Ding *Journal of Inequalities and Applications* (2016) 2016:86
DOI 10.1186/s13660-016-1029-9

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Blow-up and global existence for nonlinear reaction-diffusion equations under Neumann boundary conditions

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Abstract

In this paper, we study the blow-up and global solutions of the following nonlinear reaction-diffusion equations under Neumann boundary conditions:

$$\begin{cases} (g(u))_t = \nabla \cdot (a(u)b(x)\nabla u) + f(x,u) & \text{in } D \times (0,T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0,T), \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases}$$

where $D \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with smooth boundary ∂D. By constructing auxiliary functions and using maximum principles and a first-order differential inequality technique, sufficient conditions for the existence of the blow-up solution, an upper bound for the 'blow-up time', an upper estimate of the 'blow-up rate', sufficient conditions for the existence of global solution, and an upper estimate of the global solution are specified under some appropriate assumptions on the functions a, b, f, g, and initial value u_0.

MSC: 35K55; 35B05; 35K57

Keywords: blow-up; global existence; reaction-diffusion equation

1 Introduction

In this paper, we study the blow-up and global solutions for the following nonlinear reaction-diffusion equations under Neumann boundary conditions:

$$\begin{cases} (g(u))_t = \nabla \cdot (a(u)b(x)\nabla u) + f(x,u) & \text{in } D \times (0,T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0,T), \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases} \tag{1.1}$$

where $D \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with smooth boundary ∂D, $\partial/\partial n$ represents the outward normal derivative on ∂D, u_0 is the initial value, T is the maximal existence time of u, and $\overline{D}$ is the closure of D. In order to study the blow-up problem of (1.1) by using maximum principles, we make the following assumptions about the functions a, b, f, g, and u_0. Set $\mathbb{R}^+ := (0, +\infty)$. Throughout the paper, we assume that $a(s)$ is a positive $C^2(\mathbb{R}^+)$ function, $b(x)$ is a positive $C^1(\overline{D})$ function, $f(x,s)$ is a nonnegative $C^1(D \times \mathbb{R}^+)$ function,

$g(s)$ is a $C^3(\mathbb{R}^+)$ function, $g'(s) > 0$ for any $s \in \mathbb{R}^+$, and $u_0(x)$ is a positive $C^2(\overline{D})$ function. Under these assumptions, the classical parabolic equation theory ensures that there exists a unique classical solution $u(x,t)$ for problem (1.1) with some $T > 0$ and the solution is positive over $\overline{D} \times [0,T)$. Moreover, by regularity theorem [1], $u \in C^3(D \times (0,T)) \cap C^2(\overline{D} \times [0,T))$.

During the past decades, the problems of the blow-up and global solutions for nonlinear reaction-diffusion equations have received considerable attention. The contributions in the filed can be found in [2–8] and the references therein. Many authors discussed the blow-up and global solutions for nonlinear reaction-diffusion equations under Neumann boundary conditions and obtained a lot of interesting results; we refer the reader to [9–19]. Some particular cases of (1.1) have been investigated already. Lair and Oxley [20] studied the following problem:

$$\begin{cases} u_t = \nabla \cdot (a(u)\nabla u) + f(u) & \text{in } D \times (0,T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0,T), \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases} \tag{1.2}$$

where D is a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary ∂D. Necessary and sufficient conditions characterized by functions a and f were given for the existence of blow-up and global solutions. Zhang [21] discussed the following problem:

$$\begin{cases} (g(u))_t = \Delta u + f(u) & \text{in } D \times (0,T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0,T), \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases} \tag{1.3}$$

where D is a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary ∂D. Sufficient conditions were developed there for the existence of blow-up and global solutions. Ding and Guo [22] considered the following problem:

$$\begin{cases} (g(u))_t = \nabla \cdot (a(u)\nabla u)\Delta u + f(u) & \text{in } D \times (0,T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0,T), \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases} \tag{1.4}$$

where D is a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary ∂D. Sufficient conditions were given there for the existence of blow-up and global solutions. Meanwhile, an upper bound of the 'blow-up time', an upper estimate of 'blow-up rate', and an upper estimate of the global solution were also obtained.

