus inequality (3.4) follows from Lemma 3.7. Inequality (3.5) follows from Theorem 6 of [12] and Theorem 3.4(2). $\square$

Corollary 3.9 *Let $x, y \in \overline{\mathcal{M}}$ and $0 < r \le \infty$. Then*

$$\int_0^t \mu_s(x^*y + y^*x)\,ds \le \int_0^t \mu_s(x^*x + y^*y)\,ds, \quad t > 0, \tag{3.6}$$

and

$$\int_0^t \mu_s(yx^* + xy^*)\,ds \le \int_0^t \mu_s(x^*x + y^*y)\,ds, \quad t > 0. \tag{3.7}$$

Proof It follows from Lemma 3.7 and the proof of Proposition 3.8. $\square$

Proposition 3.10 *Let $x, y \in \overline{\mathcal{M}}$. Then*

$$\mu_t(x + y) \le \mu_t\big((|x| + |y|) \oplus (|x^*| + |y^*|)\big), \quad t > 0. \tag{3.8}$$

Proof Let $x \in \overline{\mathcal{M}}$. Note that $\begin{pmatrix} |x| & \pm x^* \\ \pm x & |x^*| \end{pmatrix} \ge 0$. Then

$$\begin{pmatrix} |x| + |y| & \pm(x + y)^* \\ \pm(x + y) & |x^*| + |y^*| \end{pmatrix} \ge 0.$$

Thus

$$\pm\begin{pmatrix} 0 & (x + y)^* \\ x + y & 0 \end{pmatrix} \le \begin{pmatrix} |x| + |y| & 0 \\ 0 & |x^*| + |y^*| \end{pmatrix}.$$

By Lemma 3.7, we obtain

$$\begin{aligned}\mu_t\big((x+y)\oplus(x+y)^*\big) &= \mu_t\left(\begin{pmatrix} 0 & (x+y)^* \\ x+y & 0\end{pmatrix}\right) \\ &\le \mu_t\left(\begin{pmatrix} |x|+|y| & 0 \\ 0 & |x^*|+|y^*|\end{pmatrix}\right. \\ &\qquad \left.\oplus \begin{pmatrix} |x|+|y| & 0 \\ 0 & |x^*|+|y^*|\end{pmatrix}\right) \\ &= \mu_{\frac{t}{2}}\left(\begin{pmatrix} |x|+|y| & 0 \\ 0 & |x^*|+|y^*|\end{pmatrix}\right), \quad t>0.\end{aligned}$$

According to Lemma 2.5 of [7] and Lemma 3.1, we get

$$\mu_t\big((x+y)\oplus(x+y)^*\big) = \mu_{\frac{t}{2}}(x+y), \quad t>0.$$

This implies that

$$\mu_t(x+y) \le \mu_t\left(\begin{pmatrix} |x|+|y| & 0 \\ 0 & |x^*|+|y^*|\end{pmatrix}\right), \quad t>0. \qquad \square$$

4 Generalized singular number inequalities for products and sums of τ-measurable operators

In this section, we establish a generalized singular number inequality for τ-measurable operators which yields the well-known arithmetic-geometric mean inequalities as special cases.

The following proposition is a refinement of the inequality in Theorem 3.4(2).

Proposition 4.1 *Let $x, y \in \overline{\mathcal{M}}$ and $0 \le z \in \mathcal{M}$. Then*

$$\mu_t(xzy^*) \le \frac{1}{2}\|z\|\mu_t(x^*x+y^*y), \quad t>0.$$

Proof According to Proposition 2.5(vi) of [7] and Theorem 3.2(2), we have

$$\begin{aligned}2\mu_t(xzy^*) &= 2\mu_t\big(xz^{\frac{1}{2}}z^{\frac{1}{2}}y^*\big) \le \mu_t\big(\big|xz^{\frac{1}{2}}\big|^2 + \big|yz^{\frac{1}{2}}\big|^2\big) \\ &= \mu_t\big(z^{\frac{1}{2}}(x^*x+y^*y)z^{\frac{1}{2}}\big) \le \|z\|\mu_t\big(x^*x+y^*y\big). \qquad \square\end{aligned}$$

From Proposition 4.1 we now obtain the promised generalized singular number inequality (1.5) for τ-measurable operators.

Proposition 4.2 *Let $x_i, y_i \in \overline{\mathcal{M}}$ and $0 \le z_i \in \mathcal{M}$ $(i = 1, 2, \ldots, n)$. Then*

$$2\mu_t\left(\sum_{i=1}^n x_i z_i y_i^*\right) \le \Big(\max_{i=1,2,\ldots,n}\|z_i\|\Big)\mu_t\left(\left(\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0\end{pmatrix}\right)^2\right), \quad t>0.$$

Proof Let

$$A = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$K = \begin{pmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & z_n \end{pmatrix}, \qquad T = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then

$$AKB^* = \begin{pmatrix} \sum_{k=1}^{n} x_i z_i y_i^* & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad A^*A + B^*B = T^*T = |T|^2.$$

From Proposition 4.1, we have

$$2\mu_t\big(AKB^*\big) \leq \|K\|\mu_t\big(A^*A + B^*B\big) = \|K\|\mu_t\big(|T|^2\big) = \|K\|\mu_t(T)^2, \quad t > 0.$$

Then the result follows from Lemma 3.1. □

Proposition 4.2 includes several generalized singular number inequalities as special cases.

