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Some new continued fraction sequence convergent to the Somos quadratic recurrence constant

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Abstract
In this paper, we provide some new continued fraction approximation and inequalities of the Somos quadratic recurrence constant, using its relation with the generalized Euler constant.

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Keywords: Somos' quadratic recurrence constant; generalized Euler constant; continued fraction; multiple-correction method

1 Introduction

Somos [1] defined the sequence $g_n = ng_{n-1}^2$, with $g_0 = 1$ in 1999. Finch [2] proved the asymptotic formula in 2003 as follows:

$$g_n = \sigma^{2^n}\left(n + 2 - \frac{1}{n} + \frac{4}{n^2} - \frac{21}{n^3} + \frac{138}{n^4} - \frac{1{,}091}{n^5} + \cdots\right)^{-1} \quad (n \to \infty),$$

where the constant $\sigma = 1.661687949\ldots$ is now known as the Somos quadratic recurrence constant. This constant appears in important problems by pure representations,

$$\sigma = \sqrt{1\sqrt{2\sqrt{3\ldots}}} = \prod_{k=1}^{\infty} k^{1/2^k} = \exp\left\{\sum_{k=1}^{\infty} \frac{\ln k}{2^k}\right\},$$

or integral representations,

$$\sigma = \exp\left\{-\int_0^1 \frac{1-x}{(2-x)\ln x}\,dx\right\} = \exp\left\{-\int_0^1\int_0^1 \frac{x}{(2-xy)\ln(xy)}\,dx\,dy\right\};$$

see [3–5].

The generalized-Euler-constant function

$$\gamma(z) = \sum_{k=1}^{\infty} z^{k-1}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right) \quad \left(|z| \le 1\right) \tag{1.1}$$

was introduced by Sondow and Hadjicostas [6] and Pilehrood and Pilehrood [7], where $\gamma(1) = 0.577215\ldots$ is the classical Euler constant.

Sondow and Hadjicostas [6] also defined the generalized Somos quadratic recurrence constant, by

$$\sigma_t = \left(\frac{t}{t-1}\right)^{1/(t-1)} \exp\left\{-\frac{1}{t(t-1)}\gamma\left(\frac{1}{t}\right)\right\}. \tag{1.2}$$

Since when we set $t = 2$ in (1.2),

$$\gamma\left(\frac{1}{2}\right) = 2\ln\frac{2}{\sigma} \quad \text{or} \quad \sigma = 2\exp\left\{-\frac{1}{2}\gamma\left(\frac{1}{2}\right)\right\}, \tag{1.3}$$

these functions are closely related to the Somos quadratic recurrence constant σ. Here we denote

$$\gamma_n(z) = \sum_{k=1}^{n} z^{k-1}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right) \quad \bigl(|z| \le 1\bigr). \tag{1.4}$$

Recently, many inspiring results of establishing more precise inequalities and more accurate approximations for the Somos quadratic recurrence constant and generalized-Euler-constant function were given. Mortici [8] provided a double inequality of the error estimate by the polynomial approximation. Lu and Song [9] gave sharper bounds.

Motivated by this important work, in this paper we will continue our previous work [10–13] and apply a *multiple-correction method* to construct some new sharper double inequality of the error estimate for the Somos quadratic recurrence constant. Moreover, we establish sharp bounds for the corresponding error terms.

Notation Throughout the paper, the notation $\Psi(k;x)$ means a polynomial of degree k in x with all of its non-zero coefficients positive, which may be different at each occurrence.

2 Estimating $\gamma(1/2)$

In order to deduce some estimates for the σ constant, we evaluate the series

$$\gamma\left(\frac{1}{2}\right) = \sum_{k=1}^{\infty}\frac{1}{2^{k-1}}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right). \tag{2.1}$$

First we need the following intermediary result.

