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# On Kantorovich modification of $(p, q)$-Baskakov operators

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**Abstract**

The concern of this paper is to introduce a Kantorovich modification of $(p,q)$-Baskakov operators and investigate their approximation behaviors. We first define a new $(p,q)$-integral and construct the operators. The rate of convergence in terms of modulus of continuities, quantitative and qualitative results in weighted spaces, and finally pointwise convergence of the operators for the functions belonging to the Lipschitz class are discussed.

**MSC:** Primary 41A25; secondary 41A36

**Keywords:** $(p,q)$-integers; $(p,q)$-integral; $(p,q)$-Baskakov-Kantorovich operators; weighted approximation; rate of convergence

## 1 Introduction

The $(p,q)$-calculus is a generalization of the well-known $q$-calculus and it is constructed by the following notations and definitions. Let $0 < q < p \leq 1$. For each nonnegative integer $n$, the $(p,q)$-number is denoted by $[n]_{p,q}$ and is given by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

For each $k, n \in \mathbb{N}$, $n \geq k \geq 0$, the $(p,q)$-factorial $[k]_{p,q}!$ and $(p,q)$-binomial are defined by

$$[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}, \qquad n \geq 1, \qquad [0]_{p,q}! = 1,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}.$$

The $(p,q)$-power basis is defined by

$$(x \oplus a)^n_{p,q} = (x+a)(px+qa)\left(p^2x+q^2a\right)\cdots\left(p^{n-1}x+q^{n-1}a\right)$$

and

$$(x \ominus a)^n_{p,q} = (x-a)(px-qa)\left(p^2x-q^2a\right)\cdots\left(p^{n-1}x-q^{n-1}a\right).$$

Let $f:\mathbb{R}\to\mathbb{R}$ then the $(p,q)$-derivative of a function $f$, denoted by $D_{p,q}f$, is defined by

$$(D_{p,q}f)(x) := \frac{f(px)-f(qx)}{(p-q)x}, \quad x\neq 0, \qquad (D_{p,q}f)(0) := f'(0)$$

provided that $f$ is differentiable at 0. The following assertions hold true:

$$\begin{aligned} D_{p,q}(x\oplus a)^n_{p,q} &= [n]_{p,q}(px\oplus a)^{n-1}_{p,q}, \quad n\geq 1, \\ D_{p,q}(a\oplus x)^n_{p,q} &= [n]_{p,q}(a\oplus qx)^{n-1}_{p,q}, \quad n\geq 1, \end{aligned}$$

and $D_{p,q}(x\oplus a)^0_{p,q}=0$. The formula for the $(p,q)$-derivative of a product is

$$D_{p,q}\big(u(x)v(x)\big) := D_{p,q}\big(u(x)\big)v(qx) + D_{p,q}\big(v(x)\big)u(px).$$

Let $f: C[0,a]\to\mathbb{R}$ $(a>0)$ then the $(p,q)$-integration of a function $f$ is defined by

$$\begin{aligned} \int_0^a f(t)\,d_{p,q}t &= (q-p)a\sum_{k=0}^{\infty} f\left(\frac{p^k}{q^{k+1}}a\right)\frac{p^k}{q^{k+1}} \quad \text{if } \left|\frac{p}{q}\right|<1, \\ \int_0^a f(t)\,d_{p,q}t &= (p-q)a\sum_{k=0}^{\infty} f\left(\frac{q^k}{p^{k+1}}a\right)\frac{q^k}{p^{k+1}} \quad \text{if } \left|\frac{p}{q}\right|>1. \end{aligned} \tag{1.1}$$

The formula of the $(p,q)$-integration by parts is given by

$$\int_a^b f(px)D_{p,q}g(x)\,d_{p,q}x = f(b)g(b)-f(a)q(a) - \int_a^b g(qx)D_{p,q}f(x)\,d_{p,q}x. \tag{1.2}$$

Here we note that all the notations mentioned above reduce to the $q$-analogs when $p=1$. For more details of the $(p,q)$-calculus, we refer the reader to [1–5].

The $(p,q)$-calculus has been used efficiently in many fields of science such as oscillator algebra, Lie group, field theory, differential equations, hypergeometric series, physical sciences. Therefore, to approximate the functions via polynomials based on $(p,q)$-integers, no doubt, would have a crucial role. To fulfill this necessity, very recently the well-known sequences of linear positive operators of approximation theory have been transferred to the $(p,q)$-calculus and the advantages of $(p,q)$ analogs of them have been intensively investigated. For some recent work devoted to $(p,q)$-operators, we refer the reader to [6–11]. Very recently, Aral and Gupta [12] introduced the $(p,q)$-analog of the well-known Baskakov operators by

$$B_{n,p,q}(f;x) = \sum_{k=0}^{\infty} b^{p,q}_{n,k}(x) f\left(\frac{p^{n-1}[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right), \tag{1.3}$$

where $x\in[0,\infty)$, $0<q<p\leq 1$, and

$$b^{p,q}_{n,k}(x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{p,q} p^{k+n(n-1)/2}q^{k(k-1)/2}\frac{x^k}{(1\oplus x)^{n+k}_{p,q}},$$

and they calculated that

$$B_{n,p,q}(1;x)=1,\qquad B_{n,p,q}(t;x)=x,\qquad B_{n,p,q}\bigl(t^2;x\bigr)=x^2+\frac{p^{n-1}x}{[n]_{p,q}}\biggl(1+\frac{p}{q}x\biggr). \tag{1.4}$$

