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Perturbation bounds for eigenvalues of diagonalizable matrices and singular values

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Abstract

Perturbation bounds for eigenvalues of diagonalizable matrices are derived.
Perturbation bounds for singular values of arbitrary matrices are also given. We
generalize some existing results.

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Keywords: perturbation bound; eigenvalue; diagonalizable matrix; singular value

1 Introduction

Many problems in science and engineering lead to eigenvalue and singular value problems for matrices. Perturbation bounds of eigenvalues and singular values play an important role in matrix computations. Let S_n be the set of all $n!$ permutations of $\{1, 2, \ldots, n\}$. If $x = (x_1, x_2, \ldots, x_n)$ and $\pi \in S_n$, then the vector x_π is defined as $(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$. A square matrix is called *doubly stochastic* if its elements are real nonnegative numbers and if the sum of the elements in each row and in each column is equal to 1. Let $\mathbb{C}^{n \times n}$ be the set of $n \times n$ complex matrices. Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, we use the notation (see [1, 2])

$$\|A\|_p = \left(\sum_{i,j=1}^{n} |a_{ij}|^p \right)^{\frac{1}{p}} \quad \text{for } p \geq 0, \tag{1.1}$$

$$\|A\|_{q,p} = \left(\sum_{j=1}^{n} \left(\sum_{k=1}^{n} |a_{kj}|^q \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \quad \text{for } p > 0, q > 0, \frac{1}{p} + \frac{1}{q} = 1. \tag{1.2}$$

Let $T \in \mathbb{C}^{n \times n}$ and assume that

$$\Lambda^{(k)} = \operatorname{diag}\left(\lambda_1^{(k)}, \lambda_2^{(k)}, \ldots, \lambda_n^{(k)} \right) \in \mathbb{C}^{n \times n}, \quad k = 1, \ldots, 4,$$

are diagonal matrices. In [3], the following classical result is given:

$$\left\| \Lambda^{(1)} T \Lambda^{(2)} - \Lambda^{(3)} T \Lambda^{(4)} \right\|_2^2 \geq s_n^2(T) \sum_{i=1}^{n} \left| \lambda_i^{(1)} \lambda_{\pi(i)}^{(2)} - \lambda_i^{(3)} \lambda_{\pi(i)}^{(4)} \right|^2 \tag{1.3}$$

for some $\pi \in S_n$, where $s_n(T)$ is the smallest singular value of T. The inequality has many applications in bounding the (relative) perturbation for eigenvalues and singular values, such as [4–6] and the references therein. We generalize (1.3) in Section 2.

Let $\lambda(A)$ denote the spectrum of matrix A. In 1970, Ikramov [7] defined the 'Hölder distance $d_p(\lambda(A), \lambda(B))$ between the spectra' of the matrices A and B, which have the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_n$, respectively, by the equation:

$$d_p\big(\lambda(A), \lambda(B)\big) = \min_{\pi \in S_n} \left(\sum_{i=1}^{n} |\lambda_i - \mu_{\pi(i)}|^p \right)^{\frac{1}{p}}. \tag{1.4}$$

If A and B are Hermitian matrices and $1 \le p < 2$, [7] obtained

$$d_p\big(\lambda(A), \lambda(B)\big) \le \|A - B\|_p, \tag{1.5}$$

which partially generalizes the Hoffman-Wielandt theorem [8]. However, for normal matrices (1.5) can no longer be valid. The purpose of this paper is to obtain several inequalities similar to (1.5) for diagonalizable matrices. We exhibit some upper bounds and lower bounds for $d_p(\lambda(A), \lambda(B))$ of diagonalizable matrices A and B in Section 2.

Majorization is one of the most powerful techniques for deriving inequalities. We use majorization to get some perturbation bounds for singular values. For simplicity of the notations, in most cases in this paper the vectors in $\mathbb{R}^n$ are regarded as row vectors, but when they are multiplied by matrices we regard them as column vectors. Given a real vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we rearrange its components as $x_{[1]} \ge x_{[2]} \ge \cdots \ge x_{[n]}$.

