Jiang et al. *Journal of Inequalities and Applications* (2016) 2016:141
DOI 10.1186/s13660-016-1082-4

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Convergence of exceedance point processes of normal sequences with a seasonal component and its applications

Yingying Jiang[1,2*], Baokun Li[1] and Fuming Lin[2]

*Correspondence: 240506274@qq.com
[1]School of Statistics, Southwestern University of Finance and Economics, Chengdu, Sichuan 611130, China
[2]School of Science, Sichuan University of Science & Engineering, Zigong, Sichuan 643000, China

Abstract

In this paper, we prove that, under some mild conditions, a time-normalized point process of exceedances by a nonstationary and strongly dependent normal sequence with a seasonal component converges in distribution to the in plane Cox process. As an application of the convergence result, we deduce two important joint limit distributions for the order statistics.

MSC: 60F05; 62E20; 62F12; 62M10

Keywords: in plane Cox process; exceedance point process; nonstationary sequences with a seasonal component; strongly dependent normal sequences; kth maxima

1 Introduction

Let $\{X_i, i \geq 1\}$ be a standardized normal sequence with correlation coefficient $r_{ij} = \mathrm{Cov}(X_i, X_j)$ satisfying the conventional assumption that $r_{ij} \to 0$ and $r_{ij}\log(|i-j|) \to \gamma$ as $j - i \to +\infty$. Normal sequences are weakly dependent if $\gamma = 0$, strongly dependent if $0 < \gamma < \infty$ and stationary if r_{ij} are related to $|i-j|$ only, and nonstationary otherwise. Denote by $M_n^{(k)}$ the kth maximum of $\{X_i, 1 \leq i \leq n\}$, whose location is denoted $L_n^{(k)}$ and may vary among $\{1, \ldots, n\}$. Leadbetter et al. [1] considered a stationary weakly dependent normal sequence $\{X_i, i \geq 1\}$ and obtained the asymptotic joint probability distribution of $M_n^{(1)}$ and $M_n^{(2)}$ and even that of $M_n^{(2)}$ and $L_n^{(2)}$. Mittal and Ylvisaker [2] proved that if $\{X_i, i \geq 1\}$ is stationary and strongly dependent, then $M_n^{(1)}$ (also called the maximum of the sequence) after normalization converges in distribution to the convolution of $\exp(-e^{-x})$ and a normal distribution function. More recent results for maxima of stationary normal sequences can be found in Ho and Hsing [3], Tan and Peng [4], and Hashorva et al. [5], among others. Meanwhile, some literature was devoted to study the maxima of nonstationary normal sequences; see Horowitz [6] and Leadbetter et al. [1] for the weakly dependence case and Zhang [7], Lin et al. [8], and Tan and Yang [9] for the strongly dependence case.

In particular, Leadbetter et al. [1] developed an important tool, the weak convergence of exceedance point processes, which is crucial to study the joint asymptotic distributions of some extremes of sequences. Due to its importance, many authors further studied the asymptotic behavior of exceedance point processes under different conditions; see Piter-

Springer

© 2016 Jiang et al. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

barg [10], Hu et al. [11], Falk et al. [12], Peng et al. [13], Hashorva et al. [14] Wiśniewski [15], Lin et al. [8], Lin et al. [16], and the references therein.

In the paper, we consider $\{X_i = Y_i + m_i, i \geq 1\}$ where Y_i is a standardized nonstationary and strongly dependent normal sequence and m_i is a trend or seasonal component. Define $\eta_n(t), t \in [0,1]$, as a continuous stochastic function such that $\eta_n(t)$ is linear on $[(i-1)/n, i/n]$, $i = 1, 2, \ldots, n$, and has the value X_i at the point i/n ($\eta_n(0) = 0$). A similar definition can also be found in Leadbetter et al. [1]. A vector point process N_n formed by exceedances of the levels $u_n^{(1)}, u_n^{(2)}, \ldots, u_n^{(r)}$ by the stochastic function is called the time-normalized one since we use 'j/n' and set $t \in [0,1]$ in the definition of $\eta_n(t)$. In the sequel, for convenience, the expression 'exceedances by $\{X_j, j \geq 1\}$' stands for 'exceedances by $\eta(t)$.' We prove that the time-normalized point process N_n converges in distribution to the in plane Cox process defined in Lin et al. [16] and extend the results in Lin et al. [16] to the case of more general normal sequences.

The remainder of the paper is structured as follows. In Section 2, we present the notation and main results. Proofs of the main results are postponed to Section 3. Throughout the paper, C stands for a constant that may vary from line to line, and '$\to$' stands for the convergence in distribution as $n \to \infty$.