Another problem in the approximation by linear positive operators is to present an approximation process for Riemann integrable functions. The main tool to solve this problem is to consider the Kantorovich modifications of the corresponding operators, which mainly depends on the replacing the sample values $f(k/n)$ by the mean values of $f$ in the intervals $[k/(n+1),(k+1)/(n+1)]$. Since the $(p,q)$-integral of $f$ over $[a,b]$ is defined as follows:

$$\int_a^b f(t)\,d_{p,q}t=\int_0^b f(t)\,d_{p,q}t-\int_0^a f(t)\,d_{p,q}t, \tag{1.5}$$

one cannot say (1.5) is positive every time unless it is assumed that $f$ is a nondecreasing function. Hence, use of (1.5) to introduce a Kantorovich modification of any $(p,q)$-operators may lead to some problem. Recently Mursaleen *et al.* [13] introduced a Kantorovich modification of $(p,q)$-Szász-Mirakjan operators using the $(p,q)$-integral (1.5) for the functions being nondecreasing. However, in this paper we define a new $(p,q)$-integral, hence we do not need to impose any condition on $f$. For the generalizations of Baskakov operators and Kantorovich operators in classical calculus and $q$-calculus, we refer the reader to some recent papers [14–19].

The aim of this paper is to introduce $(p,q)$-Baskakov-Kantorovich operators and investigate their approximation properties. In the next section, we construct the operators, calculate the moments, central moments of the operators, and give some lemmas which will be necessary to prove our main results. In Section 3, we prove a local approximation theorem for the new operators in terms of Peetre's $\mathcal{K}$-functional and its equivalent modulus of continuities. In Section 4, we investigate the uniform convergence of the operators and present the rate of convergence via the weighted modulus of continuities. In the last section, we give some pointwise estimates for the functions belonging to Lipschitz space.

## 2 Construction of operators

To present a method to solve the problem mentioned in the Introduction now we propose a new definition of the $(p,q)$-integral. Let $a,b\in\mathbb{R}$, $a<b$, and $h(x):=f(a+x)$ be an arbitrary function and $D_{p,q}H(x)=h(x)$, where $H(x):=F(a+x)$, then we can write

$$\frac{H(px)-H(qx)}{(p-q)x}=h(x),$$

that is, $H(px)-H(qx)=(p-q)xh(x)$. Hence we get

$$\begin{aligned}
&H\bigl(pq^{-1}x\bigr)-H(x)=(p-q)q^{-1}xh\bigl(q^{-1}x\bigr),\\
&H\bigl(p^2q^{-2}x\bigr)-H\bigl(pq^{-1}x\bigr)=(p-q)pq^{-2}xh\bigl(pq^{-2}x\bigr),\\
&\ldots,\\
&H\bigl(p^{n+1}q^{-(n+1)}x\bigr)-H\bigl(p^nq^{-n}x\bigr)=(p-q)p^nq^{-(n+1)}xh\bigl(p^nq^{-(n+1)}x\bigr),
\end{aligned}$$

which allows us to write

$$\begin{aligned}
&F\left(a + pq^{-1}x\right) - F(a + x) = (p - q)q^{-1}xf\left(a + q^{-1}x\right),\\
&F\left(a + p^2q^{-2}x\right) - F\left(a + pq^{-1}x\right) = (p - q)pq^{-2}xf\left(a + pq^{-2}x\right),\\
&\ldots,\\
&F\left(a + p^{n+1}q^{-(n+1)}x\right) - F\left(a + p^nq^{-n}x\right) = (p - q)p^nq^{-(n+1)}xf\left(a + p^nq^{-(n+1)}x\right).
\end{aligned}$$

Adding these formulas term by term, we have

$$F\left(a + p^{n+1}q^{-(n+1)}x\right) - F(a + x) = (p - q)x\sum_{k=0}^{n} f\left(a + p^kq^{-(k+1)}x\right)\frac{p^k}{q^{k+1}}$$

and taking the limit as $n \to \infty$ with the fact $|\frac{p}{q}| < 1$ we have

$$F(a + x) - F(a) = (q - p)x\sum_{k=0}^{\infty} f\left(a + \frac{p^k}{q^{k+1}}x\right)\frac{p^k}{q^{k+1}}.$$

Similarly we have, for $|\frac{q}{p}| < 1$,

$$F(a + x) - F(a) = (p - q)x\sum_{k=0}^{\infty} f\left(a + \frac{q^k}{p^{k+1}}x\right)\frac{q^k}{p^{k+1}}$$

and if we take $x = b - a$ then we get

$$F(b) - F(a) = (p - q)(b - a)\sum_{k=0}^{\infty} f\left(a + (b - a)\frac{q^k}{p^{k+1}}\right)\frac{q^k}{p^{k+1}}.$$

**Definition 1** Let $f$ be an arbitrary function. The $(p,q)$-integral of $f$ can be defined by

$$\begin{aligned}
&\int_a^b f(t)\,d_{p,q}t = (p - q)(b - a)\sum_{n=0}^{\infty} f\left(a + (b - a)\frac{q^n}{p^{n+1}}\right)\frac{q^n}{p^{n+1}} \quad \text{when } \left|\frac{q}{p}\right| < 1,\\
&\int_a^b f(t)\,d_{p,q}t = (q - p)(b - a)\sum_{n=0}^{\infty} f\left(a + (b - a)\frac{p^n}{q^{n+1}}\right)\frac{p^n}{q^{n+1}} \quad \text{when } \left|\frac{p}{q}\right| < 1.
\end{aligned} \tag{2.1}$$

Considering the new $(p,q)$-integral given in (2.1) we can define the Kantorovich modifications of the operators (1.3) as follows.