Definition 1.1 ([9], p.14) For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, if

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n,$$

then we say that x is weakly majorized by y and denote $x \prec_w y$. If $x \prec_w y$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, then we say that x is majorized by y and denote $x \prec y$.

Let $s_1 \ge s_2 \ge \cdots \ge s_n$ and $\delta_1 \ge \delta_2 \ge \cdots \ge \delta_n$ be the singular values of the complex matrices $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ and $B = (b_{ij}) \in \mathbb{C}^{n \times n}$, respectively. In [10], p.215, and [11], p.199, the following classical result is given:

$$\sum_{i=1}^{n} |s_i - \delta_i|^2 \le \sum_{i,j=1}^{n} |a_{ij} - b_{ij}|^2. \tag{1.6}$$

We generalize the inequality (1.6) in Section 3.

2 Perturbation bounds for eigenvalues of diagonalizable matrices

Let $A \circ B$ denote the Hadamard product of matrices A and B. $\|A\|$ denotes the spectral norm of matrix A. A^T denotes the transpose of matrix A. For two n-square real matrices A, B, we write $A \le_e B$ to mean that $B - A$ is (entrywise) nonnegative. For $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ and a real number $t > 0$, we denote $A^{|\circ|t} \equiv (|a_{ij}|^t) \in \mathbb{C}^{n \times n}$. Let $s_n(A)$ and $s_1(A)$ be the smallest and

the largest singular values of A, respectively. The following entrywise inequalities involve the smallest and the largest singular values.

Lemma 2.1 ([9], p.52) *Let $A \in \mathbb{C}^{n \times n}$ and let p, q be real numbers with $0 < p \le 2$ and $q \ge 2$. Then there exist two doubly stochastic matrices $B, C \in \mathbb{C}^{n \times n}$ such that*

$$s_n(A)^p B \le_e A^{|\circ|p} \tag{2.1}$$

and

$$A^{|\circ|q} \le_e s_1(A)^q C. \tag{2.2}$$

Theorem 2.2 *Let $T \in \mathbb{C}^{n \times n}$ and let p, q be real numbers with $0 < p \le 2$ and $q \ge 2$. Assume that $\Lambda^{(k)} = \mathrm{diag}(\lambda_1^{(k)}, \lambda_2^{(k)}, \ldots, \lambda_n^{(k)}) \in \mathbb{C}^{n \times n}$, $k = 1, \ldots, 4$, are diagonal matrices. Then there are permutations π and ν of S_n such that*

$$\left\| \Lambda^{(1)} T \Lambda^{(2)} - \Lambda^{(3)} T \Lambda^{(4)} \right\|_p^p \ge s_n^p(T) \sum_{i=1}^{n} \left| \lambda_i^{(1)} \lambda_{\pi(i)}^{(2)} - \lambda_i^{(3)} \lambda_{\pi(i)}^{(4)} \right|^p \tag{2.3}$$

and

$$\left\| \Lambda^{(1)} T \Lambda^{(2)} - \Lambda^{(3)} T \Lambda^{(4)} \right\|_q^q \le s_1^q(T) \sum_{i=1}^{n} \left| \lambda_i^{(1)} \lambda_{\nu(i)}^{(2)} - \lambda_i^{(3)} \lambda_{\nu(i)}^{(4)} \right|^q. \tag{2.4}$$

Proof Set $T = (t_{ij}) \in \mathbb{C}^{n \times n}$. Then

$$\left\| \Lambda^{(1)} T \Lambda^{(2)} - \Lambda^{(3)} T \Lambda^{(4)} \right\|_p^p = \sum_{i,j=1}^{n} |t_{ij}|^p \left| \lambda_i^{(1)} \lambda_j^{(2)} - \lambda_i^{(3)} \lambda_j^{(4)} \right|^p$$