The object of this paper is the blow-up and global solutions for problem (1.1). Since the reaction function $f(x,u)$ contains not only the concentration variable u but also the space variable x, it seems that the methods of [20–22] are not applicable to problem (1.1). In this paper, we investigate problem (1.1) by constructing auxiliary functions completely different from those in [20–22] and technically using maximum principles and a first-order differential inequality technique. We obtain some existence theorems for a blow-up solution, an upper bound of 'blow-up time', an upper estimate of 'blow-up rate', existence theorems for a global solution, and an upper estimate of the global solution. Our results can be considered as extensions and supplements of those obtained in [20–22].

We proceed as follows. In Section 2 we study the blow-up solution of problem (1.1). Section 3 is devoted to the global solution of (1.1). A few examples are presented in Section 4 to illustrate the applications of the abstract results.

2 Blow-up solution

Our main result for the blow-up solution is stated in the following theorem.

Theorem 2.1 *Let u be a solution of problem* (1.1). *Assume that the following conditions* (i)-(iv) *are satisfied*:

(i) *for any* $s \in \mathbb{R}^+$,

$$\left(\frac{a(s)}{g'(s)}\right)' \geq 0, \quad \left[\frac{1}{a(s)}\left(\frac{a(s)}{g'(s)}\right)' + \frac{1}{g'(s)}\right]' + \left[\frac{1}{a(s)}\left(\frac{a(s)}{g'(s)}\right)' + \frac{1}{g'(s)}\right] \geq 0; \tag{2.1}$$

(ii) *for any* $(x,s) \in D \times \mathbb{R}^+$,

$$\left(\frac{f(x,s)g'(s)}{a(s)}\right)_s - \frac{f(x,s)g'(s)}{a(s)} \geq 0; \tag{2.2}$$

(iii)

$$\int_{M_0}^{+\infty} \frac{g'(s)}{\mathrm{e}^s}\,\mathrm{d}s < +\infty, \qquad M_0 := \max_{\overline{D}} u_0(x); \tag{2.3}$$

(iv)

$$\beta := \min_{\overline{D}} \frac{\nabla \cdot (a(u_0)b(x)\nabla u_0) + f(x,u_0)}{\mathrm{e}^{u_0}} > 0. \tag{2.4}$$

Then the solution u to problem (1.1) *must blow up in a finite T, and*

$$T \leq \frac{1}{\beta}\int_{M_0}^{+\infty} \frac{g'(s)}{\mathrm{e}^s}\,\mathrm{d}s, \tag{2.5}$$

$$u(x,t) \leq H^{-1}\big(\beta(T-t)\big), \quad \forall (x,t) \in \overline{D} \times [0,T), \tag{2.6}$$

where

$$H(z) := \int_z^{+\infty} \frac{g'(s)}{\mathrm{e}^s}\,\mathrm{d}s, \quad z > 0, \tag{2.7}$$

and H^{-1} is the inverse function of H.

Proof Consider the auxiliary function

$$\Psi(x,t) := g'(u)u_t - \beta\mathrm{e}^u. \tag{2.8}$$

For brevity of notation, we write g in place of $g(u)$, suppressing the symbol u. We find that

$$\nabla\Psi = g''u_t\nabla u + g'\nabla u_t - \beta\mathrm{e}^u\nabla u, \tag{2.9}$$

$$\Delta\Psi = g'''u_t|\nabla u|^2 + 2g''\nabla u \cdot \nabla u_t + g''u_t\Delta u + g'\Delta u_t - \beta\mathrm{e}^u|\nabla u|^2 - \beta\mathrm{e}^u\Delta u, \tag{2.10}$$

and

$$\begin{aligned}\Psi_t &= g''(u_t)^2 + g'(u_t)_t - \beta e^u u_t \\ &= g''(u_t)^2 + g'\left(\frac{ab}{g'}\Delta u + \frac{a'b}{g'}|\nabla u|^2 + \frac{a}{g'}\nabla b\cdot\nabla u + \frac{f}{g'}\right)_t - \beta e^u u_t \\ &= g''(u_t)^2 + \left(a'b - \frac{abg''}{g'}\right)u_t\Delta u + ab\Delta u_t + \left(a''b - \frac{a'bg''}{g'}\right)u_t|\nabla u|^2 \\ &\quad + 2a'b(\nabla u\cdot\nabla u_t) + \left(a' - \frac{ag''}{g'}\right)u_t(\nabla b\cdot\nabla u) + a(\nabla b\cdot\nabla u_t) \\ &\quad + \left(f_u - \frac{fg''}{g'} - \beta e^u\right)u_t.\end{aligned} \tag{2.11}$$