Lemma 1 *For every integer positive k, we define*

$$\alpha_3(x) = \frac{1}{x^2}\left(1 + \frac{b_0}{x + \frac{c_0 x}{x + \frac{c_1 x}{x + \frac{c_2 x}{x + c_3}}}}\right), \qquad \alpha_4(x) = \frac{1}{x^2}\left(1 + \frac{b_0}{x + \frac{c_0 x}{x + \frac{c_1 x}{x + \frac{c_2 x}{x + \frac{c_3 x}{x + c_4}}}}}\right),$$

where $b_0 = -\frac{8}{3}$, $c_0 = \frac{69}{16}$, $c_1 = \frac{2{,}689}{1{,}840}$, $c_2 = \frac{1{,}759{,}264}{309{,}235}$, $c_3 = \frac{3{,}080{,}065{,}424}{1{,}034{,}832{,}071}$, $c_4 = \frac{1{,}439{,}063{,}522{,}712{,}675}{206{,}144{,}225{,}688{,}128}$. Then for every integer k, we have

$$a_3(k) - \frac{1}{2}a_3(k+1) < \frac{1}{k} - \ln\frac{k+1}{k} < a_4(k) - \frac{1}{2}a_4(k+1). \tag{2.2}$$

Proof Based on our previous work we will apply *multiple-correction method* and study the double inequality of the error estimate as follows.

(*Step* 1) *The initial correction.* Because $(\frac{1}{x} - \ln\frac{x+1}{x})' = -\frac{1}{x^2(x+1)}$, we choose $\alpha_0(x) = \frac{1}{x^2}(1 + \frac{b_0}{x+c_0})$. Then letting the coefficient of x^5, x^6 of the molecule in the following fractions equal zero, we have $b_0 = -\frac{8}{3}$, $c_0 = \frac{69}{16}$, and

$$\begin{aligned} f_0'(x) &= \left(a_0(x) - \frac{1}{2}a_0(x+1) - \left(\frac{1}{x} - \ln\frac{x+1}{x}\right)\right)' \\ &= \frac{-26{,}255{,}650 - 71{,}362{,}005x - 73{,}118{,}084x^2 - 25{,}661{,}312x^3 - 2{,}753{,}536x^4}{x^3(1+x)^3(69+16x)^2(85+16x)^2} < 0. \end{aligned}$$

As the molecule in the above fractions has all coefficients negative, we see as a result that $f_0(x)$ is strictly decreasing.

(*Step* 2) *The first correction.* We let $\alpha_1(x) = \frac{1}{x^2}(1 + \frac{b_0}{x+\frac{c_0 x}{x+c_1}})$. Then letting the coefficient of x^6 of the molecule in the following fractions equal zero, we have $c_1 = \frac{2{,}689}{1{,}840}$ and

$$\begin{aligned} f_1'(x) &= \left(a_1(x) - \frac{1}{2}a_1(x+1) - \left(\frac{1}{x} - \ln\frac{x+1}{x}\right)\right)' \\ &= \frac{\Psi(5;x)}{x^4(1+x)^4(664+115x)^2(779+115x)^2} > 0. \end{aligned}$$

As

$$\begin{aligned} \Psi(5;x) &= 541{,}756{,}089{,}068 + 2{,}164{,}112{,}814{,}170x + 3{,}349{,}014{,}987{,}656x^2 \\ &\quad + 2{,}438{,}802{,}119{,}388x^3 + 627{,}774{,}173{,}785x^4 + 50{,}167{,}886{,}925x^5 \end{aligned}$$

has all coefficients positive, we see as a result that $f_1(x)$ is strictly increasing. But $f_1(\infty) = 0$, so $f_1(x) < 0$ on $[1, \infty)$.

(*Step* 3) *The second correction.* Similarly, we let $\alpha_2(x) = \frac{1}{x^2}(1 + \frac{b_0}{x+\frac{c_0 x}{x+\frac{c_1 x}{x+c_2}}})$. Then letting the coefficient of x^7 of the molecule in the following fractions equal zero, we have $c_2 = \frac{1{,}759{,}264}{309{,}235}$ and

$$\begin{aligned} f_2'(x) &= \left(a_2(x) - \frac{1}{2}a_2(x+1) - \left(\frac{1}{x} - \ln\frac{x+1}{x}\right)\right)' \\ &= \frac{-\Psi(6;x)}{x^3(1+x)^3(329{,}862 + 154{,}120x + 13{,}445x^2)^2(497{,}427 + 181{,}010x + 13{,}445x^2)^2} \\ &< 0. \end{aligned}$$