$$= e^T \left(T^{|\circ|p} \circ M \right) e, \tag{2.5}$$

where $M = (|\lambda_i^{(1)} \lambda_j^{(2)} - \lambda_i^{(3)} \lambda_j^{(4)}|^p) \in \mathbb{C}^{n \times n}$, $e = (1, 1, \ldots, 1)^T \in \mathbb{C}^n$. Applying inequality (2.1), we have

$$T^{|\circ|p} \ge s_n^p(T) B,$$

where $B = (b_{ij})$ is a doubly stochastic matrix. Then

$$\left\| \Lambda^{(1)} T \Lambda^{(2)} - \Lambda^{(3)} T \Lambda^{(4)} \right\|_p^p \ge e^T \left(s_n^p(T) B \circ M \right) e = s_n^p(T) e^T (B \circ M) e.$$

Since B is doubly stochastic, by Birkhoff's theorem ([12, 13], p.527) B is a convex combination of permutation matrices:

$$B = \sum_{i=1}^{n!} \tau_i P_i, \quad \tau_i \ge 0, \sum_{i=1}^{n!} \tau_i = 1, P_i \text{ are permutation matrices.}$$

Suppose $e^T(B \circ P_k)e = \min\{e^T(B \circ P_i)e \mid 1 \le i \le n!\}$ and P_k corresponds to $\pi \in S_n$. Then

$$\left\| \Lambda^{(1)} T \Lambda^{(2)} - \Lambda^{(3)} T \Lambda^{(4)} \right\|_p^p \ge s_n^p(T) e^T (B \circ M)e$$

$$= s_n^p(T) e^T \left(\sum_{i=1}^{n!} \tau_i P_i \circ M \right) e$$

$$\ge s_n^p(T) \sum_{i=1}^{n!} \tau_i e^T (P_k \circ M)e$$

$$= s_n^p(T) e^T (P_k \circ M)e$$

$$= s_n^p(T) \sum_{i=1}^{n} \left| \lambda_i^{(1)} \lambda_{\pi(i)}^{(2)} - \lambda_i^{(3)} \lambda_{\pi(i)}^{(4)} \right|^p.$$

Proving (2.3).

Use (2.2), (2.5), and the Birkhoff theorem, we can deduce the inequality (2.4). □

Remark 2.3 If we take $p = 2$, we get Theorem 3.2 in [3]. So, the bound in inequality (2.3) generalizes the bound of Theorem 3.2 in [3].

Next, we apply Theorem 2.2 to get some perturbation bounds for the eigenvalues of diagonalizable matrices. Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ and $B = (b_{ij}) \in \mathbb{C}^{n \times n}$. When $p > 0$, $q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, then (see [2])

$$\|AB\|_p \le \|A\|_p \|B\|_{q,p}, \tag{2.6}$$

$$\|AB\|_p \le \left\| A^T \right\|_{q,p} \|B\|_p. \tag{2.7}$$

If B is nonsingular, then we have

$$\frac{\|A\|_p}{\|B^{-1}\|_{q,p}} \le \|AB\|_p. \tag{2.8}$$

If A is nonsingular, then we have

$$\frac{\|B\|_p}{\|(A^{-1})^T\|_{q,p}} \le \|AB\|_p. \tag{2.9}$$

For normal matrices the statement of Theorem 3 in [7] (inequality (1.5)) can no longer be valid. However, we have the following theorem.

Theorem 2.4 *Assume that both $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times n}$ are diagonalizable and admit the following decompositions:*

$$A = D_1 \Lambda_1 D_1^{-1} \quad and \quad B = D_2 \Lambda_2 D_2^{-1}, \tag{2.10}$$

where D_1 and D_2 are nonsingular, and $\Lambda_1 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\Lambda_2 = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_n)$. Then there are permutations π and ν of S_n such that

$$\left(\sum_{i=1}^{n} |\lambda_i - \mu_{\pi(i)}|^p \right)^{\frac{1}{p}} \leq \left\| \left(D_1^{-1} \right)^T \right\|_{q,p} \| D_2 \|_{q,p} \left\| D_2^{-1} \right\| \| D_1 \| \| A - B \|_p, \tag{2.11}$$

