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Some computational formulas related the Riemann zeta-function tails

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Abstract

In this paper we present two computational formulae for one kind of reciprocal sums related to the Riemann zeta-function at integer points $s = 4, 5$, which answers an open problem proposed by Lin (J. Inequal. Appl. 2016:32, 2016).

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1 Introduction and main results

Let $(a_k)_{k\geq 1}$ be a strictly increasing positive sequence such that

$$\sum_{k=1}^{\infty} \frac{1}{a_k} < +\infty. \tag{1.1}$$

Many authors study the computational formula for infinite sums of reciprocal a_k,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{a_k} \right)^{-1} \right\rfloor, \quad n \in \mathbb{N}, \tag{1.2}$$

where $\lfloor x \rfloor$ denotes the integer part of x.

For example, let (F_k) be the famous Fibonacci sequence: $F_{k+1} = F_k + F_{k-1}$ with the initial values $F_0 = 0$ and $F_1 = 1$. Ohtsuka and Nakamura [2] showed that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & \text{if } n \geq 2 \text{ is even,} \\ F_{n-2} - 1 & \text{if } n \geq 1 \text{ is odd,} \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1 & \text{if } n \geq 2 \text{ is even,} \\ F_nF_{n-1} & \text{if } n \geq 1 \text{ is odd.} \end{cases}$$

Xu and Wang [3] obtained a complex computational formula for $a_k = F_k^3$.

Zhang and Wang [4] studied this problem for the Pell numbers P_k and showed that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k} \right)^{-1} \right\rfloor = \begin{cases} P_{n-1} + P_{n-2} & \text{if } n \geq 2 \text{ is even,} \\ P_{n-1} + P_{n-2} - 1 & \text{if } n \geq 1 \text{ is odd,} \end{cases}$$

where the Pell numbers P_k are defined by $P_0 = 0$, $P_1 = 1$, and the recurrence relation $P_{k+1} = 2P_k + P_{k-1}$.

For some other results related to recursive sequences, recursive polynomials, and their promotion forms, see [5–13] and references therein.

Very recently, Lin [1] investigated the related problem for the sequence $a_k = k^s$ with integer $s \geq 2$ and showed the following two interesting identities:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{k^2} \right)^{-1} \right\rfloor = n - 1, \tag{1.3}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{k^3} \right)^{-1} \right\rfloor = 2n(n - 1). \tag{1.4}$$

This is an important problem, which has a close relationship with the Riemann zeta-function $\zeta(s)$. Lin noted that there does not exist an integer-coefficient polynomial $q(x)$ of degree 3 such that the following identity holds:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{k^4} \right)^{-1} \right\rfloor = q(n). \tag{1.5}$$

In [1], Lin declared that giving a precise calculation formula for $(\sum_{k=n}^{\infty} \frac{1}{k^s})^{-1}$ with $s = 4$ is a very complicated problem. In this paper, we tackle this open problem.

Theorem 1 *For all integer $n \geq 2$, we have the identity*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{k^4} \right)^{-1} \right\rfloor = -1 + 4n - 5n^2 + 3n^3 + \left\lfloor \frac{(2n+1)(n-1)}{4} \right\rfloor. \tag{1.6}$$

Furthermore, for $a_k = k^5$, we also have an analogous computational formula.

Theorem 2 *For all integer $n \geq 4$, we have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{k^5} \right)^{-1} \right\rfloor = -5n + 9n^2 - 8n^3 + 4n^4 + \left\lfloor \frac{(n+1)(n-2)}{3} \right\rfloor. \tag{1.7}$$

2 Proof of Theorem 1

Assume that

$$g(n) - g(n+1) < \frac{1}{n^4} < f(n) - f(n+1),$$

and $f(\infty) = g(\infty) = 0$. Summing the inequalities from n to ∞, we have

$$g(n) < \sum_{k=n}^{\infty} \frac{1}{k^4} < f(n). \tag{2.1}$$

These inequalities allow us to study the computational formulas of Theorem 1. The problem of finding the functions $f(n)$, $g(n)$ (or $F(n)$, $G(n)$ in Section 3) is transformed into solving the finite continued fraction approximation solution of difference equation for 'large' n:

$$y(n) - y(n+1) = \frac{1}{n^4}. \tag{2.2}$$

We will apply the multiple-correction method (see [14–16]) and solve it as follows.