It follows from (2.10) and (2.11) that

$$\begin{aligned}\frac{ab}{g'}\Delta\Psi - \Psi_t &= \left(\frac{abg'''}{g'} + \frac{a'bg''}{g'} - a''b\right)u_t|\nabla u|^2 + \left(2\frac{abg''}{g'} - 2a'b\right)(\nabla u\cdot\nabla u_t) \\ &\quad + \left(2\frac{abg''}{g'} - a'b\right)u_t\Delta u - \beta\frac{abe^u}{g'}|\nabla u|^2 - \beta\frac{abe^u}{g'}\Delta u - g''(u_t)^2 \\ &\quad + \left(\frac{ag''}{g'} - a'\right)u_t(\nabla b\cdot\nabla u) - a(\nabla b\cdot\nabla u_t) + \left(\frac{fg''}{g'} - f_u + \beta e^u\right)u_t.\end{aligned} \tag{2.12}$$

By (1.1) we have

$$\Delta u = \frac{g'}{ab}u_t - \frac{a'}{a}|\nabla u|^2 - \frac{1}{b}(\nabla b\cdot\nabla u) - \frac{f}{ab}. \tag{2.13}$$

Substituting (2.13) into (2.12), we get

$$\begin{aligned}\frac{ab}{g'}\Delta\Psi - \Psi_t &= \left(\frac{abg'''}{g'} - \frac{a'bg''}{g'} - a''b + \frac{(a')^2b}{a}\right)u_t|\nabla u|^2 + \left(2\frac{abg''}{g'} - 2a'b\right)(\nabla u\cdot\nabla u_t) \\ &\quad - \frac{(g')^2}{a}\left(\frac{a}{g'}\right)'(u_t)^2 - \frac{ag''}{g'}u_t(\nabla b\cdot\nabla u) + \left(\frac{a'f}{a} - \frac{fg''}{g'} - f_u\right)u_t \\ &\quad + \left(\beta\frac{a'be^u}{g'} - \beta\frac{abe^u}{g'}\right)|\nabla u|^2 + \beta\frac{ae^u}{g'}(\nabla b\cdot\nabla u) \\ &\quad + \beta\frac{fe^u}{g'} - a(\nabla b\cdot\nabla u_t).\end{aligned} \tag{2.14}$$

In view of (2.9), we have

$$\nabla u_t = \frac{1}{g'}\nabla\Psi - \frac{g''}{g'}u_t\nabla u + \beta\frac{e^u}{g'}\nabla u. \tag{2.15}$$

Substitution of (2.15) into (2.14) results in

$$\begin{aligned}&\frac{ab}{g'}\Delta\Psi + \left[2b\left(\frac{a}{g'}\right)'\nabla u + \frac{a}{g'}\nabla b\right]\cdot\nabla\Psi - \Psi_t \\ &\quad = \left(\frac{abg'''}{g'} + \frac{a'bg''}{g'} - a''b + \frac{(a')^2b}{a} - 2\frac{ab(g'')^2}{(g')^2}\right)u_t|\nabla u|^2\end{aligned}$$

$$+\left(2\beta\frac{abg''\mathrm{e}^u}{(g')^2}-\beta\frac{a'b\mathrm{e}^u}{g'}-\beta\frac{ab\mathrm{e}^u}{g'}\right)|\nabla u|^2-\frac{(g')^2}{a}\left(\frac{a}{g'}\right)'(u_t)^2$$
$$+\left(\frac{a'f}{a}-\frac{fg''}{g'}-f_u\right)u_t+\beta\frac{f\mathrm{e}^u}{g'}. \tag{2.16}$$