$$\left(\sum_{i=1}^{n} |\lambda_i - \mu_{\nu(i)}|^p \right)^{\frac{1}{p}} \leq \left\| \left(D_2^{-1} \right)^T \right\|_{q,p} \| D_1 \|_{q,p} \left\| D_1^{-1} \right\| \| D_2 \| \| A - B \|_p, \tag{2.12}$$

where $1 < p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Using (2.10), we have

$$\begin{aligned} \| A - B \|_p^p &= \left\| D_1 \Lambda_1 D_1^{-1} - D_2 \Lambda_2 D_2^{-1} \right\|_p^p \\ &= \left\| D_1 \left(\Lambda_1 D_1^{-1} D_2 - D_1^{-1} D_2 \Lambda_2 \right) D_2^{-1} \right\|_p^p \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \| A - B \|_p^p &= \left\| D_1 \Lambda_1 D_1^{-1} - D_2 \Lambda_2 D_2^{-1} \right\|_p^p \\ &= \left\| D_2 \left(D_2^{-1} D_1 \Lambda_1 - \Lambda_2 D_2^{-1} D_1 \right) D_1^{-1} \right\|_p^p. \end{aligned} \tag{2.14}$$

We give a proof of (2.11) with the help of (2.13). Similarly one can prove (2.12) using (2.14). Applying (2.8) and (2.9) to (2.13) we obtain

$$\| A - B \|_p^p \geq \frac{\left\| \Lambda_1 D_1^{-1} D_2 - D_1^{-1} D_2 \Lambda_2 \right\|_p^p}{\| (D_1^{-1})^T \|_{q,p}^p \| D_2 \|_{q,p}^p}.$$

Using inequality (2.3), there is a permutation π of S_n such that

$$\left\| \Lambda_1 D_1^{-1} D_2 - D_1^{-1} D_2 \Lambda_2 \right\|_p^p \geq s_n^p \left(D_1^{-1} D_2 \right) \sum_{i=1}^{n} |\lambda_i - \mu_{\pi(i)}|^p.$$

So we have

$$\sum_{i=1}^{n} |\lambda_i - \mu_{\pi(i)}|^p \leq \frac{\| (D_1^{-1})^T \|_{q,p}^p \| D_2 \|_{q,p}^p}{s_n^p (D_1^{-1} D_2)} \| A - B \|_p^p.$$

We use the relations

$$s_n^{-1} \left(D_1^{-1} D_2 \right) = \left\| \left(D_1^{-1} D_2 \right)^{-1} \right\| \leq \| D_1 \| \left\| D_2^{-1} \right\|$$

to get the inequality in (2.11). $\qquad \square$

Theorem 2.5 *Under the hypotheses of Theorem 2.4, there are permutations π and ν of S_n such that*

$$\left(\sum_{i=1}^{n} |\lambda_i - \mu_{\pi(i)}|^q \right)^{\frac{1}{q}} \geq \frac{\| A - B \|_q}{\| (D_1)^T \|_{p,q} \| D_2^{-1} \|_{p,q} \| D_1^{-1} \| \| D_2 \|}, \tag{2.15}$$

$$\left(\sum_{i=1}^{n}|\lambda_i - \mu_{\nu(i)}|^q\right)^{\frac{1}{q}} \geq \frac{\|A - B\|_q}{\|(D_2)^T\|_{p,q}\|D_1^{-1}\|_{p,q}\|D_1\|\|D_2^{-1}\|}, \tag{2.16}$$

where $q \geq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Using (2.10), we have

$$\begin{aligned}
\|A - B\|_q^q &= \|D_1\Lambda_1 D_1^{-1} - D_2\Lambda_2 D_2^{-1}\|_q^q \\
&= \|D_1(\Lambda_1 D_1^{-1}D_2 - D_1^{-1}D_2\Lambda_2)D_2^{-1}\|_q^q
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
\|A - B\|_q^q &= \|D_1\Lambda_1 D_1^{-1} - D_2\Lambda_2 D_2^{-1}\|_q^q \\
&= \|D_2(D_2^{-1}D_1\Lambda_1 - \Lambda_2 D_2^{-1}D_1)D_1^{-1}\|_q^q.
\end{aligned} \tag{2.18}$$