Corollary 4.3 *Let $x_i, y_i \in \overline{\mathcal{M}}$ and $0 \leq z_i \in \mathcal{M}$ $(i = 1, 2)$. Then*

$$2\mu_t\big(x_1 z_1 y_1^* + x_2 z_2 y_2^*\big) \leq \Big(\max_{i=1,2} \|z_i\|\Big)\mu_t\left(\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\right)^2, \quad t > 0.$$

In particular,

$$2\mu_t\big(xzy^* + yzx^*\big) \leq \|z\|\mu_t\left(\begin{pmatrix} x & y \\ y & x \end{pmatrix}\right)^2, \quad t > 0.$$

Proof The result follows from Proposition 4.2. □

The following inequality is an application of Corollary 4.3.

Corollary 4.4 *Let $0 \leq x, y \in \overline{\mathcal{M}}$ and $0 \leq z \in \mathcal{M}$. Then, for $t > 0$,*

$$\mu_t\big(x^{\frac{1}{2}} z x^{\frac{1}{2}} + y^{\frac{1}{2}} z y^{\frac{1}{2}}\big) \leq \|z\|\mu_t\Big(\big(x + \big|y^{\frac{1}{2}} x^{\frac{1}{2}}\big|\big) \oplus \big(y + \big|x^{\frac{1}{2}} y^{\frac{1}{2}}\big|\big)\Big).$$

In particular,

$$\mu_t(x+y) \leq \mu_t\big(\big(x + \big|y^{\frac{1}{2}}x^{\frac{1}{2}}\big|\big) \oplus \big(y + \big|x^{\frac{1}{2}}y^{\frac{1}{2}}\big|\big)\big) \quad \textit{for all } t > 0.$$

Proof Let $x_1 = y_1 = x^{\frac{1}{2}}$, $x_2 = y_2 = y^{\frac{1}{2}}$, and $z_1 = z_2 = z$ in Corollary 4.3. Then for all $t > 0$

$$\begin{aligned} 2\mu_t\big(x^{\frac{1}{2}}zx^{\frac{1}{2}} + y^{\frac{1}{2}}zy^{\frac{1}{2}}\big) &\leq \|z\|\mu_t\left(\begin{pmatrix} x^{\frac{1}{2}} & y^{\frac{1}{2}} \\ x^{\frac{1}{2}} & y^{\frac{1}{2}} \end{pmatrix}\right)^2 \\ &= 2\|z\|\mu_t\left(\begin{pmatrix} x & x^{\frac{1}{2}}y^{\frac{1}{2}} \\ y^{\frac{1}{2}}x^{\frac{1}{2}} & y \end{pmatrix}\right) \\ &= 2\|z\|\mu_t(T_1 + T_2), \end{aligned}$$

where $T_1 = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ and $T_2 = \begin{pmatrix} 0 & x^{\frac{1}{2}}y^{\frac{1}{2}} \\ y^{\frac{1}{2}}x^{\frac{1}{2}} & 0 \end{pmatrix}$. It follows from the facts that $T_2 \leq |T_2| = \begin{pmatrix} |y^{\frac{1}{2}}x^{\frac{1}{2}}| & 0 \\ 0 & |x^{\frac{1}{2}}y^{\frac{1}{2}}| \end{pmatrix}$ and $T_1 + |T_2| \geq 0$ that

$$\mu_t\big(x^{\frac{1}{2}}zx^{\frac{1}{2}} + y^{\frac{1}{2}}zy^{\frac{1}{2}}\big) \leq \|z\|\mu_t\big(T_1 + |T_2|\big), \quad t > 0.$$

This gives the desired inequality. □

The following inequality contains a generalization of the inequality in Theorem 3.4(1).

Corollary 4.5 *Let $x, y \in \overline{\mathcal{M}}$ and $0 \leq z \in \mathcal{M}$. Then*

$$\mu_t\big(xzx^* - yzy^*\big) \leq \|z\|\mu_t\big(x^*x \oplus y^*y\big) \quad \textit{for all } t > 0.$$

Proof If we replace x_1, x_2, y_1, y_2 by x, y, x, $-y$, respectively, in Corollary 4.3, we deduce

$$2\mu_t\big(xzx^* - yzy^*\big) \leq \|z\|\mu_t\left(\begin{pmatrix} 2x^*x & 0 \\ 0 & 2y^*y \end{pmatrix}\right) \quad \text{for all } t > 0.$$

□

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

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