With (2.8), we have

$$u_t=\frac{1}{g'}\Psi+\beta\frac{\mathrm{e}^u}{g'}. \tag{2.17}$$

Substituting (2.17) into (2.16), we obtain

$$\frac{ab}{g'}\Delta\Psi+\left[2b\left(\frac{a}{g'}\right)'\nabla u+\frac{a}{g'}\nabla b\right]\cdot\nabla\Psi$$
$$+\left\{ab\left[\frac{1}{a}\left(\frac{a}{g'}\right)'\right]'|\nabla u|^2+\frac{a}{(g')^2}\left(\frac{fg'}{a}\right)_u\right\}\Psi-\Psi_t$$
$$=-\beta ab\mathrm{e}^u\left\{\left[\frac{1}{a}\left(\frac{a}{g'}\right)'+\frac{1}{g'}\right]'+\left[\frac{1}{a}\left(\frac{a}{g'}\right)'+\frac{1}{g'}\right]\right\}|\nabla u|^2-\frac{(g')^2}{a}\left(\frac{a}{g'}\right)'(u_t)^2$$
$$-\beta\frac{a\mathrm{e}^u}{(g')^2}\left[\left(\frac{fg'}{a}\right)_u-\frac{fg'}{a}\right]. \tag{2.18}$$

By assumptions (2.1) and (2.2) the right-hand side of (2.18) is nonpositive, that is,

$$\frac{ab}{g'}\Delta\Psi+\left[2b\left(\frac{a}{g'}\right)'\nabla u+\frac{a}{g'}\nabla b\right]\cdot\nabla\Psi$$
$$+\left\{ab\left[\frac{1}{a}\left(\frac{a}{g'}\right)'\right]'|\nabla u|^2+\frac{a}{(g')^2}\left(\frac{fg'}{a}\right)_u\right\}\Psi-\Psi_t\le 0\quad\text{in } D\times(0,T). \tag{2.19}$$

Now by (2.4) we have

$$\begin{aligned}\min_{\overline{D}}\Psi(x,0)&=\min_{\overline{D}}\left\{g'(u_0)(u_0)_t-\beta\mathrm{e}^{u_0}\right\}\\&=\min_{\overline{D}}\left\{\nabla\cdot\left(a(u_0)b(x)\nabla u_0\right)+f(x,u_0)-\beta\mathrm{e}^{u_0}\right\}\\&=\min_{\overline{D}}\left\{\mathrm{e}^{u_0}\left[\frac{\nabla\cdot(a(u_0)b(x)\nabla u_0)+f(x,u_0)}{\mathrm{e}^{u_0}}-\beta\right]\right\}=0.\end{aligned} \tag{2.20}$$

It follows from (1.1) that

$$\frac{\partial\Psi}{\partial n}=g''u_t\frac{\partial u}{\partial n}+g'\frac{\partial u_t}{\partial n}-\beta\mathrm{e}^u\frac{\partial u}{\partial n}=g'\left(\frac{\partial u}{\partial n}\right)_t=0\quad\text{on }\partial D\times(0,T). \tag{2.21}$$

The assumptions concerning the functions a, b, f, g, and u_0 in Section 1 imply that we can use maximum principles to (2.19)-(2.21). Combining (2.19)-(2.21) and applying maximum principles [23], it follows that the minimum of Ψ in $\overline{D}\times[0,T)$ is zero. Thus, we have

$$\Psi\ge 0\quad\text{in }\overline{D}\times[0,T),$$

that is, the differential inequality

$$\frac{g'(u)}{\mathrm{e}^u}u_t\ge\beta. \tag{2.22}$$

Suppose that $x_0 \in \overline{D}$ and $u_0(x_0) = M_0$. At the x_0, integrate (2.22) over $[0, t]$ to get

$$\int_0^t \frac{g'(u)}{\mathrm{e}^u} u_t \,\mathrm{d}t = \int_{M_0}^{u(x_0,t)} \frac{g'(s)}{\mathrm{e}^s} \,\mathrm{d}s \geq \beta t, \tag{2.23}$$

which implies that u must blow up in finite time. Actually, if u is a global solution of (1.1), then for any $t > 0$, it follows from (2.23) that

$$\int_{M_0}^{+\infty} \frac{g'(s)}{\mathrm{e}^s} \,\mathrm{d}s \geq \int_{M_0}^{u(x_0,t)} \frac{g'(s)}{\mathrm{e}^s} \,\mathrm{d}s \geq \beta t. \tag{2.24}$$