Applying (2.6) and (2.7) to (2.17) we obtain

$$\|A - B\|_q^q \leq \|(D_1)^T\|_{p,q}^q\|\Lambda_1 D_1^{-1}D_2 - D_1^{-1}D_2\Lambda_2\|_q^q\|D_2^{-1}\|_{p,q}^q.$$

Using inequality (2.4), there exists a permutation π of S_n such that

$$\|\Lambda_1 D_1^{-1}D_2 - D_1^{-1}D_2\Lambda_2\|_q^q \leq s_1^q(D_1^{-1}D_2)\sum_{i=1}^{n}|\lambda_i - \mu_{\pi(i)}|^q,$$

so we have

$$\sum_{i=1}^{n}|\lambda_i - \mu_{\pi(i)}|^q \geq \frac{\|A - B\|_q^q}{\|(D_1)^T\|_{p,q}^q\|D_2^{-1}\|_{p,q}^q s_1^q(D_1^{-1}D_2)}.$$

We use the relations

$$s_1(D_1^{-1}D_2) = \|D_1^{-1}D_2\| \leq \|D_1^{-1}\|\|D_2\|$$

to get the inequality in (2.15).

The proof of inequality (2.16) is similar to the proof of (2.15) and is omitted here. □

For $1 \leq p \leq 2$, it is well known [2] that the scalar function (1.1) of a matrix A is a sub-multiplicative matrix norm. However, it is true for $0 < p \leq 2$. Actually, according to the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|AB\|_p^p &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\sum_{k=1}^{n}a_{ik}b_{kj}\right|^p \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left(\sum_{k=1}^{n}|a_{ik}|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}|b_{kj}|^2\right)^{\frac{1}{2}}\right)^p.
\end{aligned}$$

Since for fixed vector $x = (x_1, x_2, \ldots, x_n)$, the function $p \to (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$ is decreasing on $(0, \infty)$,

$$\|AB\|_p^p \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\left(\sum_{k=1}^{n} |a_{ik}|^p \right) \left(\sum_{k=1}^{n} |b_{kj}|^p \right) \right)$$

$$= \left(\sum_{i=1}^{n} \sum_{k=1}^{n} |a_{ik}|^p \right) \left(\sum_{j=1}^{n} \sum_{k=1}^{n} |b_{kj}|^p \right)$$

$$= \|A\|_p^p \|B\|_p^p.$$

That is,

$$\|AB\|_p \leq \|A\|_p \|B\|_p. \tag{2.19}$$

If B is nonsingular, then

$$\|A\|_p = \left\| ABB^{-1} \right\|_p \leq \|AB\|_p \left\| B^{-1} \right\|_p.$$

So we have

$$\frac{\|A\|_p}{\|B^{-1}\|_p} \leq \|AB\|_p. \tag{2.20}$$

Similarly, when A is nonsingular, then

$$\|B\|_p = \left\| A^{-1} AB \right\|_p \leq \left\| A^{-1} \right\|_p \|AB\|_p.$$

That is,

$$\frac{\|B\|_p}{\|A^{-1}\|_p} \leq \|AB\|_p. \tag{2.21}$$

Theorem 2.6 *Under the assumptions of Theorem 2.4, there are permutations π and v of S_n such that*

$$\left(\sum_{i=1}^{n} |\lambda_i - \mu_{\pi(i)}|^p \right)^{\frac{1}{p}} \leq \left\| D_1^{-1} \right\|_p \|D_2\|_p \left\| D_2^{-1} \right\| \|D_1\| \|A - B\|_p, \tag{2.22}$$

$$\left(\sum_{i=1}^{n} |\lambda_i - \mu_{v(i)}|^p \right)^{\frac{1}{p}} \leq \left\| D_2^{-1} \right\|_p \|D_1\|_p \left\| D_1^{-1} \right\| \|D_2\| \|A - B\|_p, \tag{2.23}$$

where $0 < p \leq 2$.