As

$$\begin{aligned} \Psi(6;x) &= 11{,}996{,}815{,}612{,}137{,}340{,}746{,}762 + 35{,}859{,}193{,}432{,}468{,}133{,}987{,}910x \\ &\quad + 43{,}146{,}717{,}746{,}712{,}804{,}785{,}684x^2 \\ &\quad + 24{,}054{,}456{,}699{,}427{,}979{,}755{,}000x^3 + 6{,}362{,}662{,}027{,}948{,}035{,}025{,}530x^4 \\ &\quad + 757{,}411{,}977{,}134{,}935{,}302{,}125x^5 + 32{,}547{,}266{,}888{,}095{,}273{,}400x^6 \end{aligned}$$

has all coefficients positive, we see as a result that $f_2(x)$ is strictly decreasing. But $f_2(\infty) = 0$, so $f_2(x) > 0$ on $[1, \infty)$.

(*Step* 4) *The third correction*. Similarly, we let $\alpha_3(x) = \frac{1}{x^2}(1 + \frac{b_0}{x+\frac{c_0x}{x+\frac{c_1x}{x+\frac{c_2x}{x+c_3}}}})$. Then letting the coefficient of x^8 of the molecule in the following fractions equal zero, we have $c_3 = \frac{3{,}080{,}065{,}424}{1{,}034{,}832{,}071}$ and

$$\begin{aligned} f_3'(x) &= \left(a_3(x) - \frac{1}{2}a_3(x+1) - \left(\frac{1}{x} - \ln\frac{x+1}{x}\right)\right)' \\ &= \frac{\Psi(7;x)}{x^4(1+x)^4(16{,}055{,}330 + 5{,}556{,}840x + 384{,}839x^2)^2(21{,}997{,}009 + 6{,}326{,}518x + 384{,}839x^2)^2} \\ &> 0, \end{aligned}$$

we see as a result that $f_3(x)$ is strictly increasing. But $f_3(\infty) = 0$, so $f_3(x) < 0$ on $[1, \infty)$. This finishes the proof of the left-hand inequality in (2.2).

(*Step* 5) *The fourth correction*. Similarly, we let $\alpha_4(x) = \frac{1}{x^2}(1 + \frac{b_0}{x+\frac{c_0x}{x+\frac{c_1x}{x+\frac{c_2x}{x+\frac{c_3x}{x+c_4}}}}})$. Then letting the coefficient of x^9 of the molecule in the following fractions equal zero, we have $c_4 = \frac{1{,}439{,}063{,}522{,}712{,}675}{206{,}144{,}225{,}688{,}128}$ and

$$\begin{aligned} f_4'(x) &= \left(a_4(x) - \frac{1}{2}a_4(x+1) - \left(\frac{1}{x} - \ln\frac{x+1}{x}\right)\right)' \\ &= \frac{-\Psi(8;x)}{x^3(1+x)^3\Psi_1^2(3;x)\Psi_2^2(3;x)} < 0, \end{aligned}$$

we see as a result that $f_4(x)$ is strictly decreasing. But $f_4(\infty) = 0$, so $f_4(x) > 0$ on $[1, \infty)$. This finishes the proof of the right-hand inequality in (2.2).

This is the end of Lemma 1. □

Remark 1 It is worth to point out that Lemma 1 provides some continued fraction inequalities by the multiple-correction method. Similarly, repeating the above approach step by step, we can get more sharp inequalities. But this maybe brings about some computation increase, the details omitted here.

By adding inequalities of the form

$$\frac{a_3(k)}{2^{k-1}} - \frac{a_3(k+1)}{2^k} < \frac{1}{2^{k-1}}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right) < \frac{a_4(k)}{2^{k-1}} - \frac{a_4(k+1)}{2^k}$$

from $k = n+1$ to $k = \infty$, we get

$$\frac{a_3(n+1)}{2^n} < \sum_{k=n+1}^{\infty}\frac{1}{2^{k-1}}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right) < \frac{a_4(n+1)}{2^n}. \tag{2.3}$$

These double inequalities give the error estimate when $\gamma(\frac{1}{2})$ is approximated by

$$\gamma_n\left(\frac{1}{2}\right) = \sum_{k=1}^{n}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right).$$

So we have the following theorem.