2 Notation and main results

Let $\{X_i = Y_i + m_i, i \geq 1\}$ be a standardized normal sequence plus a seasonal component with the correlation coefficient of $\{Y_i, i \geq 1\}$ and seasonal component satisfying the following:

$$\sup\bigl\{|r_{ij}|, i \neq j\bigr\} < 1 \quad \text{and} \quad r_{ij} \log(j-i) \to \gamma \in (0,\infty) \quad \text{as } j-i \to +\infty, \tag{2.1}$$

$$\beta_n = \max_{1 \leq i \leq n} |m_i| = o\bigl((\log n)^{1/2}\bigr) \quad \text{as } n \to +\infty, \tag{2.2}$$

$$\frac{1}{n} \sum_{i=1}^{n} \exp\biggl(a_n^*\bigl(m_i - m_n^*\bigr) - \frac{1}{2}\bigl(m_i - m_n^*\bigr)^2\biggr) \to 1 \quad \text{as } n \to \infty, \tag{2.3}$$

where $a_n^* = (2\log n)^{1/2} - \log\log n/2((2\log n)^{1/2})$, and m_n^* is a sequence of constants such that $|m_n^*| \leq \beta_n$. Condition (2.3) is the same as condition (6.2.2) in Leadbetter et al. [1]. Throughout, the standardized constants a_n and b_n are defined by

$$a_n = (2\log n)^{1/2}, \qquad b_n = a_n - (2a_n)^{-1}(\log\log n + \log 4\pi). \tag{2.4}$$

Before presenting the main results, we first give the definition of the in plane Cox process.

Definition 2.1 Let $\{\sigma_{1j}, j = 1, 2, \ldots\}$ be the points of a Cox process $N^{(r)}$ on L_r with (stochastic) intensity $\exp(-x_r - \gamma + \sqrt{2\gamma}\zeta)$, where ζ is a standard normal random variable, x_r is a constant corresponding to the $N^{(r)}$, and L_r is the in plane fixed horizontal line on which exceedances are represented as points. $N^{(r)}$ has the distribution characterized as follows:

$$P\Biggl(\bigcap_{i=1}^{I}\bigl\{N^{(r)}(B_i) = k_i\bigr\}\Biggr) = \int_{-\infty}^{\infty} \prod_{i=1}^{I} \biggl(\frac{(m(B_i)\exp(-x_r - \gamma + \sqrt{2\gamma}z))^{k_i}}{k_i!} \cdot \exp\bigl(-m(B_i)e^{-x_r-\gamma+\sqrt{2\gamma}z}\bigr)\biggr)\phi(z)\,dz, \tag{2.5}$$

where B_i are Borel sets, and $m(\cdot)$ is the Lebesgue measure. Let $\beta_j, j = 1, 2, \dots$, be independent and identically distributed (i.i.d.) random variables, independent also of the Cox process on L_r, taking the values $1, 2, \dots, r$ with conditional probabilities

$$P(\beta_j = s|\zeta = z) = \begin{cases} (\tau_{r-s+1} - \tau_{r-s})/\tau_r & \text{for } s = 1, 2, \dots, r-1, \\ \tau_1/\tau_r & \text{for } s = r, \end{cases}$$

that is, $P(\beta_j \geq s|\zeta = z) = \tau_{r-s+1}/\tau_r$ for $s = 1, 2, \dots, r$, where $\tau_i = e^{-x_i-\gamma+\sqrt{2\gamma}z}$, $i = 1, 2, \dots, r$. For each j, placing points $\sigma_{2j}, \sigma_{3j}, \dots, \sigma_{\beta_j j}$ on $\beta_j - 1$ lines $L_{r-1}, L_{r-2}, \dots, L_{r-\beta_j+1}$, vertically above σ_{1j}, we can obtain an in plane Cox process N. Specifically, the conditional probability that a point appears on L_{r-1} above σ_{1j} is $P(\beta_j \geq 2|\zeta = z) = \tau_{r-1}/\tau_r$, and the deletions are conditionally independent, so that $N^{(r-1)}$ is obtained as a conditionally independent thinning of the Cox process $N^{(r)}$. Similarly, the other $N^{(k)}$, $1 \leq k \leq r-2$, can be constructed.

In Theorem 2.1, we study a vector point process $N_n = (N_n^{(1)}, N_n^{(2)}, \dots, N_n^{(r)})$ that arises when $\{X_i, 1 \leq i \leq n\}$ exceeds the levels $u_n^{(1)}, u_n^{(2)}, \dots, u_n^{(r)}$, the structure of which is the same as that of the exceedance process on pp.111-112 in Leadbetter et al. [1]. We record the exceedance points corresponding to the levels $u_n^{(1)}, u_n^{(2)}, \dots, u_n^{(r)}$ on fixed horizontal lines $L_1, L_2, \dots, L_r$ in the plane.