**Definition 2** For $x \in [0,\infty)$, $0 < q < p \leq 1$, the $(p,q)$-analog of the Baskakov-Kantorovich operators is defined as

$$B^*_{n,p,q}(f;x) = [n]_{p,q}\sum_{k=0}^{\infty} b^{p,q}_{n,k}(x)q^{-k}\int_{p[k]_{p,q}/[n]_{p,q}}^{[k+1]_{p,q}/[n]_{p,q}} f\left(\frac{p^{n-1}t}{q^{k-1}}\right)d_{p,q}t. \tag{2.2}$$

**Lemma 1** *For* $0 < q < p \leq 1$ *and* $n \in \mathbb{N}$ *we have*

$$\int_{p[k]_{p,q}/[n]_{p,q}}^{[k+1]_{p,q}/[n]_{p,q}} d_{p,q}t = \frac{q^k}{[n]_{p,q}}, \tag{2.3}$$

$$\int_{p[k]_{p,q}/[n]_{p,q}}^{[k+1]_{p,q}/[n]_{p,q}} t\,d_{p,q}t = \frac{pq^k[k]_{p,q}}{[n]_{p,q}^2} + \frac{q^{2k}}{[n]_{p,q}^2}\frac{1}{(p+q)}, \tag{2.4}$$

$$\begin{aligned}\int_{p[k]_{p,q}/[n]_{p,q}}^{[k+1]_{p,q}/[n]_{p,q}} t^2 d_{p,q}t ={}& \frac{p^2q^k[k]_{p,q}^2}{[n]_{p,q}^3} + 2\frac{p[k]_{p,q}}{[n]_{p,q}}\frac{q^{2k}}{[n]_{p,q}^2}\frac{1}{p+q} \\ &+ \frac{q^{3k}}{[n]_{p,q}^3}\frac{1}{p^2+pq+q^2}.\end{aligned} \tag{2.5}$$

*Proof* The proof easily follows from (2.1). □

**Lemma 2** *For* $x \in [0,\infty)$, $0 < q < p \leq 1$, $n \in \mathbb{N}$, *the following hold*:

$$B_{n,p,q}^*(e_0;x) = 1, \tag{2.6}$$

$$B_{n,p,q}^*(e_1;x) = px + \frac{qp^{n-1}}{(p+q)[n]_{p,q}}, \tag{2.7}$$

$$\begin{aligned}B_{n,p,q}^*(e_2;x) ={}& \left(p^2x^2 + \frac{p^{n+1}x}{[n]_{p,q}}\left(1 + \frac{p}{q}x\right)\right) \\ &+ \frac{2qp^nx}{(p+q)[n]_{p,q}^2} + \frac{q^2p^{2n-2}}{(p^2+pq+q^2)[n]_{p,q}^2},\end{aligned} \tag{2.8}$$

*where* $e_i(x) = x^i$, $i = 0,1,2$.

*Proof* By the definition of the operators (2.2) and equality (2.3) we obtain $B_{n,p,q}^*(e_0;x) = B_{n,p,q}(1;x) = 1$. In a similar way, using (2.4) we can write

$$\begin{aligned}B_{n,p,q}^*(e_1;x) &= [n]_{p,q}\sum_{k=0}^{\infty} b_{n,k}^{p,q}(x)q^{-k}\int_{p[k]_{p,q}/[n]_{p,q}}^{[k+1]_{p,q}/[n]_{p,q}}\left(\frac{p^{n-1}t}{q^{k-1}}\right)d_{p,q}t \\ &= [n]_{p,q}\sum_{k=0}^{\infty} b_{n,k}^{p,q}(x)\frac{p^{n-1}}{q^{k-1}}q^{-k}\int_{p[k]_{p,q}/[n]_{p,q}}^{[k+1]_{p,q}/[n]_{p,q}} t\,d_{p,q}t \\ &= [n]_{p,q}\sum_{k=0}^{\infty} b_{n,k}^{p,q}(x)\frac{p^{n-1}}{q^{k-1}}q^{-k}\frac{pq^k[k]_{p,q}}{[n]_{p,q}^2} \\ &\quad + [n]_{p,q}\sum_{k=0}^{\infty} b_{n,k}^{p,q}(x)\frac{p^{n-1}}{q^{k-1}}q^{-k}\frac{q^{2k}}{[n]_{p,q}^2}\frac{1}{(p+q)} \\ &= p\sum_{k=0}^{\infty} b_{n,k}^{p,q}(x)\frac{p^{n-1}}{q^{k-1}}\frac{[k]_{p,q}}{[n]_{p,q}} \\ &\quad + \frac{qp^{n-1}}{(p+q)[n]_{p,q}}\sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) \\ &= pB_{n,p,q}(e_1;x) + \frac{qp^{n-1}}{(p+q)[n]_{p,q}}B_{n,p,q}(e_0;x).\end{aligned}$$

Using the equalities $B_{n,p,q}(e_0;x) = 1$, $B_{n,p,q}(e_1;x) = x$ we immediately have