Letting $t \to +\infty$ in (2.24) yields

$$\int_{M_0}^{+\infty} \frac{g'(s)}{\mathrm{e}^s} \,\mathrm{d}s = +\infty,$$

which contradicts with assumption (2.3). This shows that u must blow up in a finite time $t = T$. Furthermore, letting $t \to T$ in (2.23), we have

$$T \leq \frac{1}{\beta} \int_{M_0}^{+\infty} \frac{g'(s)}{\mathrm{e}^s} \,\mathrm{d}s.$$

Integrating inequality (2.22) over $[t, s]$ $(0 < t < s < T)$ yields, for each fixed x, that

$$\begin{aligned} H\bigl(u(x,t)\bigr) \geq H\bigl(u(x,t)\bigr) - H\bigl(u(x,s)\bigr) &= \int_{u(x,t)}^{+\infty} \frac{g'(s)}{\mathrm{e}^s} \,\mathrm{d}s - \int_{u(x,s)}^{+\infty} \frac{g'(s)}{\mathrm{e}^s} \,\mathrm{d}s \\ &= \int_{u(x,t)}^{u(x,s)} \frac{g'(s)}{\mathrm{e}^s} \,\mathrm{d}s = \int_t^s \frac{g'(u)}{\mathrm{e}^u} u_t \,\mathrm{d}t \geq \beta(s-t). \end{aligned}$$

Passing to the limit as $s \to T^-$ gives

$$H\bigl(u(x,t)\bigr) \geq \beta(T-t).$$

Since H is a decreasing function, we have

$$u(x,t) \leq H^{-1}\bigl(\beta(T-t)\bigr).$$

The proof is complete. □

3 Global solution

The following theorem is the main result for the global solution.

Theorem 3.1 *Let u be a solution of problem* (1.1). *Assume that the following conditions* (i)-(iv) *are satisfied*:

(i) *for any $s \in \mathbb{R}^+$*,

$$\left(\frac{a(s)}{g'(s)}\right)' \leq 0, \quad \left[\frac{1}{a(s)}\left(\frac{a(s)}{g'(s)}\right)' - \frac{1}{g'(s)}\right]' - \left[\frac{1}{a(s)}\left(\frac{a(s)}{g'(s)}\right)' - \frac{1}{g'(s)}\right] \leq 0; \tag{3.1}$$

(ii) *for any* $(x,s) \in D \times \mathbb{R}^+$,

$$\left(\frac{f(x,s)g'(s)}{a(s)}\right)_s + \frac{f(x,s)g'(s)}{a(s)} \leq 0; \tag{3.2}$$

(iii)

$$\int_{m_0}^{+\infty} \frac{g'(s)}{\mathrm{e}^{-s}}\,\mathrm{d}s = +\infty, \qquad m_0 := \min_{\overline{D}} u_0(x); \tag{3.3}$$

(iv)

$$\alpha := \max_{\overline{D}} \frac{\nabla \cdot (a(u_0)b(x)\nabla u_0) + f(x,u_0)}{\mathrm{e}^{-u_0}} > 0. \tag{3.4}$$

Then the solution u of (1.1) *must be a global solution, and*

$$u(x,t) \leq G^{-1}\big(\alpha t + G\big(u_0(x,t)\big)\big), \quad \forall (x,t) \in \overline{D} \times \overline{\mathbb{R}}^+, \tag{3.5}$$

where

$$G(z) := \int_{m_0}^{z} \frac{g'(s)}{\mathrm{e}^{-s}}\,\mathrm{d}s, \quad z \geq m_0, \tag{3.6}$$

and G^{-1} *is the inverse function of* G.

