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Integral inequalities via fractional quantum calculus

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Abstract

In this paper we prove several fractional quantum integral inequalities for the new q-shifting operator ${}_a\Phi_q(m) = qm + (1-q)a$ introduced in Tariboon *et al.* (Adv. Differ. Equ. 2015:18, 2015), such as: the q-Hölder inequality, the q-Hermite-Hadamard inequality, the q-Cauchy-Bunyakovsky-Schawrz integral inequality, the q-Grüss integral inequality, the q-Grüss-Čebyšev integral inequality, and the q-Pólya-Szegö integral inequality.

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1 Introduction

The quantum calculus is known as the calculus without limits. It substitutes the classical derivative by a difference operator, which allows one to deal with sets of nondifferentiable functions. Quantum difference operators have an interesting role due to their applications in several mathematical areas, such as orthogonal polynomials, basic hypergeometric functions, combinatorics, the calculus of variations, mechanics, and the theory of relativity. The book by Kac and Cheung [2] covers many of the fundamental aspects of quantum calculus.

In recent years, the topic of q-calculus has attracted the attention of several researchers and a variety of new results can be found in the papers [3–15] and the references cited therein.

In [16] the notions of q_k-derivative and q_k-integral of a continuous function $f : [t_k, t_{k+1}] \to \mathbb{R}$, have been introduced and their basic properties were proved. As applications existence and uniqueness results for initial value problems of first and second order impulsive q_k-difference equations were investigated. The q-calculus analogs of some classical integral inequalities, such as Hölder, Hermite-Hadamard, Trapezoid, Ostrowski, Cauchy-Bunyakovsky-Schwarz, Grüss and Grüss-Čebyšev were established in [17]. For recent results on quantum inequalities, see [18–20].

In [1] new concepts of fractional quantum calculus were defined, by defining a new q-shifting operator ${}_a\Phi_q(m) = qm + (1-q)a$. After giving the basic properties the q-derivative and q-integral were defined. New definitions of the Riemann-Liouville fractional q-integral and the q-difference on an interval $[a, b]$ were given and their basic properties were discussed. As applications of the new concepts, one proved existence and uniqueness results for first and second order initial value problems for impulsive fractional q-difference equations.

In this paper we prove several integral inequalities for the new q-shifting operator ${}_a\Phi_q(m) = qm + (1-q)a$, such as: the q-Hölder inequality, the q-Hermite-Hadamard inequality, the q-Korkine integral equality, the q-Cauchy-Bunyakovsky-Schwarz integral inequality, the q-Grüss integral inequality, the q-Grüss-Čebyšev integral inequality, and the q-Polya-Szegö integral inequality.

2 Preliminaries

To make this paper self-contained, below we recall some well-known facts on fractional q-calculus. The presentation here can be found, for example, in [7, 8].

Let us define a q-shifting operator as

$$ {}_a\Phi_q(m) = qm + (1-q)a, \tag{2.1} $$

where $0 < q < 1$, $m, a \in \mathbb{R}$. For any positive integer k, we have

$$ {}_a\Phi_q^k(m) = {}_a\Phi_q^{k-1}\big({}_a\Phi_q(m)\big) \quad \text{and} \quad {}_a\Phi_q^0(m) = m. \tag{2.2} $$

The following results can be found in [1].

Property 2.1 For any $m, n \in \mathbb{R}$ and for all positive integer k, j, the following properties hold:

(i) ${}_a\Phi_q^k(m) = {}_a\Phi_{q^k}(m)$;
(ii) ${}_a\Phi_q^j({}_a\Phi_q^k(m)) = {}_a\Phi_q^k({}_a\Phi_q^j(m)) = {}_a\Phi_q^{j+k}(m)$;
(iii) ${}_a\Phi_q(a) = a$;
(iv) ${}_a\Phi_q^k(m) - a = q^k(m-a)$;
(v) $m - {}_a\Phi_q^k(m) = (1-q^k)(m-a)$;
(vi) ${}_a\Phi_q^k(m) = m\,{}_{\frac{a}{m}}\Phi_q^k(1)$, for $m \neq 0$;
(vii) ${}_a\Phi_q(m) - {}_a\Phi_q^k(n) = q(m - {}_a\Phi_q^{k-1}(n))$.

The q-analog of the Pochhammer symbol is defined by

$$ (m;q)_0 = 1, \qquad (m;q)_k = \prod_{i=0}^{k-1}\big(1 - q^i m\big), \quad k \in \mathbb{N} \cup \{\infty\}. \tag{2.3} $$

We also define the power of the q-shifting operator as

$$ {}_a(n-m)_q^{(0)} = 1, \qquad {}_a(n-m)_q^{(k)} = \prod_{i=0}^{k-1}\big(n - {}_a\Phi_q^i(m)\big), \quad k \in \mathbb{N} \cup \{\infty\}. \tag{2.4} $$

More generally, if $\gamma \in \mathbb{R}$, then

$$ {}_a(n-m)_q^{(\gamma)} = n^{(\gamma)} \prod_{i=0}^{\infty} \frac{1 - {}_{\frac{a}{n}}\Phi_q^i(m/n)}{1 - {}_{\frac{a}{n}}\Phi_q^{\gamma+i}(m/n)}, \quad n \neq 0. \tag{2.5} $$

From the above definitions, the following results were proved in [1].