$$B^*_{n,p,q}(e_1;x) = px + \frac{qp^{n-1}}{(p+q)[n]_{p,q}}.$$

Finally, using (2.5) we have

$$\begin{aligned}
B^*_{n,p,q}(e_2;x) &= [n]_{p,q} \sum_{k=0}^{\infty} b^{p,q}_{n,k}(x) \frac{p^{2n-2}}{q^{2k-2}} q^{-k} \int_{p[k]_{p,q}/[n]_{p,q}}^{[k+1]_{p,q}/[n]_{p,q}} t^2\, d_{p,q}t \\
&= [n]_{p,q} \sum_{k=0}^{\infty} b^{p,q}_{n,k}(x) \frac{p^{2n-2}}{q^{2k-2}} q^{-k} \int_{p[k]_{p,q}/[n]_{p,q}}^{[k+1]_{p,q}/[n]_{p,q}} t^2\, d_{p,q}t \\
&= p^2 \sum_{k=0}^{\infty} b^{p,q}_{n,k}(x) \frac{p^{2n-2}}{q^{2k-2}} \frac{[k]^2_{p,q}}{[n]^2_{p,q}} \\
&\quad + \frac{2qp^n}{(p+q)[n]^2_{p,q}} \sum_{k=0}^{\infty} b^{p,q}_{n,k}(x) \frac{p^{n-1}}{q^{k-1}} \frac{[k]_{p,q}}{[n]_{p,q}} \\
&\quad + \frac{q^2 p^{2n-2}}{(p^2+pq+q^2)[n]^2_{p,q}} \sum_{k=0}^{\infty} b^{p,q}_{n,k}(x) \\
&= p^2 B_{n,p,q}(e_2;x) + \frac{2qp^n}{(p+q)[n]^2_{p,q}} B_{n,p,q}(e_1;x) \\
&\quad + \frac{q^2 p^{2n-2}}{(p^2+pq+q^2)[n]^2_{p,q}} B_{n,p,q}(e_0;x).
\end{aligned}$$

And the equalities (1.4) give us

$$B^*_{n,p,q}(e_2;x) = \left(p^2x^2 + \frac{p^{n+1}x}{[n]_{p,q}}\left(1 + \frac{p}{q}x\right)\right) + \frac{2qp^n x}{(p+q)[n]^2_{p,q}} + \frac{q^2p^{2n-2}}{(p^2+pq+q^2)[n]^2_{p,q}},$$

which completes the proof. □

**Remark 1** Using Lemma 2, we get

$$B^*_{n,p,q}\big((e_1-x)^2;x\big) = \alpha_1(n)x^2 + \alpha_2(n)x + \alpha_3(n),$$

where

$$\begin{aligned}
\alpha_1(n) &= (p-1)^2 + \frac{p^{n+2}}{q[n]_{p,q}}, \\
\alpha_2(n) &= \frac{p^{n+1}(p[n]_{p,q}+1)}{q[n]^2_{p,q}}, \\
\alpha_3(n) &= \frac{p^{2n}}{3q^2[n]^2_{p,q}}.
\end{aligned}$$

Further, choosing

$$\alpha^*(n) := \max\left\{\alpha_1(n), \frac{\alpha_2(n)}{2}, \alpha_3(n)\right\} \tag{2.9}$$

we can write

$$B^*_{n,p,q}\big((e_1-x)^2;x\big) \le \alpha^*(n)(1+x)^2.$$

**Remark 2** For $q \in (0,1)$ and $p \in (q,1]$ we easily see that $\lim_{n\to\infty}[n]_{p,q} = 1/(p-q)$. Hence, the operators (2.2) are not approximation processes with the above form. In order to study the convergence properties of the sequence of $(p,q)$-Baskakov-Durrmeyer operators, we assume that $q = (q_n)$ and $p = (p_n)$ such that $0 < q_n < p_n \le 1$ and $q_n \to 1$, $p_n \to 1$, $q_n^n \to a$, $p_n^n \to b$ as $n \to \infty$.

Here we note that with these assumptions $\alpha_1(n) \to 0$, $\alpha_2(n) \to 0$, $\alpha_3(n) \to 0$ as $n \to \infty$, hence $\alpha^*(n) \to 0$ as $n \to \infty$.

Let $C_B[0,\infty)$ denote the space of all real valued continuous and bounded functions on $[0,\infty)$. In this space we consider the norm

$$\|f\|_{C_B} = \sup_{x\in[0,\infty)} \big|f(x)\big|.$$

**Lemma 3** *Let $f \in C_B[0,\infty)$. Then for all $g \in C_B^2[0,\infty)$, we have*

$$\big|\tilde{B}^*_{n,p,q}(g;x) - g(x)\big| \le \big\|g''\big\|_{C_B}\big(\gamma^*(n)(1+x)^2 + \beta_n^2(p,q,x)\big), \tag{2.10}$$

*where $\tilde{B}^*_{n,p,q}$ is an auxiliary operator defined by*

$$\tilde{B}^*_{n,p,q}(g;x) = B^*_{n,p,q}(g;x) + g(x) - g\left(px + \frac{qp^{n-1}}{(p+q)[n]_{p,q}}\right) \tag{2.11}$$

*and*

$$\beta_n(p,q,x) = (p-1)x + \frac{p^{n-1}}{2[n]_{p,q}}.$$

*Proof* By the definition of $\tilde{B}^*_{n,p,q}$ and Lemma 2, it is obvious that

$$\tilde{B}^*_{n,p,q}(e_1 - x;x) = 0. \tag{2.12}$$

Since $g \in C_B^2[0,\infty)$, using the Taylor expansion for $x \in [0,\infty)$ we have

$$g(t) = g(x) + g(x)(t-x) + \int_x^t (t-u)g''(u)\,du.$$

Applying the operators $\tilde{B}^*_{n,p,q}$ to both sides of the above equality and considering the fact (2.12) we obtain

$$\begin{aligned}
&\tilde{B}^*_{n,p,q}(g;x) - g(x)\\
&\quad= \tilde{B}^*_{n,p,q}\left(\int_x^t (t-u)g''(u)\,du;x\right) = B^*_{n,p,q}\left(\int_x^t (t-u)g''(u)\,du;x\right)\\
&\qquad- \int_x^{px+\frac{qp^{n-1}}{(p+q)[n]_{p,q}}}\left(px + \frac{qp^{n-1}}{(p+q)[n]_{p,q}} - u\right)g''(u)\,du.
\end{aligned} \tag{2.13}$$