Theorem 1 *For every positive integer n*,

$$\begin{aligned}
&\gamma_n\left(\frac{1}{2}\right)+\frac{183{,}429{,}517+476{,}353{,}825n+170{,}553{,}590n^2+11{,}545{,}170n^3}{2^n 30(1+n)^3(21{,}997{,}009+6{,}326{,}518n+384{,}839n^2)}\\
&\quad<\gamma\left(\frac{1}{2}\right)\\
&\quad<\gamma_n\left(\frac{1}{2}\right)\\
&\qquad+\frac{13{,}772{,}456{,}102{,}417+13{,}119{,}369{,}079{,}700n+2{,}447{,}045{,}992{,}090n^2+112{,}489{,}345{,}920n^3}{30(1+n)^2(1{,}182{,}754{,}225{,}349+628{,}371{,}674{,}302n+91{,}567{,}252{,}707n^2+3{,}749{,}644{,}864n^3)}.
\end{aligned}\tag{2.4}$$

Proof The double inequality (2.3) can be equivalently written as

$$\frac{a_3(n+1)}{2^n}<\gamma\left(\frac{1}{2}\right)-\gamma_n\left(\frac{1}{2}\right)<\frac{a_4(n+1)}{2^n}$$

and the conclusion follows if we take into account that

$$\frac{a_3(n+1)}{2^n}=\frac{183{,}429{,}517+476{,}353{,}825n+170{,}553{,}590n^2+11{,}545{,}170n^3}{2^n 30(1+n)^3(21{,}997{,}009+6{,}326{,}518n+384{,}839n^2)}$$

and

$$\begin{aligned}
&\frac{a_4(n+1)}{2^n}\\
&\quad=\frac{13{,}772{,}456{,}102{,}417+13{,}119{,}369{,}079{,}700n+2{,}447{,}045{,}992{,}090n^2+112{,}489{,}345{,}920n^3}{30(1+n)^2(1{,}182{,}754{,}225{,}349+628{,}371{,}674{,}302n+91{,}567{,}252{,}707n^2+3{,}749{,}644{,}864n^3)}.
\end{aligned}$$

This is the end of Theorem 1. □

Remark 2 In fact, the upper and lower bounds in (2.4) are sharper than the ones in (2.3) of Mortici [8] and (2.8) of Lu and Song [9] for every positive integer n.

From (2.4) we can provide the following result which has a simpler form than (2.4), although it is weaker than (2.4).

Corollary 1 *For every positive integer* $n\geq 1$, *we have*

$$\gamma_n\left(\frac{1}{2}\right)+\frac{1}{2^n(n+\frac{8}{3})^2}<\gamma\left(\frac{1}{2}\right)<\gamma_n\left(\frac{1}{2}\right)+\frac{1}{2^n(n+\frac{3}{2})^2}.\tag{2.5}$$

Proof We take into account that

$$\begin{aligned}
&a_3(n+1)-\frac{1}{(n+\frac{8}{3})^2}\\
&\quad=\frac{5{,}800{,}296{,}658+19{,}765{,}524{,}466n+12{,}385{,}315{,}613n^2+1{,}837{,}256{,}025n^3+69{,}271{,}020n^4}{30(1+n)^3(8+3n)^2(21{,}997{,}009+6{,}326{,}518n+384{,}839n^2)}\\
&\quad>0
\end{aligned}$$

for $n \geq 1$, and

$$\begin{aligned}
&\frac{1}{(n+\frac{3}{2})^2} - a_4(n+1) \\
&\quad = \bigl(17{,}978{,}402{,}120{,}127 + 75{,}921{,}820{,}053{,}696n + 69{,}182{,}111{,}904{,}322n^2 \\
&\qquad + 14{,}976{,}266{,}612{,}440n^3 + 749{,}928{,}972{,}800n^4\bigr) \\
&\qquad /\bigl(30(1+n)^2(3+2n)^2\bigl(1{,}182{,}754{,}225{,}349 + 628{,}371{,}674{,}302n \\
&\qquad + 91{,}567{,}252{,}707n^2 + 3{,}749{,}644{,}864n^3\bigr)\bigr) \\
&\quad > 0
\end{aligned}$$

for $n \geq 1$. Combining with Theorem 1, the conclusion follows.