Proof The proof is similar to the proof of Theorem 2.4 and is omitted here. $\quad\square$

Remark 2.7 Since

$$\|A\|_{q,p} = \left(\sum_{j=1}^{n} \left(\sum_{k=1}^{n} |a_{kj}|^q \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^{n} \left(\sum_{k=1}^{n} |a_{kj}|^p \right)^{\frac{p}{p}} \right)^{\frac{1}{p}} = \|A\|_p$$

and

$$\left\|A^T\right\|_{q,p} \le \left\|A^T\right\|_p = \|A\|_p$$

for $1 < p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, the bounds in (2.11) and (2.12) are always sharper than those in (2.22) and (2.23), respectively.

When $p = q = 2$, then $\|AB\|_2 \le \min\{\|A\|_2\|B\|, \|A\|\|B\|_2\}$. We obtain

$$\left(\sum_{i=1}^n |\lambda_i - \mu_{\pi(i)}|^2\right)^{\frac{1}{2}} \le \left\|D_1^{-1}\right\|\|D_2\|\left\|D_2^{-1}D_1\right\|\|A - B\|_2$$

$$\le \left\|D_1^{-1}\right\|\|D_2\|\left\|D_2^{-1}\right\|\|D_1\|\|A - B\|_2,$$

$$\left(\sum_{i=1}^n |\lambda_i - \mu_{\nu(i)}|^2\right)^{\frac{1}{2}} \le \left\|D_2^{-1}\right\|\|D_1\|\left\|D_1^{-1}D_2\right\|\|A - B\|_2$$

$$\le \left\|D_2^{-1}\right\|\|D_1\|\left\|D_1^{-1}\right\|\|D_2\|\|A - B\|_2.$$

Since

$$\|A\|_{p,q} = \left(\sum_{j=1}^n \left(\sum_{k=1}^n |a_{kj}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \le \left(\sum_{j=1}^n \left(\sum_{k=1}^n |a_{kj}|^p\right)^{\frac{p}{p}}\right)^{\frac{1}{p}} = \|A\|_p$$

and

$$\left\|A^T\right\|_{p,q} \le \left\|A^T\right\|_p = \|A\|_p$$

for $q \ge 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have the following corollary.

Corollary 2.8 *Under the same conditions as in Theorem 2.4, there are permutations π and ν of S_n such that*

$$\left(\sum_{i=1}^n |\lambda_i - \mu_{\pi(i)}|^q\right)^{\frac{1}{q}} \ge \frac{\|A - B\|_q}{\|D_1\|_p\|D_2^{-1}\|_p\|D_1^{-1}\|\|D_2\|},$$

$$\left(\sum_{i=1}^n |\lambda_i - \mu_{\nu(i)}|^q\right)^{\frac{1}{q}} \ge \frac{\|A - B\|_q}{\|D_1^{-1}\|_p\|D_2\|_p\|D_1\|\|D_2^{-1}\|},$$

where $q \ge 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 2.9 When $p = q = 2$, we obtain

$$\left(\sum_{i=1}^n |\lambda_i - \mu_{\pi(i)}|^2\right)^{\frac{1}{2}} \ge \frac{\|A - B\|_2}{\|D_1\|\|D_2^{-1}\|\|D_1^{-1}D_2\|} \ge \frac{\|A - B\|_2}{\|D_1\|\|D_2^{-1}\|\|D_1^{-1}\|\|D_2\|},$$

$$\left(\sum_{i=1}^n |\lambda_i - \mu_{\nu(i)}|^2\right)^{\frac{1}{2}} \ge \frac{\|A - B\|_2}{\|D_1^{-1}\|\|D_2\|\|D_2^{-1}D_1\|} \ge \frac{\|A - B\|_2}{\|D_1^{-1}\|\|D_2\|\|D_1\|\|D_2^{-1}\|}.$$

3 Perturbation bounds for singular values

For brevity we only consider square matrices. The generalizations from square matrices to rectangular matrices are obvious, and usually problems on singular values of rectangular matrices can be converted to the case of square matrices by adding zero rows or zero columns.