Property 2.2 For any $\gamma, m, n \in \mathbb{R}$ with $n \neq a$ and $k \in \mathbb{N} \cup \{\infty\}$, the following properties hold:

(i) ${}_a(n-m)_q^{(k)} = (n-a)^k (\frac{m-a}{n-a}; q)_k$;

(ii) ${}_a(n-m)_q^{(\gamma)} = (n-a)^\gamma \prod_{i=0}^{\infty} \frac{1-\frac{m-a}{n-a}q^i}{1-\frac{m-a}{n-a}q^{\gamma+i}} = (n-a)^\gamma \frac{(\frac{m-a}{n-a};q)_\infty}{(\frac{m-a}{n-a}q^\gamma;q)_\infty}$;

(iii) ${}_a(n - {}_a\Phi_q^k(n))_q^{(\gamma)} = (n-a)^\gamma \frac{(q^k;q)_\infty}{(q^{\gamma+k};q)_\infty}$.

The q-number is defined by

$$[m]_q = \frac{1-q^m}{1-q}, \quad m \in \mathbb{R}. \tag{2.6}$$

If $a = 0$ and $m = n = 1$, then (2.5) is reduced to

$${}_0\big(1 - {}_0\Phi_q(1)\big)_q^{(\gamma)} = \prod_{i=0}^{\infty} \frac{1-q^{i+1}}{1-q^{\gamma+i+1}}. \tag{2.7}$$

The q-gamma function is defined by

$$\Gamma_q(t) = \frac{{}_0(1 - {}_0\Phi_q(1))_q^{(t-1)}}{(1-q)^{t-1}}, \quad t \in \mathbb{R}\backslash\{0,-1,-2,\ldots\}. \tag{2.8}$$

Obviously, $\Gamma_q(t+1) = [t]_q\Gamma_q(t)$. For any $s, t > 0$, the q-beta function is defined by

$$B_q(s,t) = \int_0^1 u^{(s-1)}\big(1 - {}_0\Phi_q(u)\big)^{(t-1)}\, d_qu. \tag{2.9}$$

The q-beta function in terms of the q-gamma function can be written as

$$B_q(s,t) = \frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(s+t)}. \tag{2.10}$$

Let us give the definitions of Riemann-Liouville fractional q-integral and the q-derivative on the dense interval $[a,b]$.

Definition 2.3 Let $\alpha \geq 0$ and f be a continuous function defined on $[a,b]$. The fractional q-integral of Riemann-Liouville type is given by $({}_aI_q^0 f)(t) = f(t)$ and

$$\begin{aligned}\big({}_aI_q^\alpha f\big)(t) &= \frac{1}{\Gamma_q(\alpha)}\int_a^t {}_a\big(t - {}_a\Phi_q(s)\big)_q^{(\alpha-1)} f(s)\,{}_a d_qs \\ &= \frac{(1-q)(t-a)}{\Gamma_q(\alpha)} \sum_{i=0}^{\infty} q^i\,{}_a\big(t - {}_a\Phi_q^{i+1}(t)\big)_q^{(\alpha-1)} f\big({}_a\Phi_q^i(t)\big).\end{aligned}$$

Definition 2.4 The fractional q-derivative of Riemann-Liouville type of order $\alpha \geq 0$ of a continuous function f on the interval $[a,b]$ is defined by $({}_aD_q^0 f)(t) = f(t)$ and

$$\big({}_aD_q^\alpha f\big)(t) = \big({}_aD_q^\upsilon\,{}_aI_q^{\upsilon-\alpha} f\big)(t), \quad \alpha > 0,$$

where υ is the smallest integer greater than or equal to α.

Lemma 2.5 [1] *Let $\alpha,\beta \geq 0$, and f be a continuous function on $[a,b]$. The Riemann-Liouville fractional q-integral has the following semi-group properties*:

$$ {}_aI_q^{\beta}\big({}_aI_q^{\alpha}f\big)(t) = {}_aI_q^{\alpha}\big({}_aI_q^{\beta}f\big)(t) = \big({}_aI_q^{\alpha+\beta}f\big)(t). \tag{2.11}$$

Throughout this paper, in some places, the variable s will be shown inside the fractional integral notation as $({}_aI_q^{\alpha}f(s))(t)$, which means

$$\big({}_aI_q^{\alpha}f(s)\big)(t) = \frac{1}{\Gamma_q(\alpha)}\int_a^t {}_a\big(t - {}_a\Phi_q(s)\big)_q^{(\alpha-1)}f(s)_a\,d_qs.$$