Also we get

$$\begin{aligned}\left|\int_x^t (t-u)g''(u)\,du\right| &\leq \left|\int_x^t |t-u|\left|g''(u)\right|du\right| \\ &\leq \left\|g''\right\|_{C_B}\left|\int_x^t |t-u|\,du\right| \leq \left\|g''\right\|_{C_B}(t-x)^2 \end{aligned} \tag{2.14}$$

and

$$\begin{aligned}&\left|\int_x^{px+\frac{qp^{n-1}}{(p+q)[n]_{p,q}}}\left(px+\frac{qp^{n-1}}{(p+q)[n]_{p,q}}-u\right)g''(u)\,du\right| \\ &\leq \left(px+\frac{qp^{n-1}}{(p+q)[n]_{p,q}}-x\right)^2\left\|g''\right\|_{C_B} = \left((p-1)x+\frac{qp^{n-1}}{(p+q)[n]_{p,q}}\right)^2\left\|g''\right\|_{C_B} \\ &\leq \left((p-1)x+\frac{p^{n-1}}{2[n]_{p,q}}\right)^2\left\|g''\right\|_{C_B} \\ &:= \beta_n^2(p,q,x)\left\|g''\right\|_{C_B}. \end{aligned} \tag{2.15}$$

Using the inequalities (2.14) and (2.15) in (2.13) we immediately have

$$\left|\tilde{B}^*_{n,p,q}(g;x)-g(x)\right| \leq \left\|g''\right\|_{C_B}\left(\alpha^*(n)(1+x)^2+\beta_n^2(p,q,x)\right). \qquad \square$$

## 3 Local approximation

Let us consider the following $\mathcal{K}$ functional:

$$\mathcal{K}_2(f,\delta) = \inf_{g\in W^2}\left\{\|f-g\|_{C_B}+\delta\left\|g''\right\|_{C_B}\right\},$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By [20], p.177, Theorem 2.4, there exists an absolute constant $C > 0$ such that

$$\mathcal{K}_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta}), \tag{3.1}$$

where

$$\omega_2(f,\delta) = \sup_{0<h\leq\sqrt{\delta}}\sup_{x\in[0,\infty)}\left|f(x+2h)-2f(x+h)+f(x)\right|$$

is the second order modulus of smoothness of $f \in C_B[0,\infty)$. The usual modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$\omega(f,\delta) = \sup_{0<h\leq\delta}\sup_{x\in[0,\infty)}\left|f(x+h)-f(x)\right|.$$

Let us recall the definitions of the weighted spaces and corresponding modulus of continuity. Let $C[0,\infty)$ be the set of all continuous functions $f$ defined on $[0,\infty)$ and $B_2[0,\infty)$ the set of all functions $f$ defined on $[0,\infty)$ satisfying the condition $|f(x)| \leq M(1+x^2)$ with some positive constant $M$ which may depend only on $f$. $C_2[0,\infty)$ denotes the subspace of all continuous functions in $B_2[0,\infty)$. By $C_2^*[0,\infty)$, we denote the subspace of all functions $f \in C_2[0,\infty)$ for which $\lim_{x\to\infty} \frac{f(x)}{1+x^2}$ is finite. $B_2[0,\infty)$ is a linear normed space with the norm $\|f\|_2 = \sup_{x\geq 0} \frac{|f(x)|}{1+x^2}$.

**Theorem 1** *Let $f \in C_B[0,\infty)$. Then for every $x \in [0,\infty)$, there exists a constant $L > 0$ such that*

$$\left|B_{n,p,q}^*(f;x) - f(x)\right| \leq L\omega_2\left(f;\sqrt{\alpha^*(n)(1+x)^2 + \beta_n^2(p,q,x)}\right) + \omega\bigl(f;\beta_n(p,q,x)\bigr).$$

*Proof* By (2.11), for every $g \in C_B^2[0,\infty)$ one can obtain

$$\begin{aligned}
&\left|B_{n,p,q}^*(f;x) - f(x)\right| \\
&\quad\leq \left|\tilde{B}_{n,p,q}^*(f;x) - f(x)\right| + \left|f(x) - f\left(px + \frac{qp^{n-1}}{(p+q)[n]_{p,q}}\right)\right| \\
&\quad\leq \left|\tilde{B}_{n,p,q}^*(f-g;x) - (f-g)(x)\right| \\
&\qquad+ \left|f(x) - f\left(px + \frac{qp^{n-1}}{(p+q)[n]_{p,q}}\right)\right| + \left|\tilde{B}_{n,p,q}^*(g;x) - g(x)\right|.
\end{aligned}$$

Taking into account (2.2), (2.6), and (2.11) we have

$$\left|\tilde{B}_{n,p,q}^*(f;x)\right| \leq 4\|f\|_{C_B}.$$

Using this inequality and Lemma 3 we get

$$\begin{aligned}
\left|B_{n,p,q}^*(f;x) - f(x)\right| &\leq 4\|f-g\|_{C_B} + \left|f(x) - f\left(px + \frac{qp^{n-1}}{(p+q)[n]_{p,q}}\right)\right| \\
&\quad+ \left\|g''\right\|_{C_B}\bigl(\alpha^*(n)(1+x)^2 + \beta_n^2(p,q,x)\bigr)
\end{aligned}$$

and taking the infimum on the right-hand side over all $g \in C_B^2[0,\infty)$ and using (3.1), we deduce