Theorem 2.1 *Suppose that* $\{X_i = Y_i + m_i, i \geq 1\}$ *satisfies conditions* (2.1)-(2.3), *and let* $u_n^{(k)} = x_k/a_n + b_n + m_n^*$ $(1 \leq k \leq r)$ *satisfy* $u_n^{(1)} \geq u_n^{(2)} \geq \cdots \geq u_n^{(r)}$. *Then the time-normalized exceedance point process* N_n *of levels* $u_n^{(1)}, u_n^{(2)}, \dots, u_n^{(r)}$ *by* $\{X_i, 1 \leq i \leq n\}$ *converges in distribution to the before-mentioned in plane Cox process.*

Corollary 2.1 *Let* $\{X_i, i \geq 1\}$ *satisfy the conditions of Theorem* 2.1. *Let* $B_1, B_2, \dots, B_s$ *be Borel subsets of the unit interval whose boundaries have zero Lebesgue measures. Then, for integers* $m_j^{(k)}$,

$$\begin{aligned} &P\big(N_n^{(k)}(B_j) = m_j^{(k)}, j = 1, 2, \dots, s; k = 1, 2, \dots, r\big) \\ &\quad \to P\big(N^{(k)}(B_j) = m_j^{(k)}, j = 1, 2, \dots, s; k = 1, 2, \dots, r\big). \end{aligned}$$

Theorem 2.2 *Suppose that the levels* $u_n^{(k)}$ $(1 \leq k \leq r)$ *satisfy*

$$P\Big(\max_{1\leq i\leq n} X_i \leq u_n^{(k)}\Big) \to \int_{-\infty}^{+\infty} \exp\big(-e^{-x_k-\gamma+\sqrt{2\gamma}z}\big)\phi(z)\,dz \quad \textit{as } n \to \infty,$$

with $u_n^{(1)} \geq u_n^{(2)} \geq \cdots \geq u_n^{(r)}$. *Let* $S_n^{(k)}$ *be the numbers of exceedances of* $u_n^{(k)}$ *by* $\{X_i, 1 \leq i \leq n\}$ *that satisfy the conditions of Theorem* 2.1. *Then, for* $k_1 \geq 0$, $k_2 \geq 0, \dots, k_r \geq 0$,

$$\begin{aligned} &P\big(S_n^{(1)} = k_1, S_n^{(2)} = k_1 + k_2, \dots, S_n^{(r)} = k_1 + k_2 + \cdots + k_r\big) \\ &\quad \longrightarrow \frac{\tau_1^{k_1}(\tau_2 - \tau_1)^{k_2} \cdots (\tau_r - \tau_{r-1})^{k_r}}{k_1! k_2! \cdots k_r!} \\ &\qquad \cdot \int_{-\infty}^{+\infty} \big(\exp(\sqrt{2\gamma}z - \gamma)\big)^{k_1+k_2+\cdots+k_r} \cdot \exp\big(-e^{-x_k-\gamma+\sqrt{2\gamma}z}\big)\phi(z)\,dz. \end{aligned} \tag{2.6}$$

Theorem 2.3 *Let* $\{X_i, i \geq 1\}$ *be a normal sequence satisfying the conditions of Theorem* 2.1. *Let* $u_n^{(k)} = x_k/a_n + b_n + m_n^*$. *Then, for* $x_1 > x_2$, *as* $n \to \infty$,

$$\begin{aligned} &P\bigl(a_n\bigl(M_n^{(1)} - b_n - m_n^*\bigr) \leq x_1, a_n\bigl(M_n^{(2)} - b_n - m_n^*\bigr) \leq x_2\bigr) \\ &\quad\longrightarrow \int_{-\infty}^{+\infty} \bigl(\exp(-x_2 - \gamma + \sqrt{2\gamma}z) - \exp(-x_1 - \gamma + \sqrt{2\gamma}z) + 1\bigr) \\ &\qquad\times \exp\bigl(-e^{-x_2-\gamma+\sqrt{2\gamma}z}\bigr)\phi(z)\,dz \end{aligned} \tag{2.7}$$

and

$$P\biggl(\frac{1}{n}L_n^{(2)} \leq t, a_n\bigl(M_n^{(2)} - b_n - m_n^*\bigr) \leq x\biggr) \longrightarrow \int_{-\infty}^{x} H(y,t)\,dy, \tag{2.8}$$

where

$$\begin{aligned} H(y,t) = {}& \int_{-\infty}^{+\infty} (1-t)\exp(-y - \gamma + \sqrt{2\gamma}z)\exp\bigl(-(1-t)e^{-y-\gamma+\sqrt{2\gamma}z}\bigr)\phi(z)\,dz \\ &\cdot \int_{-\infty}^{+\infty} t\exp(-y - \gamma + \sqrt{2\gamma}z)\exp\bigl(-te^{-y-\gamma+\sqrt{2\gamma}z}\bigr)\phi(z)\,dz \\ &+ \int_{-\infty}^{+\infty} \exp\bigl(-(1-t)e^{-y-\gamma+\sqrt{2\gamma}z}\bigr)\phi(z)\,dz \\ &\cdot \int_{-\infty}^{+\infty} t^2\exp(-2y - 2\gamma + 2\sqrt{2\gamma}z)\exp\bigl(-te^{-y-\gamma+\sqrt{2\gamma}z}\bigr)\phi(z)\,dz. \end{aligned}$$

3 The proofs of main results

The proof of Theorem 2.1 will use the famous Berman inequality, which was first presented by Slepian [17] and Berman [18] and then polished up by Li and Shao [19]. For the latest results related to Berman's inequality, we refer the reader to Hashorva and Weng [20] and Lu and Wang [21]. The upper bound of Berman's inequality gives an estimate of the difference between two standardized n-dimensional distribution functions by a convenient function of their covariances. According to Hashorva and Weng [20], some results for normal sequences may be extended to nonnormal cases. The proof of Theorem 2.1 also depends on the following lemma of Zhang [7].