Proof Construct the auxiliary function

$$\Phi(x,t) := g'(u)u_t - \alpha \mathrm{e}^{-u}. \tag{3.7}$$

By using the same reasoning process with that of (2.9)-(2.18), we have

$$\begin{aligned}
&\frac{ab}{g'}\Delta\Phi + \left[2b\left(\frac{a}{g'}\right)'\nabla u + \frac{a}{g'}\nabla b\right]\cdot\nabla\Phi \\
&\quad + \left\{ab\left[\frac{1}{a}\left(\frac{a}{g'}\right)'\right]'|\nabla u|^2 - \frac{a}{(g')^2}\left(\frac{fg'}{a}\right)_u\right\}\Phi - \Phi_t \\
&= -\alpha ab\mathrm{e}^{-u}\left\{\left[\frac{1}{a}\left(\frac{a}{g'}\right)' - \frac{1}{g'}\right]' - \left[\frac{1}{a}\left(\frac{a}{g'}\right)' - \frac{1}{g'}\right]\right\}|\nabla u|^2 - \frac{(g')^2}{a}\left(\frac{a}{g'}\right)'(u_t)^2 \\
&\quad - \alpha\frac{a\mathrm{e}^{-u}}{(g')^2}\left[\left(\frac{fg'}{a}\right)_u + \frac{fg'}{a}\right].
\end{aligned} \tag{3.8}$$

From assumptions (3.1) and (3.2) we see that the right-hand side of (3.8) is nonnegative, that is,

$$\begin{aligned}
&\frac{ab}{g'}\Delta\Phi + \left[2b\left(\frac{a}{g'}\right)'\nabla u + \frac{a}{g'}\nabla b\right]\cdot\nabla\Phi \\
&\quad + \left\{ab\left[\frac{1}{a}\left(\frac{a}{g'}\right)'\right]'|\nabla u|^2 - \frac{a}{(g')^2}\left(\frac{fg'}{a}\right)_u\right\}\Phi - \Phi_t \geq 0 \quad \text{in } D \times (0,T).
\end{aligned} \tag{3.9}$$

By (3.4) we have

$$\begin{aligned}\max_{\overline{D}} \Phi(x,0) &= \max_{\overline{D}}\left\{g'(u_0)(u_0)_t - \alpha e^{-u_0}\right\} \\ &= \max_{\overline{D}}\left\{\nabla\cdot\big(a(u_0)b(x)\nabla u_0\big) + f(x,u_0) - \alpha e^{-u_0}\right\} \\ &= \max_{\overline{D}}\left\{e^{-u_0}\left[\frac{\nabla\cdot(a(u_0)b(x)\nabla u_0) + f(x,u_0)}{e^{-u_0}} - \alpha\right]\right\} = 0. \end{aligned} \tag{3.10}$$

Repeating the arguments for (2.21), we have

$$\frac{\partial\Phi}{\partial n} = 0 \quad \text{on } \partial D \times (0,T). \tag{3.11}$$

Combining (3.9)-(3.11) and applying the maximum principles again, we get that the maximum of Φ in $\overline{D}\times[0,T)$ is zero. Hence, we have

$$\Phi \le 0 \quad \text{in } \overline{D}\times[0,T),$$

that is, the differential inequality

$$\frac{g'(u)}{e^{-u}} u_t \le \alpha. \tag{3.12}$$

For each fixed $x \in \overline{D}$, integrate (3.12) over $[0,t]$ to produce

$$\int_0^t \frac{g'(u)}{e^{-u}} u_t \,dt = \int_{u_0(x)}^{u(x,t)} \frac{g'(s)}{e^{-s}}\,ds \le \alpha t, \tag{3.13}$$

which shows that u must be a global solution. In fact, suppose that u blows up at finite time T, that is,

$$\lim_{t\to T^-} u(x,t) = +\infty.$$

Passing to the limit as $t \to T^-$ in (3.13) gives

$$\int_{u_0(x)}^{+\infty} \frac{g'(s)}{e^{-s}}\,ds \le \alpha T$$

and

$$\int_{m_0}^{+\infty} \frac{g'(s)}{e^{-s}}\,ds = \int_{m_0}^{u_0(x)} \frac{g'(s)}{e^{-s}}\,ds + \int_{u_0(x)}^{+\infty} \frac{g'(s)}{e^{-s}}\,ds \le \int_{m_0}^{u_0(x)} \frac{g'(s)}{e^{-s}}\,ds + \alpha T < +\infty,$$