$$\begin{aligned}
&\left|B_{n,p,q}^*(f;x) - f(x)\right| \\
&\quad\leq 4K_2\bigl(f;\alpha^*(n)(1+x)^2 + \beta_n^2(p,q,x)\bigr) + \omega\bigl(f;\beta_n(p,q,x)\bigr) \\
&\quad\leq 4\omega_2\left(f;\sqrt{\alpha^*(n)(1+x)^2 + \beta_n^2(p,q,x)}\right) + \omega\bigl(f;\beta_n(p,q,x)\bigr) \\
&\quad= L\omega_2\left(f;\sqrt{\alpha^*(n)(1+x)^2 + \beta_n^2(p,q,x)}\right) + \omega\bigl(f;\beta_n(p,q,x)\bigr),
\end{aligned}$$

where $L = 4M > 0$. □

**Theorem 2** *Let $f \in C_2[0,\infty)$, $p_n, q_n \in (0,1)$ such that $0 < q_n < p_n \leq 1$ and $\omega_{a+1}(f,\delta)$ be its modulus of continuity on the finite interval $[0,a+1] \subset [0,\infty)$, where $a > 0$. Then the inequality*

$$\left|B_{n,p,q}^*(f;x) - f(x)\right| \leq 4M_f\bigl(1+a^2\bigr)\gamma^*(n)(1+x)^2 + 2\omega_{a+1}\left(f,(1+x)\sqrt{\alpha^*(n)}\right)$$

*holds, where $M_f$ is positive constant independent of $n$ and $\alpha^*(n)$ is as indicated in* (2.9).

*Proof* By [21], $\omega_{a+1}(\cdot,\delta)$ has the property

$$\left|f(t)-f(x)\right| \leq 4M_f\left(1+a^2\right)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta), \quad \delta > 0.$$

Applying the Cauchy-Schwarz inequality and choosing $\delta = \sqrt{\alpha^*(n)(1+x)^2}$, we have

$$\begin{aligned}
&\left|B^*_{n,p,q}(f;x)-f(x)\right| \\
&\quad\leq 4M_f\left(1+a^2\right)B^*_{n,p,q}\left((t-x)^2;x\right) \\
&\qquad + \omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\left(B^*_{n,p,q}\left((t-x)^2;x\right)\right)^{1/2}\right) \\
&\quad\leq 4M_f\left(1+a^2\right)\alpha^*(n)(1+x)^2 + 2\omega_{a+1}\left(f,(1+x)\sqrt{\alpha^*(n)}\right),
\end{aligned}$$

which completes the proof. □

## 4 Weighted approximation

**Theorem 3** *Let $q = q_n \in (0,1)$, $p = p_n \in (q,1]$ such that $q_n \to 1$, $p_n \to 1$ as $n \to \infty$. Then for each function $f \in C_2^*[0,\infty)$ we get*

$$\lim_{n\to\infty}\left\|B^*_{n,p_n,q_n}f - f\right\|_2 = 0.$$

*Proof* According to the weighted Korovkin theorem proved in [22], it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty}\left\|B^*_{n,p_n,q_n}e_i - e_i\right\|_2 = 0, \quad i = 0,1,2. \tag{4.1}$$

By (2.6), (4.1) holds for $i = 0$. By (2.7) and (2.8) we have

$$\begin{aligned}
\left\|B^*_{n,p_n,q_n}e_1 - e_1\right\|_2 &= \sup_{x\geq 0}\frac{\beta_n(p_n,q_n,x)}{1+x^2} \\
&\leq (1-p_n)\sup_{x\geq 0}\frac{x}{1+x^2} + \frac{qp_n^{n-1}}{(p_n+q)[n]_{p,q}} \\
&\leq (1-p_n) + \frac{p_n^{n-1}}{2[n]_{p,q}}
\end{aligned}$$

and by a similar consideration we have

$$\begin{aligned}
\left\|B^*_{n,p_n,q_n}e_2 - e_2\right\|_2 &\leq \left(1-p_n^2+\frac{p_n^{n+2}}{q_n[n]_{p_n,q_n}}\right)\sup_{x\geq 0}\frac{x^2}{1+x^2} \\
&\quad + \left(\frac{2q_np_n^n}{(p_n+q_n)[n]^2_{p_n,q_n}} + \frac{p_n^{n+1}}{[n]_{p_n,q_n}}\right)\sup_{x\geq 0}\frac{x}{1+x^2} \\
&\quad + \frac{q_n^2p_n^{2n-2}}{(p_n^2+p_nq_n+q_n^2)[n]^2_{p_n,q_n}}\sup_{x\geq 0}\frac{1}{1+x^2} \\
&\leq \left(1-p_n^2+\frac{p_n^{n+2}}{q_n[n]_{p_n,q_n}}\right) + \left(\frac{p_n^n}{[n]^2_{p_n,q_n}} + \frac{p_n^{n+1}}{[n]_{p_n,q_n}}\right) + \frac{p_n^{2n-2}}{3[n]^2_{p_n,q_n}}.
\end{aligned}$$

The last two inequalities mean that (4.1) holds for $i = 1, 2$. Hence, the proof is completed. □

To obtain the rate of convergence, we consider the weighted modulus of continuity defined by

$$\Omega_2(f;\delta) = \sup_{x\geq 0, 0<h\leq\delta} \frac{|f(x+h)-f(x)|}{1+(x+h)^2}$$

for $f \in C^*_{x^2}[0,\infty)$, and $\Omega_2(\cdot;\cdot)$ has the following properties.