Lemma 3.1 *Suppose that* $\{X_i, i \geq 1\}$ *is a standardized normal sequence with correlation coefficient* r_{ij} *satisfying* (2.1). *Define* $u_n = u_n(x) = x/a_n + b_n$ *and* $\rho_n = \gamma/\log n$. *Then*

(i) $r_{ij} \to 0$ *as* $j - i \to +\infty$,

(ii) $\sum_{1\leq i<j\leq nb} |r_{ij} - \rho_n| \exp(-\frac{u_n^2}{1+w_{ij}}) \to 0$ *as* $n \to +\infty$,

where $0 < b < +\infty$ *and* $w_{ij} = \max\{|r_{ij}|, \rho_n\}$.

Proof of Theorem 2.1 It is sufficient to show that, as n goes to ∞,

(a) $E(N_n(B)) \to E(N(B))$ for all sets B of the form $(c,d] \times (r,\delta]$, $r < \delta$, $0 < c < d$, where $d \leq 1$, and $E(\cdot)$ is the expectation, and

(b) $P(N_n(B) = 0) \to P(N(B) = 0)$ for all sets B that are finite unions of disjoint sets of this form.

First, consider (a). If $B = (c,d] \times (r,\delta]$ intersects any of the lines, suppose that these are $L_s, L_{s+1}, \dots, L_t$ $(1 \le s < t \le r)$. Then

$$N_n(B) = \sum_{k=s}^{t} N_n^{(k)}\big((c,d]\big), \qquad N(B) = \sum_{k=s}^{t} N^{(k)}\big((c,d]\big),$$

and the number of points j/n in $(c,d]$ is $([nd] - [nc])$. As in the proof Theorem 5.5.1 on p.113 in Leadbetter et al. [1], we have $E(N_n(B)) = ([nd] - [nc]) \sum_{k=s}^{t}(1 - F(u_n^{(k)}))$ and

$$1 - F\big(u_n^{(k)}\big) = 1 - \Phi\big(u_n^{(k)} - m_j\big), \quad 1 \le j \le n.$$

Using conditions (2.2) and (2.3) yields

$$n\big(1 - \Phi\big(u_n^{(k)} - m_j\big)\big) = n\big(1 - \Phi(x_k/a_n + b_n + m_n - m_j)\big) \sim e^{-x_k} \quad \text{as } n \to \infty, \tag{3.1}$$

where the last '$\sim$' attributes to the well-known fact that $n(1 - \Phi(x_k/a_n + b_n)) \sim e^{-x}$ implies $n(1 - \Phi(x_k/\alpha_n + \beta_n)) \sim e^{-x}$ if $\alpha_n/a_n \to 1$ and $(\beta_n - b_n)/a_n \to 0$. Thus, we have $E(N_n(B)) \sim n(d-c)\sum_{k=s}^{t}(\frac{e^{-x_k}}{n} + o(\frac{1}{n})) \to (d-c)\sum_{k=s}^{t} e^{-x_k}$. So, since

$$\begin{aligned} E\big(N(B)\big) &= \sum_{k=s}^{t} E\big((d-c)\exp(-x_k - \gamma + \sqrt{2\gamma}\zeta)\big) \\ &= \sum_{k=s}^{t}(d-c)e^{-x_k-\gamma} \cdot e^{\frac{(\sqrt{2\gamma})^2}{2}} = \sum_{k=s}^{t}(d-c)e^{-x_k}, \end{aligned}$$

the first result follows. In order to prove (b), we must prove that $P(N_n(B) = 0) \to P(N(B) = 0)$, where $B = \bigcup_1^m C_k$ with disjoint $C_k = (c_k, d_k] \times (r_k, s_k]$. It is convenient to neglect any set C_k that does not intersect any of the lines $L_1, L_2, \dots, L_r$. Because there are intersections and differences of the intervals $(c_k, d_k]$, we may write B in the form $\bigcup_{k=1}^{s}(c_k, d_k] \times E_k$, where $(c_k, d_k]$ are disjoint, and E_k is a finite union of semiclosed intervals. So we have

$$\big\{N_n(B) = 0\big\} = \bigcap_{k=1}^{s}\big\{N_n(F_k) = 0\big\}, \tag{3.2}$$

where $F_k = (c_k, d_k] \times E_k$. L_{l_k} stands for the lowest L_j intersecting F_k. The aforementioned thinning property induces

$$\begin{aligned} \big\{N_n(F_k) = 0\big\} &= \big\{N_n^{(l_k)}\big((c_k, d_k]\big) = 0\big\} \\ &= \big\{M_n(c_k, d_k) \le u_n^{(l_k)}\big\}, \end{aligned} \tag{3.3}$$

where $M_n(c_k, d_k)$ stands for the maximum of $\{X_k\}$ with index k $([cn] < k \le [dn])$. Calculating the probabilities of (3.2) and (3.3), we obtain

$$P\big(N_n(B) = 0\big) = P\left(\bigcap_{k=1}^{s}\big\{M_n(c_k, d_k) \le u_n^{(l_k)}\big\}\right). \tag{3.4}$$