This is the end of Corollary 1. □

Combining (1.3) and Corollary 1, we obtain the following estimates for the Somos quadratic recurrence constant.

Corollary 2 *For every positive integer $n \geq 1$, we have*

$$2\exp\left\{-\frac{1}{2}\gamma_n\left(\frac{1}{2}\right) - \frac{1}{2^{n+1}(n+\frac{3}{2})^2}\right\} < \sigma < 2\exp\left\{-\frac{1}{2}\gamma_n\left(\frac{1}{2}\right) - \frac{1}{2^{n+1}(n+\frac{5}{2})^2}\right\}. \tag{2.6}$$

3 Estimating $\gamma(1/3)$

Mortici [8] and Lu and Song [9] have provided a double inequality for the error estimate of $\gamma(1/3)$. In order to give the new error estimate for $\gamma(1/3)$, we need the following intermediary result.

Lemma 2 *For every integer positive k, we define*

$$b_3(k) = -\frac{1}{2x^3}\left(1 + \frac{d_0}{x + \frac{k_0 x}{x + \frac{k_1 x}{x + \frac{k_2 x}{x + k_3}}}}\right),$$

$$b_4(k) = -\frac{1}{2x^3}\left(1 + \frac{d_0}{x + \frac{k_0 x}{x + \frac{k_1 x}{x + \frac{k_2 x}{x + \frac{k_3 x}{x + k_4}}}}}\right),$$

where $d_0 = -\frac{9}{4}$, $k_0 = \frac{18}{5}$, $k_1 = \frac{392}{405}$, $k_2 = \frac{1{,}880{,}525}{444{,}528}$, $k_3 = \frac{905{,}967{,}495}{412{,}812{,}848}$, $k_4 = \frac{33{,}856{,}400{,}240{,}124}{7{,}571{,}975{,}660{,}155}$. Then, for every integer k, we have

$$b_4(k) - \frac{1}{3}b_4(k+1) < \frac{1}{k} - \ln\frac{k+1}{k} - \frac{1}{2k^2} < b_3(k) - \frac{1}{3}b_3(k+1). \tag{3.1}$$

Proof Based on our previous work we will apply *multiple-correction method* to study the double inequality of the error estimate as follows.

(*Step* 1) *The initial correction*. Because $(\frac{1}{x} - \ln\frac{x+1}{x} - \frac{1}{2x^2})' = \frac{1}{x^3(x+1)}$, we choose $b_0(x) = -\frac{1}{2x^3}(1 + \frac{d_0}{x+k_0})$. Then letting the coefficient of x^6, x^7 of the molecule in the following fractions equal zero, we have $d_0 = -\frac{9}{4}$, $k_0 = \frac{18}{5}$, and

$$\begin{aligned} g_0'(x) &= \left(b_0(x) - \frac{1}{3}b_0(x+1) - \left(\frac{1}{x} - \ln\frac{x+1}{x} - \frac{1}{2x^2}\right)\right)' \\ &= \frac{385{,}641 + 1{,}357{,}920x + 1{,}824{,}369x^2 + 1{,}223{,}822x^3 + 369{,}865x^4 + 39{,}200x^5}{4x^4(1+x)^4(18+5x)^2(23+5x)^2} \\ &> 0, \end{aligned}$$

we see as a result that $g_0(x)$ is strictly increasing.

(*Step* 2) *The first correction.* We let $b_1(x) = -\frac{1}{2x^3}(1 + \frac{d_0}{x+\frac{k_0 x}{x+k_1}})$. Then letting the coefficient of x^7 of the molecule in the following fractions equal to zero, we have $k_1 = \frac{392}{405}$ and

$$\begin{aligned} g_1'(x) &= \left(b_1(x) - \frac{1}{3}b_1(x+1) - \left(\frac{1}{x} - \ln\frac{x+1}{x} - \frac{1}{2x^2}\right)\right)' \\ &= \frac{-\Psi(6;x)}{4x^5(1+x)^5(370+81x)^2(451+81x)^2}. \end{aligned}$$

As $\Psi(6;x)$ has all coefficients positive, we see as a result that $g_1(x)$ is strictly decreasing.