Step 1 (The initial correction) Choosing $\eta_0(n) = \frac{b}{n^3+a_2n^2+a_1n+a_0}$ and developing the expression $\eta_0(n) - \eta_0(n+1) - \frac{1}{n^4}$ into power series expansion in $1/n$, we easily obtain

$$\begin{aligned} &\eta_0(n) - \eta_0(n+1) - \frac{1}{n^4} \\ &\quad = (3b-1)\frac{1}{n^4} + (-6b - 4a_2b)\frac{1}{n^5} + \bigl(10b - 5a_1b + 10a_2b + 5a_2^2b\bigr)\frac{1}{n^6} \\ &\qquad + \bigl(-15b - 6a_0b + 15a_1b - 20a_2b + 12a_1a_2b - 15a_2^2b - 6a_2^3b\bigr)\frac{1}{n^7} + O\left(\frac{1}{n^8}\right). \end{aligned} \tag{2.3}$$

If $b = \frac{1}{3}$, $a_2 = -\frac{3}{2}$, $a_1 = \frac{5}{4}$, $a_0 = -\frac{3}{8}$, then we can get the approximation solution

$$g(n) = \frac{b}{n^3 + a_2n^2 + a_1n + a_0}$$

of difference equation (2.2), which is the best possible rational approximation solution of such structure as n tends to infinity.

Step 2 (The first correction) Choose $\eta_1(n) = \frac{b}{n^3+a_2n^2+a_1n+a_0+\frac{u}{x+v}}$ and developing the expression $\eta_1(n) - \eta_1(n+1) - \frac{1}{n^4}$ into power series expansion in $1/n$, we easily obtain

$$\eta_1(n) - \eta_1(n+1) - \frac{1}{n^4} = \left(-\frac{7}{16} - \frac{7}{3}u\right)\frac{1}{n^8} + \frac{4}{3}(u + 2uv)\frac{1}{n^9} + O\left(\frac{1}{n^{10}}\right). \tag{2.4}$$

If $u = -\frac{3}{16}$, $v = -\frac{1}{2}$, then we can get the approximation solution

$$f(n) = \frac{b}{n^3 + a_2n^2 + a_1n + a_0 + \frac{u}{x+v}}$$

of difference equation (2.2), which has a better approximation rate than $g(n)$ for 'large' n.

So we can get following inequalities necessary in the proofs of our theorems.

Lemma 1 *Let*

$$f(n) = \frac{\frac{8}{3}}{-1 + 10n - 12n^2 + 8n^3 - \frac{3}{2n-1}}. \tag{2.5}$$

Then, for $n \geq 2$,

$$f(n) - f(n+1) > \frac{1}{n^4}. \tag{2.6}$$

Proof We easily check that

$$f(n) - f(n+1) - \frac{1}{n^4} = \frac{1}{n^4(1+n)(1+n+n^2)(-1+2n-2n^2+n^3)}.$$

Note that $-1 + 2n - 2n^2 + n^3 = (-2 + n)(2 + n^2) + 3$, so the above polynomial is positive for $n \geq 2$. Then, for $n \in \mathbb{N}$,

$$f(n) - f(n+1) - \frac{1}{n^4} > 0.$$ □

Lemma 2 *Let*

$$g(n) = \frac{8}{3(-3+10n-12n^2+8n^3)}. \tag{2.7}$$

Then, for $n \in \mathbb{N}$,

$$g(n) - g(n+1) < \frac{1}{n^4}. \tag{2.8}$$

Proof We have

$$g(n) - g(n+1) - \frac{1}{n^4} = \frac{-28n^2+9}{n^4(-3+10n-12n^2+8n^3)(3+10n+12n^2+8n^3)},$$

where $h(n) := -3 + 10n - 12n^2 + 8n^3 = (-2 + n)(18 + 4n + 8n^2) + 33 > 0$ for $n \geq 2$, and $h(1) = 3 > 0$. So $h(n) > 0$ for $n \in \mathbb{N}$. This completes the proof of Lemma 2. □