**Lemma 4** ([23]) *If $f \in C^*_{x^2}[0,\infty)$ then*
(i) *$\Omega_2(f;\delta)$ is monotone increasing function of $\delta$,*
(ii) $\lim_{\delta\to 0^+}\Omega_2(f;\delta) = 0$,
(iii) *for any $\lambda \in [0,\infty)$, $\Omega_2(f;\lambda\delta) \leq (1+\lambda)\Omega_2(f;\delta)$.*

**Theorem 4** *Let $p = p_n$ and $q = q_n$ satisfy $0 < q_n < p_n \leq 1$ and for n sufficiently large $p_n \to 1$, $q_n \to 1$, and $q_n^n \to 1$ and $p_n^n \to 1$. If $f \in C^*_{x^2}[0,\infty)$, then for sufficiently large n we have*

$$\left|B^*_{n,p_n,q_n}(f;x) - f(x)\right| \leq K\left(1+x^{2+\lambda}\right)\Omega_2\left(f;\sqrt{\alpha^*(n)}\right), \tag{4.2}$$

*where $\lambda \geq 1$ and K is a positive constant independent of f and n, $\alpha^*(n)$ is as indicated in* (2.9).

*Proof* By the definition of the weighted modulus of continuity and Lemma 4, we can write

$$\begin{aligned}\left|f(t)-f(x)\right| &\leq \left(1+\left(x+|t-x|\right)^2\right)\left(1+\frac{|t-x|}{\delta}\right)\Omega_2(f;\delta)\\ &\leq \left(1+(2x+t)^2\right)\left(1+\frac{|t-x|}{\delta}\right)\Omega_2(f;\delta).\end{aligned}$$

The above inequality allows us to write

$$\begin{aligned}\left|B^*_{n,p_n,q_n}(f;x)-f(x)\right| \leq{}& \left(B^*_{n,p_n,q_n}\left(1+(2x+t)^2;x\right) + B^*_{n,p_n,q_n}\left(\left(1+(2x+t)^2\right)\frac{|t-x|}{\delta};x\right)\right)\\ &\times \Omega_2(f;\delta).\end{aligned}$$

Using the Cauchy-Schwarz inequality we have

$$\begin{aligned}\left|B^*_{n,p_n,q_n}(f;x)-f(x)\right| \leq{}& \left(B^*_{n,p_n,q_n}\left(1+(2x+t)^2;x\right) + \frac{1}{\delta_n}\sqrt{B^*_{n,p_n,q_n}\left(\left(1+(2x+t)^2\right)^2;x\right)}\right.\\ &\left.\times\sqrt{B^*_{n,p_n,q_n}\left((t-x)^2;x\right)}\right)\Omega_2(f;\delta).\end{aligned}$$

On the other hand, by (2.8) we get

$$\begin{aligned}&\frac{1}{1+x^2}B^*_{n,p_n,q_n}\left(1+t^2;x\right)\\ &\quad= \left(p_n^2+\frac{p_n^{n+2}}{q_n[n]_{p_n,q_n}}\right)\frac{x^2}{1+x^2}\end{aligned}$$

$$\begin{aligned}
&+\left(\frac{2q_np_n^n}{(p_n+q_n)[n]^2_{p_n,q_n}}+\frac{p_n^{n+1}}{[n]_{p_n,q_n}}\right)\frac{x}{1+x^2}\\
&+\left(\frac{q_n^2p_n^{2n-2}+(p_n^2+p_nq_n+q_n^2)[n]^2_{p_n,q_n}}{(p_n^2+p_nq_n+q_n^2)[n]^2_{p_n,q_n}}\right)\frac{1}{1+x^2}\\
&\le 1+K_1 \qquad (4.3)
\end{aligned}$$

for sufficiently large $n$, where $K_1$ is a positive constant. From (4.3), there exists $K_2 > 0$ such that $B^*_{n,p_n,q_n}(1+(2x+t)^2;x) \le K_2(1+x^2)$, for sufficiently large $n$. In a similar way we get

$$\frac{1}{1+x^4}B^*_{n,p_n,q_n}\bigl(1+t^4;x\bigr) \le 1+K_3,$$

where $K_3$ is a positive constant. Hence we have $\sqrt{B^*_{n,p_n,q_n}((1+(2x+t)^2)^2;x)} \le K_4(1+x^2)$, for sufficiently large $n$. Hence we have

$$\bigl|B^*_{n,p_n,q_n}(f;x)-f(x)\bigr| \le \bigl(1+x^2\bigr)\left(K_2+\frac{1}{\delta_n}K_4(1+x)\sqrt{\alpha^*(n)}\right)\Omega_2(f;\delta).$$

Hence choosing $\delta_n = \sqrt{\alpha^*(n)}$ we have

$$\begin{aligned}
\bigl|B^*_{n,p_n,q_n}(f;x)-f(x)\bigr| &\le \bigl(1+x^2\bigr)\bigl(K_2+K_4(1+x)\bigr)\Omega_2\bigl(f;\sqrt{\alpha^*(n)}\bigr)\\
&\le K\bigl(1+x^{2+\lambda}\bigr)\Omega_2\bigl(f;\sqrt{\alpha^*(n)}\bigr)
\end{aligned}$$

for sufficiently large $n$ and $x \in [0,\infty)$, where $K := K_2 + K_4$. □

**Corollary 1** *With the assumptions of Theorem* 4, *if we take the limit as $n \to \infty$ in* (4.2) *we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|B^*_{n,p_n,q_n}(f,x)-f(x)|}{(1+x^2)^{1+\lambda}} = 0.$$