Lemma 2.6 *If $\alpha,\beta \geq 0$, then, for $t \in [a,b]$, the following relation holds*:

$$\big({}_aI_q^{\alpha}(s-a)^{\beta}\big)(t) = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}(t-a)^{\beta+\alpha}. \tag{2.12}$$

Proof From Definition 2.3 and applying Property 2.1(iv), Property 2.2(iii), it follows that

$$\begin{aligned}
\big({}_aI_q^{\alpha}(s-a)^{\beta}\big)(t) &= \frac{1}{\Gamma_q(\alpha)}\int_a^t {}_a\big(t - {}_a\Phi_q(s)\big)_q^{(\alpha-1)}(s-a)^{\beta}{}_a\,d_qs \\
&= \frac{(1-q)(t-a)}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty} q^i{}_a\big(t - {}_a\Phi_q^{i+1}(t)\big)_q^{(\alpha-1)}\big({}_a\Phi_q^{i}(t)-a\big)^{\beta} \\
&= \frac{(1-q)(t-a)}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty} q^i(t-a)^{\alpha-1}\frac{(q^{i+1};q)_\infty}{(q^{\alpha+i};q)_\infty}\big(q^i(t-a)\big)^{\beta} \\
&= \frac{(1-q)(t-a)^{\beta+\alpha}}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty} q^i\frac{(q^{i+1};q)_\infty}{(q^{\alpha+i};q)_\infty}q^{\beta i} \\
&= \frac{(1-q)(t-a)^{\beta+\alpha}}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty} q^i\prod_{i=0}^{\infty}\frac{(1-q^{i+1}q^i)}{(1-q^{i+1}q^{\alpha-1+i})}q^{\beta i} \\
&= \frac{(1-q)(t-a)^{\beta+\alpha}}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty} q^i{}_0\big(1 - {}_0\Phi_q^{i+1}(1)\big)_q^{(\alpha-1)}q^{\beta i} \\
&= \frac{(t-a)^{\beta+\alpha}}{\Gamma_q(\alpha)}(1-q)\sum_{i=0}^{\infty} q^i{}_0\big(1 - {}_0\Phi_q^{i+1}(1)\big)_q^{(\alpha-1)}\big(q^i\big)^{(\beta)} \\
&= \frac{(t-a)^{\beta+\alpha}}{\Gamma_q(\alpha)}\int_0^1 s^{(\beta)}\big(1 - {}_0\Phi_q(s)\big)^{(\alpha-1)}{}_0\,d_qs \\
&= \frac{(t-a)^{\beta+\alpha}}{\Gamma_q(\alpha)}B_q(\beta+1,\alpha) \\
&= \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}(t-a)^{\beta+\alpha},
\end{aligned}$$

which leads to (2.12) as required. □

Corollary 2.7 *Let $f(t) = t$ and $g(t) = t^2$ for $t \in [a,b]$, and $\alpha > 0$. Then we have*

(i) $({}_aI_q^{\alpha}f(s))(t) = \frac{(t-a)^{\alpha}}{\Gamma_q(\alpha+2)}(t + ([\alpha+1]_q - 1)a)$;

(ii) $({}_aI_q^{\alpha}g(s))(t) = \frac{(t-a)^{\alpha}}{\Gamma_q(\alpha+3)}((1+q)(t-a)^2 + 2a(t-a)[\alpha+2]_q + a^2[\alpha+1]_q[\alpha+2]_q)$.

3 Main results

Let us start with the fractional q-Hölder inequality on the interval $[a,b]$.

Theorem 3.1 *Let* $0<q<1$, $\alpha>0$, $p_1,p_2>1$, *such that* $\frac{1}{p_1}+\frac{1}{p_2}=1$. *Then for* $t\in[a,b]$ *we have*

$$\left({}_aI_q^\alpha |f(s)||g(s)|\right)(t)\leq\left(\left({}_aI_q^\alpha |f(s)|^{p_1}\right)(t)\right)^{\frac{1}{p_1}}\left(\left({}_aI_q^\alpha |g(s)|^{p_2}\right)(t)\right)^{\frac{1}{p_2}}. \tag{3.1}$$