(*Step* 3) *The second correction.* Similarly, we let $b_2(x) = -\frac{1}{2x^3}(1+\frac{d_0}{x+\frac{k_0 x}{x+\frac{k_1 x}{x+k_2}}})$. Then we let the coefficient of x^8 of the molecule in the following fractions equal zero, we have $k_2 = \frac{1{,}880{,}525}{444{,}528}$ and

$$\begin{aligned} g_2'(x) &= \left(b_2(x) - \frac{1}{3}b_2(x+1) - \left(\frac{1}{x} - \ln\frac{x+1}{x} - \frac{1}{2x^2}\right)\right)' \\ &= \frac{\Psi(7;x)}{4x^4(1+x)^4(752{,}210 + 434{,}565x + 49{,}392x^2)^2(1{,}236{,}167 + 533{,}349x + 49{,}392x^2)^2} \\ &> 0, \end{aligned}$$

we see as a result that $g_2(x)$ is strictly increasing.

(*Step* 4) *The third correction.* We let $b_3(x) = -\frac{1}{2x^3}(1 + \frac{d_0}{x+\frac{k_0 x}{x+\frac{k_1 x}{x+\frac{k_2 x}{x+k_3}}}})$. Then letting the coefficient of x^9 of the molecule in the following fractions equal zero, we have $k_3 = \frac{905{,}967{,}495}{412{,}812{,}848}$ and

$$\begin{aligned} g_3'(x) &= \left(b_3(x) - \frac{1}{3}b_3(x+1) - \left(\frac{1}{x} - \ln\frac{x+1}{x} - \frac{1}{2x^2}\right)\right)' \\ &= -\Psi(8;x)/\big(4x^5(1+x)^5\big(26{,}595{,}045 + 11{,}576{,}565x + 1{,}053{,}094x^2\big)^2 \\ &\quad\times \big(39{,}224{,}704 + 13{,}682{,}753x + 1{,}053{,}094x^2\big)^2\big). \end{aligned}$$

As $\Psi(8;x)$ has all coefficients positive, we see as a result that $g_3(x)$ is strictly decreasing. But $g_3(\infty) = 0$, so $g_3(x) > 0$ on $[1,\infty)$. This finishes the proof of the right-hand inequality in (3.1).

(*Step* 5) *The fourth correction*. Similarly, we let $b_4(x) = -\frac{1}{2x^3}(1 + \frac{d_0}{x+\frac{k_0x}{x+\frac{k_1x}{x+\frac{k_2x}{x+\frac{k_3x}{x+k_4}}}}})$. Then letting the coefficient of x^{10} of the molecule in the following fractions equal zero, we have $k_4 = \frac{33{,}856{,}400{,}240{,}124}{7{,}571{,}975{,}660{,}155}$ and

$$\begin{aligned} g_4'(x) &= \left(b_4(x) - \frac{1}{3}b_4(x+1) - \left(\frac{1}{x} - \ln\frac{x+1}{x} - \frac{1}{2x^2}\right)\right)' \\ &= \frac{\Psi(9;x)}{4x^4(1+x)^4\Psi_1^2(3;x)\Psi_2^2(3;x)} > 0, \end{aligned}$$

we see as a result that $g_4(x)$ is strictly increasing. But $g_4(\infty) = 0$, so $g_4(x) < 0$ on $[1,\infty)$. This finishes the proof of the left-hand inequality in (3.1).