In order to get the limit of the right-hand side of (3.4), we first prove the following result. Define a sequence $\{\bar{X}_i = \bar{Y}_i + m_i, i \geq 1\}$, where $\{\bar{Y}_i, i \geq 1\}$ is a standardized normal sequence with correlation coefficient ρ, and $\{m_i, i \geq 1\}$ is the same as that in $\{X_i, i \geq 1\}$. $M_n(c, d; \rho)$ stands for the maximum of $\{\bar{X}_k\}$ with index k ($[cn] < k \leq [dn]$). It is well known that $M_n(c_1, d_1; \rho), \dots, M_n(c_k, d_k; \rho)$ have the same distribution as $(1-\rho)^{1/2}M_n(c_1, d_1; 0) + \rho^{1/2}\zeta, \dots, (1-\rho)^{1/2}M_n(c_k, d_k; 0) + \rho^{1/2}\zeta$, where $c = c_1 < d_1 < \cdots < c_k < d_k = d$, and ζ is a standard normal variable. In the following, we estimate the bound of

$$\left| P\left(\bigcap_{k=1}^{s}\{M_n(c_k, d_k) \leq u_n^{(l_k)}\}\right) - P\left(\bigcap_{k=1}^{s}\{M_n(c_k, d_k, \rho_n) \leq u_n^{(l_k)}\}\right)\right|, \tag{3.5}$$

where $\rho_n = \gamma / \log n$.

Using Berman's inequality, the bound of (3.5) does not exceed

$$\begin{aligned}
&\frac{1}{2\pi}\sum |r_{ij} - \rho_n|\left(1-\rho_n^2\right)^{-1/2}\exp\left(-\frac{\frac{1}{2}((u_n^i)^2 + (u_n^j)^2)}{1+\omega_{ij}}\right) \\
&\quad \leq C\sum_{1\leq i<j\leq n}|r_{ij} - \rho_n|\exp\left(-\frac{\frac{1}{2}((x_i/a_n + b_n)^2 + (x_j/a_n + b_n)^2)}{1+\omega_{ij}}\right) \\
&\quad\quad \cdot\exp\Bigl(\bigl((x_i/a_n + b_n)\bigl(m_i - m_n^*\bigr) + (x_i/a_n + b_n)\bigl(m_i - m_n^*\bigr) \\
&\quad\quad - \frac{1}{2}\bigl(\bigl(m_n^* - m_i\bigr)^2 + \bigl(m_n^* - m_j\bigr)^2\bigr)\bigr)/(1+\omega_{ij})\Bigr),
\end{aligned} \tag{3.6}$$

where the first sum is carried out over $i, j \in \bigcup_{k=1}^{s}([c_k n], [d_k n]]$, $i < j$, u_n^i or u_n^j stands for $u_n^{(l_k)} - m_i$ or $u_n^{(l_k)} - m_j$ when i or $j \in ([c_k n], [d_k n]]$, and $\omega_{ij} = \max\{|r_{ij}|, \rho_n\}$. Using the proof of Theorem 6.2.1 on p.129 in Leadbetter et al. [1], (2.3) implies that

$$\frac{1}{n}\sum_{i=1}^{n}\exp\left((x_i/a_n + b_n)\bigl(m_i - m_n^*\bigr) - \frac{1}{2}\bigl(m_i - m_n^*\bigr)^2\right) \to 1.$$

Since ω_{ij} is bounded, we further get

$$\sup_{1\leq i<j\leq n}\exp\left(\frac{(x_i/a_n + b_n)(m_i - m_n^*) - \frac{1}{2}(m_i - m_n^*)^2}{1+\omega_{ij}}\right) < C.$$

So, (3.6) does not exceed

$$\begin{aligned}
&C\sum_{1\leq i<j\leq n}|r_{ij} - \rho_n|\exp\left(-\frac{\frac{1}{2}((x_i/a_n + b_n)^2 + (x_j/a_n + b_n)^2)}{1+\omega_{ij}}\right) \\
&\quad < C\sum_{1\leq i<j\leq n}|r_{ij} - \rho_n|\exp\left(-\frac{((\min_{1\leq i\leq n} x_i)/a_n + b_n)^2}{1+\omega_{ij}}\right) \\
&\quad \to 0.
\end{aligned}$$

The last '→' attributes to Lemma 3.1. In order to get the desired limit of (3.4), we only need to prove

$$P\left(\bigcap_{k=1}^{s}\left\{M_n(c_k,d_k,\rho_n)\le u_n^{(l_k)}\right\}\right)\to P\bigl(N(B)=0\bigr).$$