Proof From Definition 2.3 and the discrete Hölder inequality, we have

$$\begin{aligned}
&\left({}_aI_q^\alpha |f(s)||g(s)|\right)(t)\\
&=\frac{1}{\Gamma_q(\alpha)}\int_a^t {}_a\left(t-{}_a\Phi_q(s)\right)_q^{(\alpha-1)}|f(s)||g(s)|\,{}_ad_qs\\
&=\frac{(1-q)(t-a)}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty}q^i\,{}_a\left(t-{}_a\Phi_q^{i+1}(t)\right)_q^{(\alpha-1)}\left|f\left({}_a\Phi_q^i(t)\right)\right|\left|g\left({}_a\Phi_q^i(t)\right)\right|\\
&=\frac{(1-q)(t-a)}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty}{}_a\left(t-{}_a\Phi_q^{i+1}(t)\right)_q^{(\alpha-1)}\left(q^i\right)^{\frac{1}{p_1}}\left|f\left({}_a\Phi_q^i(t)\right)\right|\left(q^i\right)^{\frac{1}{p_2}}\left|g\left({}_a\Phi_q^i(t)\right)\right|\\
&\leq\left(\frac{(1-q)(t-a)}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty}q^i\,{}_a\left(t-{}_a\Phi_q^{i+1}(t)\right)_q^{(\alpha-1)}\left|f\left({}_a\Phi_q^i(t)\right)\right|^{p_1}\right)^{\frac{1}{p_1}}\\
&\quad\times\left(\frac{(1-q)(t-a)}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty}q^i\,{}_a\left(t-{}_a\Phi_q^{i+1}(t)\right)_q^{(\alpha-1)}\left|g\left({}_a\Phi_q^i(t)\right)\right|^{p_2}\right)^{\frac{1}{p_2}}\\
&=\left(\frac{1}{\Gamma_q(\alpha)}\int_a^t {}_a\left(t-{}_a\Phi_q(s)\right)_q^{(\alpha-1)}|f(s)|^{p_1}\,{}_ad_qs\right)^{\frac{1}{p_1}}\\
&\quad\times\left(\frac{1}{\Gamma_q(\alpha)}\int_a^t {}_a\left(t-{}_a\Phi_q(s)\right)_q^{(\alpha-1)}|g(s)|^{p_2}\,{}_ad_qs\right)^{\frac{1}{p_2}}\\
&=\left(\left({}_aI_q^\alpha |f(s)|^{p_1}\right)(t)\right)^{\frac{1}{p_1}}\left(\left({}_aI_q^\alpha |g(s)|^{p_2}\right)(t)\right)^{\frac{1}{p_2}}.
\end{aligned}$$

Therefore, inequality (3.1) holds. □

Remark 3.2 If $\alpha=1$ and $a=0$, then (3.1) is reduced to the q-Hölder inequality in [21].

The fractional q-Hermite-Hadamard integral inequality on the interval $[a,b]$ will be proved as follows.

Theorem 3.3 *Let* $f:[a,b]\to\mathbb{R}$ *be a convex continuous function*, $0<q<1$ *and* $\alpha>0$. *Then we have*

$$\begin{aligned}
&\frac{2}{\Gamma_q(\alpha+1)}f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)^\alpha}\left({}_aI_q^\alpha f(a+b-s)\right)(b)\\
&\leq\frac{1}{(b-a)^\alpha}\left({}_aI_q^\alpha f(s)\right)(b)\\
&\leq\frac{1}{\Gamma_q(\alpha+2)}\left(\left([\alpha+1]_q-1\right)f(a)+f(b)\right).
\end{aligned}\tag{3.2}$$

Proof The convexity of f on $[a,b]$ means that

$$f\big((1-s)a+sb\big) \le (1-s)f(a)+sf(b), \quad s\in[0,1]. \tag{3.3}$$

Multiplying both sides of (3.3) by ${}_0(1-{}_0\Phi_q(s))_q^{(\alpha-1)}/\Gamma_q(\alpha)$, $s\in(0,1)$, we get

$$\begin{aligned}&\frac{1}{\Gamma_q(\alpha)}{}_0\big(1-{}_0\Phi_q(s)\big)_q^{(\alpha-1)}f\big((1-s)a+sb\big)\\&\quad\le\frac{f(a)}{\Gamma_q(\alpha)}{}_0\big(1-{}_0\Phi_q(s)\big)_q^{(\alpha-1)}(1-s)+\frac{f(b)}{\Gamma_q(\alpha)}{}_0\big(1-{}_0\Phi_q(s)\big)_q^{(\alpha-1)}s.\end{aligned} \tag{3.4}$$

Taking q-integration of order $\alpha>0$ for (3.4) with respect to s on $[0,1]$, we have

$$\begin{aligned}&\frac{1}{\Gamma_q(\alpha)}\int_0^1{}_0\big(1-{}_0\Phi_q(s)\big)_q^{(\alpha-1)}f\big((1-s)a+sb\big)_0\,d_qs\\&\quad\le\frac{f(a)}{\Gamma_q(\alpha)}\int_0^1{}_0\big(1-{}_0\Phi_q(s)\big)_q^{(\alpha-1)}(1-s)_0\,d_qs+\frac{f(b)}{\Gamma_q(\alpha)}\int_0^1{}_0\big(1-{}_0\Phi_q(s)\big)_q^{(\alpha-1)}s_0\,d_qs,\end{aligned} \tag{3.5}$$

which means that

$$\big({}_0I_q^\alpha f\big((1-s)a+sb\big)\big)(1)\le f(a)\big({}_0I_q^\alpha(1-s)\big)(1)+f(b)\big({}_0I_q^\alpha s\big)(1). \tag{3.6}$$