For a Hermitian matrix $G \in \mathbb{C}^{n \times n}$, we always denote $\lambda(G) = (\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G))$, where $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ are the eigenvalues of G in decreasing order.

Lemma 3.1 (Lidskii [14], Lemma 3.18 [9]) *If $G, H \in \mathbb{C}^{n \times n}$ are Hermitian matrices, then*

$$\lambda(G) - \lambda(H) \prec \lambda(G - H).$$

Lemma 3.2 ([9], p.18) *Let $f(t)$ be a convex function, $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Then*

$$x \prec y \quad \Longrightarrow \quad \big(f(x_1), f(x_2), \ldots, f(x_n)\big) \prec_w \big(f(y_1), f(y_2), \ldots, f(y_n)\big).$$

Theorem 3.3 *Let $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A)$, $\sigma_1(B) \geq \sigma_2(B) \geq \cdots \geq \sigma_n(B)$ and $\sigma_1(A - B) \geq \sigma_2(A - B) \geq \cdots \geq \sigma_n(A - B)$ be the singular values of the complex matrices $A = (a_{ij})$, $B = (b_{ij})$ and $A - B$, respectively. Then*

$$\sum_{i=1}^{n} \big|\sigma_i(A) - \sigma_i(B)\big|^p \leq \sum_{i,j=1}^{n} |a_{ij} - b_{ij}|^p, \tag{3.1}$$

$$\sum_{i=1}^{n} \big|\sigma_i(A) - \sigma_i(B)\big|^q \geq \sum_{i=1}^{n} \sigma_i^q(A - B), \tag{3.2}$$

where $1 \leq p \leq 2$, $0 < q \leq 1$.

Proof Let $\varphi(A) \triangleq \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$, $\varphi(B) \triangleq \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix}$. Then

$$\varphi(A - B) = \begin{pmatrix} 0 & (A - B)^* \\ A - B & 0 \end{pmatrix}.$$

$\varphi(A), \varphi(B), \varphi(A - B)$ are three Hermitian matrices. Assume that $U_1^* A V_1 = \operatorname{diag}(\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A))$, $U_2^* B V_2 = \operatorname{diag}(\sigma_1(B), \sigma_2(B), \ldots, \sigma_n(B))$ and $U_3^*(A - B) V_3 = \operatorname{diag}(\sigma_1(A - B), \sigma_2(A - B), \ldots, \sigma_n(A - B))$ are singular value decompositions with $U_1, U_2, U_3, V_1, V_2, V_3$ unitary. Then

$$Q_1 \triangleq \frac{1}{\sqrt{2}} \begin{pmatrix} V_1 & V_1 \\ U_1 & -U_1 \end{pmatrix}, \quad Q_2 \triangleq \frac{1}{\sqrt{2}} \begin{pmatrix} V_2 & V_2 \\ U_2 & -U_2 \end{pmatrix} \quad \text{and} \quad Q_3 \triangleq \frac{1}{\sqrt{2}} \begin{pmatrix} V_3 & V_3 \\ U_3 & -U_3 \end{pmatrix}$$

are unitary matrices and

$$Q_1^* \varphi(A) Q_1 = \operatorname{diag}\big(\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A), -\sigma_1(A), -\sigma_2(A), \ldots, -\sigma_n(A)\big),$$

$$Q_2^* \varphi(B) Q_2 = \operatorname{diag}\big(\sigma_1(B), \sigma_2(B), \ldots, \sigma_n(B), -\sigma_1(B), -\sigma_2(B), \ldots, -\sigma_n(B)\big)$$

and

$$Q_3^* \varphi(A-B)Q_3 = \mathrm{diag}\big(\sigma_1(A-B), \sigma_2(A-B), \ldots, \sigma_n(A-B),$$
$$-\sigma_1(A-B), -\sigma_2(A-B), \ldots, -\sigma_n(A-B)\big).$$