By the definition of $M_n(c_k,d_k,\rho_n)$ it clearly follows that

$$\begin{aligned}
&P\left(\bigcap_{k=1}^{s}\left\{M_n(c_k,d_k,\rho_n)\le u_n^{(l_k)}\right\}\right)\\
&\quad=P\left(\bigcap_{k=1}^{s}\left\{(1-\rho_n)^{\frac{1}{2}}M_n(c_k,d_k,0)+\rho_n^{\frac{1}{2}}\zeta\le u_n^{(l_k)}\right\}\right)\\
&\quad=\int_{-\infty}^{+\infty}P\left(\bigcap_{k=1}^{s}\left\{M_n(c_k,d_k,0)\le(1-\rho_n)^{-\frac{1}{2}}\bigl(u_n^{(l_k)}-\rho_n^{\frac{1}{2}}z\bigr)\right\}\right)\phi(z)\,dz,
\end{aligned}$$

where the proof of the last '=' is the same as the argument on the first line from the bottom on p.136 in Leadbetter et al. [1]. Since $a_n=(2\log n)^{\frac{1}{2}}$, $b_n=a_n+O(a_n^{-1}\log\log n)$, and $\rho_n=\gamma/\log n$, it is easy to show that

$$(1-\rho_n)^{-\frac{1}{2}}\bigl(u_n^{(l_k)}-\rho_n^{\frac{1}{2}}z\bigr)=\frac{x_{l_k}+\gamma-\sqrt{2\gamma}z}{a_n}+b_n+o\bigl(a_n^{-1}\bigr).$$

Furthermore, we have

$$\begin{aligned}
&P\left(\bigcap_{k=1}^{s}\left\{M_n(c_k,d_k,0)\le(1-\rho_n)^{-\frac{1}{2}}\bigl(u_n^{(l_k)}-\rho_n^{\frac{1}{2}}z\bigr)\right\}\right)\\
&\quad=P\left(\bigcap_{k=1}^{s}\left\{\tilde{\xi}_{[c_kn]+1}\le(1-\rho_n)^{-\frac{1}{2}}\bigl(u_n^{(l_k)}-\rho_n^{\frac{1}{2}}z\bigr)-m_{[c_kn]+1},\ldots,\tilde{\xi}_{[d_kn]}\right.\right.\\
&\qquad\left.\left.\le(1-\rho_n)^{-\frac{1}{2}}\bigl(u_n^{(l_k)}-\rho_n^{\frac{1}{2}}z\bigr)-m_{[d_kn]}\right\}\right)\\
&\quad\to\prod_{k=1}^{s}\exp\bigl(-(d_k-c_k)e^{-x_{l_k}-\gamma+\sqrt{2\gamma}z}\bigr),
\end{aligned}$$

where $\tilde{\xi}_k$ stands for independent standard normal variables, and the proof of the last '→' is the same as that of (3.1). Using the dominated convergence theorem yields that

$$\begin{aligned}
&\int_{-\infty}^{+\infty}P\left(\bigcap_{k=1}^{s}\left\{M_n(c_k,d_k,0)\le(1-\rho_n)^{-\frac{1}{2}}\bigl(u_n^{(l_k)}-\rho_n^{\frac{1}{2}}z\bigr)\right\}\right)\phi(z)\,dz\\
&\quad\to\int_{-\infty}^{+\infty}\prod_{k=1}^{s}\exp\bigl(-(d_k-c_k)e^{-x_{l_k}-\gamma+\sqrt{2\gamma}z}\bigr)\phi(z)\,dz\\
&\quad=P\bigl(N(B)=0\bigr).
\end{aligned}$$

The proof of (b) is completed. □

Proof of Corollary 2.1 Using Theorem 2.1, the proof is similar to that of Corollary 5.5.2 in Leadbetter et al. [1]. So we omit it. □

Proof of Theorem 2.2 By Corollary 2.1 the left-hand side of (2.6) converges to

$$P\big(S^{(1)} = k_1, S^{(2)} = k_1 + k_2, \dots, S^{(r)} = k_1 + k_2 + \cdots + k_r\big), \tag{3.7}$$

where $S^{(i)} = N^{(i)}([0,1])$ is the ith component of N. In our paper, the structure of the Cox process is similar to that of the Poisson process in plane in Leadbetter et al. [1]. So we can refer to the proof of Theorem 5.6.1 in Leadbetter et al. [1], and hence (3.7) equals

$$\begin{aligned}&\frac{(k_1 + k_2 + \cdots + k_r)!}{k_1!k_2!\cdots k_r!}\left(\frac{\tau_1}{\tau_r}\right)^{k_1}\left(\frac{\tau_2-\tau_1}{\tau_r}\right)^{k_2}\left(\frac{\tau_r-\tau_{r-1}}{\tau_r}\right)^{k_r}\\ &\quad\cdot P\big(N^{(r)}\big((0,1]\big) = k_1 + k_2 + \cdots + k_r\big).\end{aligned}$$