From Corollary 2.7(i), we have

$$\big({}_0I_q^\alpha s\big)(1)=\frac{1}{\Gamma_q(\alpha+2)}\quad\text{and}\quad\big({}_0I_q^\alpha(1-s)\big)(1)=\frac{1}{\Gamma_q(\alpha+1)}-\frac{1}{\Gamma_q(\alpha+2)}.$$

Using the definition of fractional q-integration on $[a,b]$, we have

$$\begin{aligned}&\big({}_0I_q^\alpha f\big((1-s)a+sb\big)\big)(1)\\&\quad=\frac{1}{\Gamma_q(\alpha)}\int_0^1{}_0\big(1-{}_0\Phi_q(s)\big)_q^{(\alpha-1)}f\big((1-s)a+sb\big)_0\,d_qs\\&\quad=\frac{1-q}{\Gamma_q(\alpha)}\sum_{i=0}^\infty q^i{}_0\big(1-{}_0\Phi_q^{i+1}(1)\big)_q^{(\alpha-1)}f\big(\big(1-q^i\big)a+q^ib\big)\\&\quad=\frac{1-q}{\Gamma_q(\alpha)}\sum_{i=0}^\infty q^i\prod_{i=0}^\infty\frac{1-q^iq^{i+1}}{1-q^iq^{\alpha+i}}f\big({}_a\Phi_q^i(b)\big)\\&\quad=\frac{1-q}{(b-a)^{\alpha-1}\Gamma_q(\alpha)}\sum_{i=0}^\infty q^i(b-a)^{\alpha-1}\frac{(q^{i+1};q)_\infty}{(q^{i+\alpha};q)_\infty}f\big({}_a\Phi_q^i(b)\big)\\&\quad=\frac{1}{(b-a)^\alpha}\bigg(\frac{1}{\Gamma_q(\alpha)}\int_a^b{}_a\big(b-{}_a\Phi_q(s)\big)_q^{(\alpha-1)}f(s)_a\,d_qs\bigg)\\&\quad=\frac{1}{(b-a)^\alpha}\big({}_aI_q^\alpha f\big)(b),\end{aligned}$$

which gives the second part of (3.2) by using (3.6).

To prove the first part of (3.2), we use the convex property of f as follows:

$$\begin{aligned}\frac{1}{2}\big[f\big((1-s)a+sb\big)+f\big(sa+(1-s)b\big)\big] &\geq f\left(\frac{(1-s)a+sb+sa+(1-s)b}{2}\right)\\ &= f\left(\frac{a+b}{2}\right).\end{aligned} \tag{3.7}$$

Multiplying both sides of (3.7) by ${}_0(1-{}_0\Phi_q(s))_q^{(\alpha-1)}/\Gamma_q(\alpha)$, $s\in(0,1)$, we get

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\frac{1}{\Gamma_q(\alpha)}{}_0\big(1-{}_0\Phi_q(s)\big)_q^{(\alpha-1)}\\ &\quad\leq \frac{1}{2\Gamma_q(\alpha)}{}_0\big(1-{}_0\Phi_q(s)\big)_q^{(\alpha-1)}f\big((1-s)a+sb\big)\\ &\qquad+\frac{1}{2\Gamma_q(\alpha)}{}_0\big(1-{}_0\Phi_q(s)\big)_q^{(\alpha-1)}f\big(sa+(1-s)b\big).\end{aligned}$$

Again on fractional q-integration of order $\alpha>0$ to the above inequality with respect to t on $[0,1]$ and changing variables, we get

$$f\left(\frac{a+b}{2}\right)\leq \frac{\Gamma_q(\alpha+1)}{2(b-a)^\alpha}\big({}_aI_q^\alpha f(s)\big)(b)+\frac{\Gamma_q(\alpha+1)}{2(b-a)^\alpha}\big({}_aI_q^\alpha f(a+b-s)\big)(b). \tag{3.8}$$

By a direct computation, we have

$$\begin{aligned}&\big({}_0I_q^\alpha f\big((1-s)b+sa\big)\big)(1)\\ &\quad=\frac{1-q}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty}q^i{}_0\big(1-q^{i+1}\big)_q^{(\alpha-1)}f\big(\big(1-q^i\big)b+q^ia\big)\\ &\quad=\frac{1-q}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty}q^i{}_0\big(1-q^{i+1}\big)_q^{(\alpha-1)}f\big(a+b-{}_a\Phi_q^i(b)\big)\\ &\quad=\frac{1-q}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty}q^i\frac{(q^{i+1};q)_\infty}{(q^{\alpha+i};q)_\infty}f\big(a+b-{}_a\Phi_q^i(b)\big)\\ &\quad=\frac{1}{(b-a)^\alpha}\left(\frac{1}{\Gamma_q(\alpha)}\int_a^b{}_a\big(b-{}_a\Phi_q(s)\big)_q^{(\alpha-1)}f(a+b-s)_a\,d_qs\right)\\ &\quad=\frac{1}{(b-a)^\alpha}\big({}_aI_q^\alpha f(a+b-s)\big)(b),\end{aligned}$$

together with (3.8), we derive the first part of inequality (3.2) as requested. The proof is completed. □