Proof of Theorem 1 Summing the inequalities of the form

$$g(n) - g(n+1) < \frac{1}{n^4} < f(n) - f(n+1)$$

from n to ∞ and noting that $f(\infty) = g(\infty) = 0$, we have

$$g(n) < \sum_{k=n}^{\infty} \frac{1}{k^4} < f(n), \quad n \geq 2. \tag{2.9}$$

Then, for $n \geq 2$,

$$\frac{3}{8}\left(\frac{-3}{2n-1} - 3 + 10n - 12n^2 + 8n^3\right) < \left(\sum_{k=n}^{\infty} \frac{1}{k^4}\right)^{-1} < \frac{3}{8}\left(-3 + 10n - 12n^2 + 8n^3\right). \tag{2.10}$$

Note that

$$\frac{3}{8}\left(-3 + 10n - 12n^2 + 8n^3\right) = 3n^3 - 5n^2 + 4n - 1 + \left(\frac{n(2n-1)}{4} - \frac{1}{8}\right)$$

and

$$\frac{3}{8}\left(\frac{-3}{2n-1} - 3 + 10n - 12n^2 + 8n^3\right) = 3n^3 - 5n^2 + 4n - 1 + \left(\frac{n(2n-1)}{4} - \frac{9}{8(2n-1)} - \frac{1}{8}\right).$$

For $n \geq 5$, we have $\frac{9}{8(2n-1)} \leq \frac{1}{8}$. Then

$$\left\lfloor \frac{n(2n-1)}{4} - \frac{1}{8} \right\rfloor \geq \left\lfloor \frac{n(2n-1)}{4} - \frac{9}{8(2n-1)} - \frac{1}{8} \right\rfloor \geq \left\lfloor \frac{n(2n-1)}{4} - \frac{1}{4} \right\rfloor$$
$$= \left\lfloor \frac{(2n+1)(n-1)}{4} \right\rfloor$$

and, for $n \in \mathbb{N}$,

$$\left\lfloor \frac{n(2n-1)}{4} - \frac{1}{8} \right\rfloor = \left\lfloor \frac{(2n+1)(n-1)}{4} \right\rfloor.$$

It follows that, for $n \geq 5$,

$$\left\lfloor \frac{n(2n-1)}{4} - \frac{1}{8} \right\rfloor = \left\lfloor \frac{n(2n-1)}{4} - \frac{9}{8(2n-1)} - \frac{1}{8} \right\rfloor = \left\lfloor \frac{(2n+1)(n-1)}{4} \right\rfloor. \tag{2.11}$$

Finally, we note that the above identities hold for $n = 2, 3, 4$. Combining (2.10) and (2.11), we prove Theorem 1. □

3 Proof of Theorem 2

Similarly to Section 2, by the multiple-correction method we can solve the finite continued fraction approximation solution $F(n)$, $G(n)$ of the differential equation

$$y(n) - y(n+1) = \frac{1}{n^5}. \tag{3.1}$$

So we have the following inequalities.

Lemma 3 *Let*

$$F(n) = \frac{9}{-2 - 48n + 84n^2 - 72n^3 + 36n^4}. \tag{3.2}$$

Then, for $n \geq 2$,

$$F(n) - F(n+1) > \frac{1}{n^5}. \tag{3.3}$$

Proof

$$F(n) - F(n+1) - \frac{1}{n^5}$$
$$= \frac{-1 + 660n^2}{n^5(-1 - 24n + 42n^2 - 36n^3 + 18n^4)(-1 + 24n + 42n^2 + 36n^3 + 18n^4)}. \tag{3.4}$$

Note that

$$-1 - 24n + 42n^2 - 36n^3 + 18n^4 = (n-2)\big(60 + 42n + 18n^3\big) + 119. \tag{3.5}$$