The proof is completed since

$$\begin{aligned}&P\big(N^{(r)}\big((0,1]\big) = k_1 + k_2 + \cdots + k_r\big)\\ &\quad= \int_{-\infty}^{+\infty}\frac{(\exp(-x_r-\gamma+\sqrt{2\gamma}z))^{k_1+k_2+\cdots+k_r}}{(k_1 + k_2 + \cdots + k_r)!}\cdot\exp\big(-e^{-x_r-\gamma+\sqrt{2\gamma}z}\big)\phi(z)\,dz\\ &\quad= \frac{(\exp(-x_r))^{k_1+k_2+\cdots+k_r}}{(k_1 + k_2 + \cdots + k_r)!}\int_{-\infty}^{+\infty}\big(\exp(-\gamma+\sqrt{2\gamma}z)\big)^{k_1+k_2+\cdots+k_r}\\ &\qquad\times\exp\big(-e^{-x_r-\gamma+\sqrt{2\gamma}z}\big)\phi(z)\,dz.\end{aligned}$$

□

Proof of Theorem 2.3 Clearly, the left-hand side of (2.7) is equal to

$$\begin{aligned}&P\big(a_n\big(M_n^{(1)} - b_n - m_n^*\big) \le x_1, a_n\big(M_n^{(2)} - b_n - m_n^*\big) \le x_2\big)\\ &\quad= P\big(S_n^{(2)} = 0\big) + P\big(S_n^{(1)} = 0, S_n^{(2)} = 1\big),\end{aligned}$$

where $S_n^{(i)}$ is the number of exceedances of $u_n^{(i)}$ by $X_1, X_2, \dots, X_n$. Using Theorem 2.2 can complete the proof. In order to prove (2.8), write I, J for intervals $\{1, 2, \dots, [nt]\}$, $\{[nt] + 1, \dots, n\}$, respectively, and $M^{(1)}(I)$, $M^{(2)}(I)$, $M^{(1)}(J)$, $M^{(2)}(J)$ for the maxima and the second largest of $\{X_i, 1 \le i \le n\}$ in the intervals I, J. Let $K_n(x_1, x_2, x_3, x_4)$ be the joint d.f. of the normalized r.v.s

$$\begin{aligned}X_n^{(1)} &= a_n\big(M_n^{(1)}(I) - b_n - m_n^*\big), & X_n^{(2)} &= a_n\big(M_n^{(2)}(I) - b_n - m_n^*\big),\\ Y_n^{(1)} &= a_n\big(M_n^{(1)}(J) - b_n - m_n^*\big), & Y_n^{(2)} &= a_n\big(M_n^{(2)}(J) - b_n - m_n^*\big).\end{aligned}$$

Consider an interesting case of $x_1 > x_2$ and $x_3 > x_4$, that is,

$$\begin{aligned}&H_n(x_1, x_2, x_3, x_4)\\ &\quad= P\big(M_n^{(1)}(I) \le u_n^{(1)}, M_n^{(2)}(I) \le u_n^{(2)}, M_n^{(1)}(J) \le u_n^{(3)}, M_n^{(2)}(J) \le u_n^{(4)}\big)\\ &\quad= P\big(N_n^{(1)}\big(I'\big) = 0, N_n^{(2)}\big(I'\big) \le 1, N_n^{(3)}\big(J'\big) = 0, N_n^{(4)}\big(J'\big) \le 1\big),\end{aligned}$$

where $I' = (0, t]$ and $J' = (t, 1]$. By Corollary 2.1 with $B_1 = I'$ and $B_2 = J'$ we have

$$
\begin{aligned}
&\lim_{n\to\infty} H_n(x_1, x_2, x_3, x_4) \\
&\quad = \lim_{n\to\infty} P\bigl(N_n^{(1)}(I') = 0, N_n^{(2)}(I') \le 1\bigr) \cdot P\bigl(N_n^{(3)}(J') = 0, N_n^{(4)}(J') \le 1\bigr) \\
&\quad = \int_{-\infty}^{+\infty} \bigl(\bigl(e^{-x_2} - e^{-x_1}\bigr) t \exp(\sqrt{2\gamma} - \gamma) + 1\bigr) \exp\bigl(-te^{-x_2-\gamma+\sqrt{2\gamma}z}\bigr)\phi(z)\,dz \\
&\qquad \times \int_{-\infty}^{+\infty} \bigl(\bigl(e^{-x_4} - e^{-x_3}\bigr)(1-t) \exp(\sqrt{2\gamma} - \gamma) + 1\bigr) \exp\bigl(-(1-t)e^{-x_4-\gamma+\sqrt{2\gamma}z}\bigr)\phi(z)\,dz \\
&\quad = H_t(x_1, x_2) H_{1-t}(x_3, x_4) = H(x_1, x_2, x_3, x_4).
\end{aligned}
$$

Now the left-hand side of (2.8) is equal to

$$
\begin{aligned}
&P\bigl(M_n^{(2)}(I) \le u_n^{(2)}, M_n^{(2)}(I) \ge M_n^{(1)}(J)\bigr) \\
&\quad + P\bigl(M_n^{(1)}(I) \le u_n^{(2)}, M_n^{(1)}(J) > M_n^{(1)}(I) \ge M_n^{(2)}(J)\bigr).
\end{aligned}
\tag{3.8}
$$