Remark 3.4 If $\alpha=1$ and $q\to1$, then inequality (3.2) is reduced to the classical Hermite-Hadamard integral inequality as

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(s)\,ds\leq\frac{f(a)+f(b)}{2}.$$

See also [22, 23].

Let us prove the fractional q-Korkine equality on the interval $[a,b]$.

Lemma 3.5 *Let $f,g:[a,b]\to\mathbb{R}$ be continuous functions, $0<q<1$, and $\alpha>0$. Then we have*

$$\begin{aligned}&\frac{1}{2}\big({}_aI_q^{2\alpha}\big(f(s)-f(r)\big)\big(g(s)-g(r)\big)\big)(b)\\&\quad=\frac{(b-a)^\alpha}{\Gamma_q(\alpha+1)}\big({}_aI_q^\alpha f(s)g(s)\big)(b)-\big({}_aI_q^\alpha f(s)\big)(b)\big({}_aI_q^\alpha g(s)\big)(b).\end{aligned}\tag{3.9}$$

Proof From Definition 2.3, we have

$$\begin{aligned}&\big({}_aI_q^{2\alpha}\big(f(s)-f(r)\big)\big(g(s)-g(r)\big)\big)(b)\\&\quad=\frac{1}{\Gamma_q^2(\alpha)}\int_a^b\int_a^b{}_a\big(b-{}_a\Phi_q(s)\big)_q^{(\alpha-1)}{}_a\big(b-{}_a\Phi_q(r)\big)_q^{(\alpha-1)}\\&\qquad\times\big(f(s)-f(r)\big)\big(g(s)-g(r)\big)_a\,d_qs_a\,d_qr\\&\quad=\frac{1}{\Gamma_q^2(\alpha)}\int_a^b\int_a^b\big(b-{}_a\Phi_q(s)\big)_a^{(\alpha-1)}\big(b-{}_a\Phi_q(r)\big)_a^{(\alpha-1)}\\&\qquad\times\big(f(s)g(s)-f(s)g(r)-f(r)g(s)+f(r)g(r)\big)_a\,d_qs_a\,d_qr\\&\quad=\frac{(b-a)^\alpha}{\Gamma_q(\alpha+1)}\left(\frac{(1-q)(b-a)}{\Gamma_q(\alpha)}\sum_{i=0}^\infty q^i{}_a\big(b-{}_a\Phi_q^{i+1}(b)\big)_q^{(\alpha-1)}f\big({}_a\Phi_q^i(b)\big)g\big({}_a\Phi_q^i(b)\big)\right)\\&\qquad-\left(\frac{(1-q)(b-a)}{\Gamma_q(\alpha)}\sum_{i=0}^\infty q^i{}_a\big(b-{}_a\Phi_q^{i+1}(b)\big)_q^{(\alpha-1)}f\big({}_a\Phi_q^i(b)\big)\right)\\&\qquad\times\left(\frac{(1-q)(b-a)}{\Gamma_q(\alpha)}\sum_{i=0}^\infty q^i{}_a\big(b-{}_a\Phi_q^{i+1}(b)\big)_q^{(\alpha-1)}g\big({}_a\Phi_q^i(b)\big)\right)\\&\qquad-\left(\frac{(1-q)(b-a)}{\Gamma_q(\alpha)}\sum_{i=0}^\infty q^i{}_a\big(b-{}_a\Phi_q^{i+1}(b)\big)_q^{(\alpha-1)}g\big({}_a\Phi_q^i(b)\big)\right)\\&\qquad\times\left(\frac{(1-q)(b-a)}{\Gamma_q(\alpha)}\sum_{i=0}^\infty q^i{}_a\big(b-{}_a\Phi_q^{i+1}(b)\big)_q^{(\alpha-1)}f\big({}_a\Phi_q^i(b)\big)\right)\\&\qquad+\frac{(b-a)^\alpha}{\Gamma_q(\alpha+1)}\left(\frac{(1-q)(b-a)}{\Gamma_q(\alpha)}\sum_{i=0}^\infty q^i{}_a\big(b-{}_a\Phi_q^{i+1}(b)\big)_q^{(\alpha-1)}f\big({}_a\Phi_q^i(b)\big)g\big({}_a\Phi_q^i(b)\big)\right)\\&\quad=\frac{2(b-a)^\alpha}{\Gamma_q(\alpha+1)}\left(\frac{(1-q)(b-a)}{\Gamma_q(\alpha)}\sum_{i=0}^\infty q^i{}_a\big(b-{}_a\Phi_q^{i+1}(b)\big)_q^{(\alpha-1)}f\big({}_a\Phi_q^i(b)\big)g\big({}_a\Phi_q^i(b)\big)\right)\\&\qquad-2\left(\frac{(1-q)(b-a)}{\Gamma_q(\alpha)}\sum_{i=0}^\infty q^i{}_a\big(b-{}_a\Phi_q^{i+1}(b)\big)_q^{(\alpha-1)}f\big({}_a\Phi_q^i(b)\big)\right)\\&\qquad\times\left(\frac{(1-q)(b-a)}{\Gamma_q(\alpha)}\sum_{i=0}^\infty q^i{}_a\big(b-{}_a\Phi_q^{i+1}(b)\big)_q^{(\alpha-1)}g\big({}_a\Phi_q^i(b)\big)\right)\\&\quad=\frac{2(b-a)^\alpha}{\Gamma_q(\alpha+1)}\left(\frac{1}{\Gamma_q(\alpha)}\int_a^b{}_a\big(b-{}_a\Phi_q(s)\big)_q^{(\alpha-1)}f(s)g(s)_a\,d_qs\right)\end{aligned}$$