By Lemma 3.1, we have

$$\big(\sigma_1(A) - \sigma_1(B), \sigma_2(A) - \sigma_2(B), \ldots, \sigma_n(A) - \sigma_n(B),$$
$$\sigma_n(B) - \sigma_n(A), \ldots, \sigma_2(B) - \sigma_2(A), \sigma_1(B) - \sigma_1(A)\big)$$
$$\prec \big(\sigma_1(A-B), \sigma_2(A-B), \ldots, \sigma_n(A-B),$$
$$-\sigma_n(A-B), \ldots, -\sigma_2(A-B), -\sigma_1(A-B)\big). \tag{3.3}$$

First consider the case $1 \le p \le 2$. Since the function $f(t) = |t|^p$ is convex on $(-\infty, +\infty)$, applying Lemma 3.2 with $f(t)$ to the majorization (3.3) yields

$$\big(\big|\sigma_1(A) - \sigma_1(B)\big|^p, \big|\sigma_2(A) - \sigma_2(B)\big|^p, \ldots, \big|\sigma_n(A) - \sigma_n(B)\big|^p,$$
$$\big|\sigma_n(B) - \sigma_n(A)\big|^p, \ldots, \big|\sigma_2(B) - \sigma_2(A)\big|^p, \big|\sigma_1(B) - \sigma_1(A)\big|^p\big)$$
$$\prec_w \big(\sigma_1^p(A-B), \sigma_2^p(A-B), \ldots, \sigma_n^p(A-B), \sigma_n^p(A-B), \ldots, \sigma_2^p(A-B), \sigma_1^p(A-B)\big).$$

In particular,

$$\big(\big|\sigma_1(A) - \sigma_1(B)\big|^p, \big|\sigma_2(A) - \sigma_2(B)\big|^p, \ldots, \big|\sigma_n(A) - \sigma_n(B)\big|^p\big)$$
$$\prec_w \big(\sigma_1^p(A-B), \sigma_2^p(A-B), \ldots, \sigma_n^p(A-B)\big). \tag{3.4}$$

According to Theorem 1 of [15] or Theorem 3.32 of [9], we have

$$\sum_{i=1}^n \sigma_i^p(A-B) \le \sum_{i,j=1}^n |a_{ij} - b_{ij}|^p \tag{3.5}$$

for $1 \le p \le 2$. Combining (3.4) and (3.5), we obtain (3.1).

When $0 < q \le 1$, by considering the convex function $g(t) = -|t|^q$, on $(-\infty, +\infty)$, applying Lemma 3.2 with $g(t)$ to the majorization (3.3) yields

$$\big(-\big|\sigma_1(A) - \sigma_1(B)\big|^q, \ldots, -\big|\sigma_n(A) - \sigma_n(B)\big|^q, -\big|\sigma_n(B) - \sigma_n(A)\big|^q, \ldots, -\big|\sigma_1(B) - \sigma_1(A)\big|^q\big)$$
$$\prec_w \big(-\sigma_1^q(A-B), \ldots, -\sigma_n^q(A-B), -\sigma_n^q(A-B), \ldots, -\sigma_1^q(A-B)\big).$$

In particular,

$$\big(-\big|\sigma_1(A) - \sigma_1(B)\big|^q, -\big|\sigma_2(A) - \sigma_2(B)\big|^q, \ldots, -\big|\sigma_n(A) - \sigma_n(B)\big|^q\big)$$
$$\prec_w \big(-\sigma_1^q(A-B), -\sigma_2^q(A-B), \ldots, -\sigma_n^q(A-B)\big). \tag{3.6}$$

From (3.6), we get (3.2). □

Remark 3.4 From inequality (3.1), if $p = 2$, we get the inequality (1.6). So the inequality (3.1) generalizes the inequality (3.5.33) of [10], p.215, and Theorem 3.12 of [11], p.199.

Authors' contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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