This is the end of Lemma 2. □

By adding inequalities of the form

$$\frac{b_4(k)}{3^{k-1}} - \frac{b_4(k+1)}{3^k} < \frac{1}{3^{k-1}}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right) - \frac{1}{2(3^{k-1})k^2} < \frac{b_3(k)}{3^{k-1}} - \frac{b_3(k+1)}{3^k}$$

from $k = n+1$ to $k = \infty$, we get

$$\frac{b_4(n+1)}{3^n} < \sum_{k=n+1}^{\infty}\frac{1}{3^{k-1}}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right) - \sum_{k=n+1}^{\infty}\frac{1}{2(3^{k-1})k^2} < \frac{b_3(n+1)}{3^n}. \tag{3.2}$$

Combining equations (1.1) and (1.4), we have

$$\gamma\left(\frac{1}{3}\right) - \gamma_n\left(\frac{1}{3}\right) = \sum_{k=n+1}^{\infty}\frac{1}{3^{k-1}}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right). \tag{3.3}$$

Using inequality (3.2) and equality (3.3), we have the following theorem.

Theorem 2 *For every positive integer n,*

$$\begin{aligned} &\gamma_n\left(\frac{1}{3}\right) - \frac{\Psi_1(3;n)}{3^n 40(1+n)^3\Psi_2(3;n)} + \frac{3}{2}\sum_{k=n+1}^{\infty}\frac{1}{3^k k^2} \\ &\quad < \gamma\left(\frac{1}{3}\right) \\ &\quad < \gamma_n\left(\frac{1}{3}\right) - \frac{286{,}097{,}598 + 613{,}026{,}483n + 247{,}327{,}710n^2 + 21{,}061{,}880n^3}{3^n 40(1+n)^4(39{,}224{,}704 + 13{,}682{,}753n + 1{,}053{,}094n^2)} \\ &\qquad + \frac{3}{2}\sum_{k=n+1}^{\infty}\frac{1}{3^k k^2}, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \Psi_1(3;n) &= 10{,}676{,}544{,}766{,}567 + 11{,}386{,}057{,}364{,}010n \\ &\quad + 2{,}742{,}044{,}504{,}220n^2 + 169{,}113{,}932{,}400n^3, \end{aligned}$$

$$\Psi_2(3;n) = 1{,}261{,}190{,}652{,}791 + 833{,}073{,}379{,}047n + 156{,}127{,}542{,}606n^2 + 8{,}455{,}696{,}620n^3.$$

Proof If we take into account that $\frac{b_3(n+1)}{3^n}$ and $\frac{b_4(n+1)}{3^n}$, combining equations (3.2) and (3.3), the conclusion follows.

This is the end of Theorem 2. □

From (3.4) we can provide another result, which has a simpler form than (3.4), although it is weaker than (3.4).

Corollary 3 *For every positive integer $n \geq 1$, we have*

$$\gamma_n\left(\frac{1}{3}\right) - \frac{1}{3^n 4(n+1)^3} + \frac{1}{23^{n-1}(n+1)(3n+4)} < \gamma\left(\frac{1}{3}\right). \tag{3.5}$$

Proof We use the bounds

$$b_4(n+1) + \frac{1}{(n+1)^3} = \frac{\Psi_1(3;x)}{40((1+n)^3)\Psi_2(3;x)} > 0$$

for $n \geq 1$, where

$$\Psi_1(3;x) = 39{,}771{,}081{,}345{,}073 + 21{,}936{,}877{,}797{,}870n + 3{,}503{,}057{,}200{,}020n^2 + 169{,}113{,}932{,}400n^3,$$

$$\Psi_2(3;x) = 1{,}261{,}190{,}652{,}791 + 833{,}073{,}379{,}047n + 156{,}127{,}542{,}606n^2 + 8{,}455{,}696{,}620n^3,$$

and the telescoping inequalities

$$\frac{1}{3^{n-1}(3n^2+n)} - \frac{1}{3^n(3(n+1)^2)+n+1} < \frac{1}{3^n n^2} < \frac{1}{3^{n-1}(2n^2+n)} - \frac{1}{3^n(2(n+1)^2)+n+1}.$$

Combining Theorem 2, the conclusion follows.

This is the end of Corollary 3. □

Remark 3 It is worth to point out that the multiple-correction method provides a general way to find some continued fraction approximation of σ_t for $t > 3$. Similarly, repeating the above approach step by step, we can get more sharp inequalities. But this maybe brings about some computation increase, the details omitted here.

Competing interests
The authors declared that they have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

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