Obviously, H is absolutely continuous, and the boundaries of sets in R^4 such as $\{(w_1, w_2, w_3, w_4) : w_2 \le x_2, w_2 > w_3\}$ and $\{(w_1, w_2, w_3, w_4) : w_1 \le x_2, w_3 > w_1 \ge w_4\}$ have zero Lebesgue measure. Thus, by Corollary 2.1, (3.8) converges to

$$
P(X_2 \le x_2, X_2 \ge Y_1) + P(X_1 \le x_2, Y_1 > X_1 \ge Y_2).
$$

Using the joint distribution $H(x_1, x_2, x_3, x_4)$ and a simple evaluation complete the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

Acknowledgements
The research of the authors was supported by the Scientific Research Fund of Sichuan Provincial Education Department under Grant 12ZB082, the Scientific research cultivation project of Sichuan University of Science & Engineering under Grant 2013PY07, and the Scientific Research Fund of Sichuan University of Science & Engineering under Grant 2013KY03.

Received: 22 April 2015 Accepted: 12 May 2016 Published online: 26 May 2016

References
1. Leadbetter, MR, Lindgren, G, Rootzen, H: Extremes and Related Properties of Sequences and Processes. Springer, New York (1983)
2. Mittal, Y, Ylvisaker, D: Limit distributions for the maxima of stationary Gaussian processes. Stoch. Process. Appl. **3**, 1-18 (1975)
3. Ho, HC, Hsing, T: On the asymptotic joint distribution of the sum and maximum of stationary normal random variables. J. Appl. Probab. **33**, 138-145 (1996)
4. Tan, Z, Peng, Z: Joint asymptotic distribution of exceedances point process and partial sum of stationary Gaussian sequence. Appl. Math. J. Chin. Univ. Ser. B **26**, 319-326 (2011)
5. Hashorva, E, Peng, Z, Weng, Z: On Piterbarg theorem for maxima of stationary Gaussian sequences. Lith. Math. J. **53**, 280-292 (2013)
6. Horowitz, J: Extreme values from a nonstationary stochastic process: an application to air quality analysis. Technometrics **22**, 469-478 (1980)
7. Zhang, L: Weak convergence of high level exceedances by a nonstationary Gaussian sequence with strong dependence. Acta Math. Appl. Sin. **26**, 56-61 (2003) (in Chinese)
8. Lin, F, Shi, D, Jiang, Y: Some distributional limit theorems for the maxima of Gaussian vector sequences. Comput. Math. Appl. **64**, 2497-2506 (2012)
9. Tan, Z, Yang, Y: The maxima and sums of multivariate non-stationary Gaussian sequences. Appl. Math. J. Chin. Univ. **30**, 197-209 (2015)
10. Piterbarg, VI: Asymptotic Methods in the Theory of Gaussian Processes and Fields. Translations of Mathematical Monographs, vol. 148. Am. Math. Soc., Providence (1996)

11. Hu, A, Peng, Z, Qi, Y: Limit laws for maxima of contracted stationary Gaussian sequences. Metrika **70**, 279-295 (2009)
12. Falk, M, Hüsler, J, Reiss, RD: Laws of Small Numbers: Extremes and Rare Events, 3rd edn. In DMV Seminar, vol. 23. Birkhäuser, Basel (2010)
13. Peng, Z, Tong, J, Weng, Z: Joint limit distributions of exceedances point processes and partial sums of Gaussian vector sequence. Acta Math. Sin. Engl. Ser. **28**, 1647-1662 (2012)
14. Hashorva, E, Peng, Z, Weng, Z: Limit properties of exceedances point processes of scaled stationary Gaussian sequences. Probab. Math. Stat. **34**, 45-59 (2014)
15. Wiśiewski, M: Multidimensional point processes of extreme order statistics. Demonstr. Math. **27**, 475-485 (1994)
16. Lin, F, Shi, D, Jiang, Y: Limit properties of exceedance point processes of strongly dependent normal sequences. J. Inequal. Appl. **2015**, 63 (2015)
17. Slepian, D: The one sided barrier problem for Gaussian noise. Bell Syst. Tech. J. **41**, 463-501 (1962)
18. Berman, SM: Limit theorems for the maximum term in stationary sequences. Ann. Math. Stat. **35**, 502-516 (1964)
19. Li, WV, Shao, QM: A normal comparison inequality and its applications. Probab. Theory Relat. Fields **122**, 494-508 (2002)
20. Hashorva, E, Weng, Z: Berman's inequality under random scaling. Stat. Interface **7**, 339-349 (2014)
21. Lu, D, Wang, X: Some new normal comparison inequalities related to Gordon's inequality. Stat. Probab. Lett. **88**, 133-140 (2014)

Submit your manuscript to a SpringerOpen journal and benefit from:

- ► Convenient online submission
- ► Rigorous peer review
- ► Immediate publication on acceptance
- ► Open access: articles freely available online
- ► High visibility within the field
- ► Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com