$$-2\left(\frac{1}{\Gamma_q(\alpha)}\int_a^b {}_a\big(b-{}_a\Phi_q(s)\big)_q^{(\alpha-1)}f(s)_a\,d_qs\right)$$
$$\times\left(\frac{1}{\Gamma_q(\alpha)}\int_a^b {}_a\big(b-{}_a\Phi_q(s)\big)_q^{(\alpha-1)}g(s)_a\,d_qs\right)$$
$$=\frac{2(b-a)^\alpha}{\Gamma_q(\alpha+1)}\big({}_aI_q^\alpha fg\big)(b)-2\big({}_aI_q^\alpha f\big)(b)\big({}_aI_q^\alpha g\big)(b),$$

from which one deduces (3.9). □

Remark 3.6 If $\alpha = 1$, then Lemma 3.5 is reduced to Lemma 3.1 in [17].

Next, we will prove the fractional q-Cauchy-Bunyakovsky-Schwarz integral inequality on the interval $[a,b]$.

Theorem 3.7 *Let $f,g:[a,b]\to\mathbb{R}$ be continuous functions, $0<q<1$, and $\alpha,\beta>0$. Then we have*

$$\big|\big({}_aI_q^{\beta+\alpha}f(s,r)g(s,r)\big)(b)\big|\le\sqrt{\big({}_aI_q^{\beta+\alpha}f^2(s,r)\big)(b)}\sqrt{\big({}_aI_q^{\beta+\alpha}g^2(s,r)\big)(b)}.\tag{3.10}$$

Proof From Definition 2.3, we have

$$\begin{aligned}
&\big({}_aI_q^{\beta+\alpha}f(s,r)\big)(b)\\
&\quad=\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_a^b\int_a^b {}_a\big(b-{}_a\Phi_q(s)\big)_q^{(\alpha-1)}{}_a\big(b-{}_a\Phi_q(r)\big)_q^{(\beta-1)}f(s,r)_a\,d_qs_a\,d_qr\\
&\quad=\frac{(1-q)^2(b-a)^2}{\Gamma_q(\alpha)\Gamma_q(\beta)}\sum_{i=0}^\infty\sum_{n=0}^\infty q^{i+n}{}_a\big(b-{}_a\Phi_q^{i+1}(b)\big)_q^{(\alpha-1)}{}_a\big(b-{}_a\Phi_q^{i+1}(b)\big)_q^{(\beta-1)}\\
&\qquad\times f\big({}_a\Phi_q^i(b),{}_a\Phi_q^n(b)\big).
\end{aligned}$$