which is a contradiction. This shows that u is global. Moreover, (3.13) implies that

$$\int_{u_0(x)}^{u(x,t)} \frac{g'(s)}{e^{-s}}\,ds = \int_{m_0}^{u(x,t)} \frac{g'(s)}{e^{-s}}\,ds - \int_{m_0}^{u_0(x)} \frac{g'(s)}{e^{-s}}\,ds = G\big(u(x,t)\big) - G\big(u_0(x)\big) \le \alpha t.$$

Since G is an increasing function, we have

$$u(x,t) \le G^{-1}\big(\alpha t + G\big(u_0(x)\big)\big).$$

The proof is complete. □

4 Applications

When $g(u) \equiv u$, $b(x) \equiv 1$, and $f(x,u) \equiv f(u)$, problem (1.1) is problem (1.2) studied by Lair and Oxley [20]. When $a(u) \equiv 1$, $b(x) \equiv 1$, and $f(x,u) \equiv f(u)$, problem (1.1) is problem (1.3) discussed by Zhang [21]. When $b(x) \equiv 1$ and $f(x,u) \equiv f(u)$, problem (1.1) is problem (1.4) considered by Ding and Guo [22]. In these three cases, the conclusions of Theorems 2.1 and 3.1 still hold. In this sense, our results extend and supplement the results of [20–22].

In what follows, we present several examples to demonstrate applications of Theorems 2.1 and 3.1.

Example 4.1 Let u be a solution of the following problem:

$$\begin{cases} (2\mathrm{e}^{\frac{u}{2}} + u)_t = \nabla \cdot ((1 + \mathrm{e}^{\frac{u}{2}})(1 + \|x\|^2)\nabla u) + 7\mathrm{e}^u - \|x\|^2 & \text{in } D \times (0,T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0,T), \\ u(x,0) = 1 + (1 - \|x\|^2)^2 & \text{in } \overline{D}, \end{cases}$$

where $D = \{x = (x_1, x_2, x_3) \mid \|x\|^2 = x_1^2 + x_2^2 + x_3^2 < 1\}$ is the unit ball of $\mathbb{R}^3$. Now we have

$$g(u) = 2\mathrm{e}^{\frac{u}{2}} + u, \qquad a(u) = 1 + \mathrm{e}^{\frac{u}{2}}, \qquad b(x) = 1 + \|x\|^2,$$
$$f(x,u) = 7\mathrm{e}^u - \|x\|^2, \qquad u_0(x) = 1 + \left(1 - \|x\|^2\right)^2.$$

In order to determine the constant β, we assume that

$$s = \|x\|^2.$$

Then $0 \le s \le 1$ and

$$\begin{aligned} \beta &= \min_{\overline{D}} \frac{\nabla \cdot (a(u_0)b(x)\nabla u_0) + f(x,u_0)}{\mathrm{e}^{u_0}} \\ &= \min_{\overline{D}} \Big\{ \big(\mathrm{e}^{-1-(1-\|x\|^2)^2} + \mathrm{e}^{-\frac{1}{2}-\frac{1}{2}(1-\|x\|^2)^2}\big)\big(-12 + 28\|x\|^2\big) \\ &\quad + 8\mathrm{e}^{-\frac{1}{2}-\frac{1}{2}(1-\|x\|^2)^2}\|x\|^2\big(1 + \|x\|^2\big)\big(1 - \|x\|^2\big)^2 + 7 - \|x\|^2\mathrm{e}^{-1-(1-\|x\|^2)^2} \Big\} \\ &= \min_{0 \le s \le 1} \Big\{ \big(\mathrm{e}^{-1-(1-s)^2} + \mathrm{e}^{-\frac{1}{2}-\frac{1}{2}(1-s)^2}\big)(-12 + 28s) \\ &\quad + 8\mathrm{e}^{-\frac{1}{2}-\frac{1}{2}(1-s)^2}s(1+s)(1-s)^2 + 7 - s\mathrm{e}^{-1-(1-s)^2} \Big\} \\ &= 0.9614. \end{aligned}$$