Then, for $n \geq 2$, we have

$$F(n) - F(n+1) - \frac{1}{n^5} > 0.$$

□

Lemma 4 *Let*

$$G(n) = \frac{9}{-1/4 - 48n + 84n^2 - 72n^3 + 36n^4}. \tag{3.6}$$

Then, for $n \geq 5$,

$$G(n) - G(n+1) < \frac{1}{n^5}. \tag{3.7}$$

Proof Similarly to the proof of Lemma 3, we have

$$\begin{aligned} &G(n) - G(n+1) - \frac{1}{n^5} \\ &\quad = -\frac{1 - 37{,}536n^2 + 2{,}016n^4}{n^5(-1 - 192n + 336n^2 - 288n^3 + 144n^4)(-1 + 192n + 336n^2 + 288n^3 + 144n^4)}. \end{aligned}$$

Note that

$$1 - 37{,}536n^2 + 2{,}016n^4 = (-5 + n)\bigl(64{,}320 + 12{,}864n + 10{,}080n^2 + 2{,}016n^3\bigr) + 321{,}601$$

and

$$-1 - 192n + 336n^2 - 288n^3 + 144n^4 = (-4 + n)\bigl(5{,}760 + 1{,}488n + 288n^2 + 144n^3\bigr) + 23{,}039.$$

Then, for $n \geq 5$, the inequality $G(n) - G(n+1) - \frac{1}{n^5} < 0$ holds. □

Proof of Theorem 2 We assume that $n \geq 5$ in the following proof. Summing the inequalities of the form

$$G(n) - G(n+1) < \frac{1}{n^4} < F(n) - F(n+1)$$

from n to ∞ and noting that $F(\infty) = G(\infty) = 0$, we have

$$\begin{aligned} &-5n + 9n^2 - 8n^3 + 4n^4 + \left(-\frac{2}{9} - \frac{n}{3} + \frac{n^2}{3}\right) \\ &\quad < \left(\sum_{k=n}^{\infty} \frac{1}{k^5}\right)^{-1} \\ &\quad < -5n + 9n^2 - 8n^3 + 4n^4 + \left(-\frac{1}{36} - \frac{n}{3} + \frac{n^2}{3}\right). \end{aligned} \tag{3.8}$$

Next, for $n \geq 3$, we will prove the following identities:

$$\left\lfloor -\frac{2}{9} - \frac{n}{3} + \frac{n^2}{3} \right\rfloor = \left\lfloor -\frac{1}{36} - \frac{n}{3} + \frac{n^2}{3} \right\rfloor = \left\lfloor \frac{(n+1)(n-2)}{3} \right\rfloor. \tag{3.9}$$

Since

$$\frac{(n+1)(n-2)}{3} < -\frac{2}{9} - \frac{n}{3} + \frac{n^2}{3} < -\frac{1}{36} - \frac{n}{3} + \frac{n^2}{3},$$

it suffices to prove that

$$\left\lfloor -\frac{1}{36} - \frac{n}{3} + \frac{n^2}{3} \right\rfloor = \left\lfloor \frac{(n+1)(n-2)}{3} \right\rfloor. \tag{3.10}$$

We will consider three cases.

Case 1. If $n = 3m$, $m \in \mathbb{N}$, then we have

$$\left\lfloor -\frac{1}{36} - \frac{n}{3} + \frac{n^2}{3} \right\rfloor = \left\lfloor \frac{(n+1)(n-2)}{3} \right\rfloor = -1 - m + 3m^2.$$

Case 2. If $n = 3m + 1$, $m \in \mathbb{N}$, then we have

$$\left\lfloor -\frac{1}{36} - \frac{n}{3} + \frac{n^2}{3} \right\rfloor = \left\lfloor \frac{(n+1)(n-2)}{3} \right\rfloor = -1 + m + 3m^2.$$

Case 3. If $n = 3m + 2$, $m \in \mathbb{N}$, then we have

$$\left\lfloor -\frac{1}{36} - \frac{n}{3} + \frac{n^2}{3} \right\rfloor = \left\lfloor \frac{(n+1)(n-2)}{3} \right\rfloor = 3m + 3m^2.$$

This proves that (3.10) holds. Finally, combining (3.8) and (3.9), we prove Theorem 2. □

Competing interests

The author declares that she has no competing interests.

Author's contributions

The author read and approved the final manuscript.

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