## 5 Pointwise estimates

**Theorem 5** *Let $0 < \alpha \le 1$ and $E$ be any subset of the interval $[0,\infty)$. Then, if $f \in C_B[0,\infty)$ is locally in* $\mathrm{Lip}(\alpha)$, *i.e., the condition*

$$\bigl|f(y)-f(x)\bigr| \le L|y-x|^\alpha, \quad y \in E \text{ and } x \in [0,\infty) \qquad (5.1)$$

*holds, then, for each $x \in [0,\infty)$, we have*

$$\bigl|B^*_{n,p,q}(f;x)-f(x)\bigr| \le L\bigl\{\bigl(\alpha^*(n)\bigr)^{\alpha/2}(1+x)^\alpha + 2\bigl(d(x,E)\bigr)^\alpha\bigr\},$$

*where $L$ is a constant depending on $\alpha$ and $f$; and $d(x,E)$ is the distance between $x$ and $E$ defined by*

$$d(x,E) = \inf\bigl\{|t-x| : t \in E\bigr\}.$$

*Proof* Let $\bar{E}$ denote the closure of $E$ in $[0,\infty)$. Then there exists a point $x_0 \in \bar{E}$ such that $|x - x_0| = d(x,E)$. Using the triangle inequality

$$\bigl|f(t) - f(x)\bigr| \le \bigl|f(t) - f(x_0)\bigr| + \bigl|f(x) - f(x_0)\bigr|$$

we immediately have by (5.1)

$$\begin{aligned}
\bigl|B^*_{n,p,q}(f;x) - f(x)\bigr| &\le B^*_{n,p,q}\bigl(\bigl|f(t) - f(x_0)\bigr|;x\bigr) + B^*_{n,p,q}\bigl(\bigl|f(x) - f(x_0)\bigr|;x\bigr) \\
&\le L\bigl\{B^*_{n,p,q}\bigl(|t - x_0|^\alpha;x\bigr) + |x - x_0|^\alpha\bigr\} \\
&\le L\bigl\{B^*_{n,p,q}\bigl(|t - x|^\alpha + |x - x_0|^\alpha;x\bigr) + |x - x_0|^\alpha\bigr\} \\
&= L\bigl\{B^*_{n,p,q}\bigl(|t - x|^\alpha;x\bigr) + 2|x - x_0|^\alpha\bigr\}.
\end{aligned}$$

Using the Hölder inequality with $p = 2/\alpha$, $q = 2/(2 - \alpha)$, we obtain

$$\begin{aligned}
\bigl|B^*_{n,p,q}(f;x) - f(x)\bigr| &\le L\bigl\{\bigl[B^*_{n,p,q}\bigl(|t - x|^{\alpha p};x\bigr)\bigr]^{\frac{1}{p}} + 2\bigl(d(x,E)\bigr)^\alpha\bigr\} \\
&= L\bigl\{\bigl[B^*_{n,p,q}\bigl(|t - x|^2;x\bigr)\bigr]^{\frac{\alpha}{2}} + 2\bigl(d(x,E)\bigr)^\alpha\bigr\} \\
&\le L\bigl\{\bigl(\alpha^*(n)(1 + x)^2\bigr)^{\frac{\alpha}{2}} + 2\bigl(d(x,E)\bigr)^\alpha\bigr\} \\
&= L\bigl\{\bigl(\alpha^*(n)\bigr)^{\alpha/2}(1 + x)^\alpha + 2\bigl(d(x,E)\bigr)^\alpha\bigr\}.
\end{aligned}$$ □

Next we obtain the local direct estimate for the operators $B^*_{n,p,q}$, using the Lipschitz type maximal function of order $\alpha$ introduced by Lenze [24]:

$$\tilde{\omega}_a(f,x) = \sup_{t\neq x, t\in[0,\infty)} \frac{|f(t) - f(x)|}{|t - x|^\alpha}, \quad x \in [0,\infty) \text{ and } \alpha \in (0,1]. \tag{5.2}$$

**Theorem 6** *Let $f \in C_B[0,\infty)$ and $0 < \alpha \le 1$. Then, for all $x \in [0,\infty)$ we have*

$$\bigl|B^*_{n,p,q}(f;x) - f(x)\bigr| \le \tilde{\omega}_a(f,x)\bigl(\alpha^*(n)\bigr)^{\alpha/2}(1 + x)^\alpha.$$

*Proof* From equation (5.2), we have

$$\bigl|B^*_{n,p,q}(f;x) - f(x)\bigr| \le \tilde{\omega}_a(f,x)B^*_{n,p,q}\bigl(|t - x|^\alpha;x\bigr).$$

Applying the Hölder inequality with $p = 2/\alpha$, $q = 2/(2 - \alpha)$, we get

$$\begin{aligned}
\bigl|B^*_{n,p,q}(f;x) - f(x)\bigr| &\le \tilde{\omega}_a(f,x)\bigl[B^*_{n,p,q}\bigl(|t - x|^2;x\bigr)\bigr]^{\frac{\alpha}{2}} \\
&\le \tilde{\omega}_a(f,x)\bigl(\alpha^*(n)\bigr)^{\alpha/2}(1 + x)^\alpha.
\end{aligned}$$ □

**Remark 3** The further properties of the operators such as convergence properties via summability methods (see, for example, [25, 26]) can be studied.

**Conclusion 1** To introduce Kantorovich modifications of the approximation operators in $(p,q)$-calculus, the existing $(p,q)$-integral did not meet the purposes since the positivity of the operators was not guaranteed. In this paper, we solved this problem and presented a new Riemann type $(p,q)$-integral. As an application, we introduced the $(p,q)$-Baskakov-Kantorovich operators and investigated their approximation properties. Using the new $(p,q)$-integral, one can introduce Kantorovich modifications of other well-known operators.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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