It is easy to check that (2.1)-(2.3) hold. By Theorem 2.1, u must blow up in a finite time T, and

$$T \le \frac{1}{\beta}\int_{M_0}^{+\infty} \frac{g'(s)}{\mathrm{e}^s}\,\mathrm{d}s = \frac{1}{0.9614}\int_1^{+\infty} \frac{\mathrm{e}^{\frac{s}{2}} + 1}{\mathrm{e}^s}\,\mathrm{d}s = 1.4025,$$
$$u(x,t) \le H^{-1}\big(\beta(T-t)\big) = \ln \frac{1}{(\sqrt{1 + 0.9614(T-t)} - 1)^2}.$$

Example 4.2 Let u be a solution of the following problem:

$$\begin{cases} (\ln(e^u - 1) - u)_t = \nabla \cdot (\frac{1}{e^u - 1}(1 + \|x\|^2)\nabla u) + e^{-u}(1 + \|x\|^2) & \text{in } D \times (0, T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0, T), \\ u(x, 0) = 1 + (1 - \|x\|^2)^2 & \text{in } \overline{D}, \end{cases}$$

where $D = \{x = (x_1, x_2, x_3) \mid \|x\|^2 = x_2^2 + x_2^2 + x_3^2 < 1\}$ is the unit ball of $\mathbb{R}^3$. Now we have

$$g(u) = \ln\bigl(e^u - 1\bigr) - u, \qquad a(u) = \frac{1}{e^u - 1}, \qquad b(x) = 1 + \|x\|^2,$$
$$f(x, u) = e^{-u}\bigl(1 + \|x\|^2\bigr), \qquad u_0(x) = 1 + \bigl(1 - \|x\|^2\bigr)^2.$$

By setting

$$s = \|x\|^2,$$

we have $0 \le s \le 1$ and

$$\begin{aligned} \alpha &= \min_{\overline{D}} \frac{\nabla \cdot (a(u_0)b(x)\nabla u_0) + f(x, u_0)}{e^{-u_0}} \\ &= \min_{\overline{D}} \biggl\{ \frac{1}{(e^{1+(1-\|x\|^2)^2} - 1)^2} \bigl[\bigl(-12 + 28\|x\|^2\bigr)e^{1+(1-\|x\|^2)^2}\bigl(e^{1+(1-\|x\|^2)^2} - 1\bigr) \\ &\quad - 16\|x\|^2\bigl(1 + \|x\|^2\bigr)\bigl(1 - \|x\|^2\bigr)^2 e^{2+2(1-\|x\|^2)^2} + \bigl(1 + \|x\|^2\bigr)\bigl(e^{1+(1-\|x\|^2)^2} - 1\bigr)^2\bigr] \biggr\} \\ &= \min_{0 \le s \le 1} \biggl\{ \frac{1}{(e^{1+(1-s)^2} - 1)^2} \bigl[(-12 + 28s)e^{1+(1-s)^2}\bigl(e^{1+(1-s)^2} - 1\bigr) \\ &\quad - 16s(1 + s)(1 - s)^2 e^{2+2(1-s)^2} + (1 + s)\bigl(e^{1+(1-s)^2} - 1\bigr)^2\bigr] \biggr\} \\ &= 27.3116. \end{aligned}$$

Again, it is easy to check that (3.1)-(3.3) hold. By Theorem 3.1, u must be a global solution, and

$$u(x, t) \le G^{-1}\bigl(\alpha t + G\bigl(u_0(x)\bigr)\bigr) = \ln\bigl[1 + e^{27.3116t}\bigl(e^{1+(1-\|x\|^2)^2} - 1\bigr)\bigr].$$

Competing interests
The author declares that he has no competing interests.

Acknowledgements
This work was supported by the National Natural Science Foundation of China (Nos. 61473180 and 61174082).

Received: 13 June 2015 Accepted: 19 February 2016 Published online: 03 March 2016

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