Using the classical discrete Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\big(\big({}_aI_q^{\beta+\alpha}f(s,r)g(s,r)\big)(b)\big)^2\\
&\quad=\Bigg(\frac{(1-q)^2(b-a)^2}{\Gamma_q(\alpha)\Gamma_q(\beta)}\sum_{i=0}^\infty\sum_{n=0}^\infty q^{i+n}{}_a\big(b-{}_a\Phi_q^{i+1}(b)\big)_q^{(\alpha-1)}{}_a\big(b-{}_a\Phi_q^{i+1}(b)\big)_q^{(\beta-1)}\\
&\qquad\times f\big({}_a\Phi_q^i(b),{}_a\Phi_q^n(b)\big)g\big({}_a\Phi_q^i(b),{}_a\Phi_q^n(b)\big)\Bigg)^2\\
&\quad\le\Bigg(\frac{(1-q)^2(b-a)^2}{\Gamma_q(\alpha)\Gamma_q(\beta)}\sum_{i=0}^\infty\sum_{n=0}^\infty q^{i+n}{}_a\big(b-{}_a\Phi_q^{i+1}(b)\big)_q^{(\alpha-1)}{}_a\big(b-{}_a\Phi_q^{i+1}(b)\big)_q^{(\beta-1)}\\
&\qquad\times f^2\big({}_a\Phi_q^i(b),{}_a\Phi_q^n(b)\big)\Bigg)\Bigg(\frac{(1-q)^2(b-a)^2}{\Gamma_q(\alpha)\Gamma_q(\beta)}\sum_{i=0}^\infty\sum_{n=0}^\infty q^{i+n}{}_a\big(b-{}_a\Phi_q^{i+1}(b)\big)_q^{(\alpha-1)}\\
&\qquad\times{}_a\big(b-{}_a\Phi_q^{i+1}(b)\big)_q^{(\beta-1)}g^2\big({}_a\Phi_q^i(b),{}_a\Phi_q^n(b)\big)\Bigg)\\
&\quad=\big(\big({}_aI_q^{\beta+\alpha}f^2(s,r)\big)(b)\big)\big(\big({}_aI_q^{\beta+\alpha}g^2(s,r)\big)(b)\big).
\end{aligned}$$

Therefore, inequality (3.10) holds. □

Remark 3.8 If $\alpha = 1$, then inequality (3.10) is reduced to the q-Cauchy-Bunyakovsky-Schwarz integral inequality in [17].

Now, we will prove the fractional q-Grüss integral inequality on the interval $[a,b]$.

Theorem 3.9 *Let* $f,g:[a,b]\to\mathbb{R}$ *be continuous functions satisfying*

$$\phi \le f(s) \le \Phi, \qquad \psi \le g(s) \le \Psi, \quad \text{for all } s\in[a,b], \phi,\Phi,\psi,\Psi\in\mathbb{R}. \tag{3.11}$$

For $0<q<1$ *and* $\alpha>0$, *we have the inequality*

$$\begin{aligned}&\left|\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\big({}_aI_q^\alpha f(s)g(s)\big)(b) - \left(\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\big({}_aI_q^\alpha f(s)\big)(b)\right)\right.\\ &\qquad\left.\times\left(\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\big({}_aI_q^\alpha g(s)\big)(b)\right)\right| \le \frac{1}{4}(\Phi-\phi)(\Psi-\psi).\end{aligned} \tag{3.12}$$

Proof Applying Theorem 3.7, we have

$$\begin{aligned}&\big|\big({}_aI_q^{2\alpha}\big(f(s)-f(r)\big)\big(g(s)-g(r)\big)\big)(b)\big|\\ &\quad\le \Big(\big({}_aI_q^{2\alpha}\big(f(s)-f(r)\big)^2\big)(b)\Big)^{\frac{1}{2}}\Big(\big({}_aI_q^{2\alpha}\big(g(s)-g(r)\big)^2\big)(b)\Big)^{\frac{1}{2}}.\end{aligned} \tag{3.13}$$

From Lemma 3.5, it follows that

$$\frac{1}{2}\big({}_aI_q^{2\alpha}\big(f(s)-f(r)\big)^2\big)(b) = \frac{(b-a)^\alpha}{\Gamma_q(\alpha+1)}\big({}_aI_q^\alpha f^2(s)\big)(b) - \Big(\big({}_aI_q^\alpha f(s)\big)(b)\Big)^2. \tag{3.14}$$

By a simple computation, we have

$$\begin{aligned}&\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\big({}_aI_q^\alpha f^2(s)\big)(b) - \left(\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\big({}_aI_q^\alpha f(s)\big)(b)\right)^2\\ &\quad= \left(\Phi - \frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\big({}_aI_q^\alpha f(s)\big)(b)\right)\left(\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\big({}_aI_q^\alpha f(s)\big)(b)-\phi\right)\\ &\qquad- \frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\big({}_aI_q^\alpha\big(f(s)-\phi\big)\big(\Phi-f(s)\big)\big)(b),\end{aligned} \tag{3.15}$$

and an analogous identity for g.

By assumption (3.11) we have $(f(s)-\phi)(\Phi-f(s))\ge 0$ for all $s\in[a,b]$, which implies

$$\big({}_aI_q^\alpha\big(f(s)-\phi\big)\big(\Phi-f(s)\big)\big)(b)\ge 0.$$

From (3.15) and using the fact that $(\frac{A+B}{2})^2\ge AB$, $A,B\in\mathbb{R}$, we have

$$\begin{aligned}&\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\big({}_aI_q^\alpha f^2(s)\big)(b) - \left(\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\big({}_aI_q^\alpha f(s)\big)(b)\right)^2\\ &\quad\le \left(\Phi - \frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\big({}_aI_q^\alpha f(s)\big)(b)\right)\left(\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\big({}_aI_q^\alpha f(s)\big)(b)-\phi\right)\end{aligned}$$

$$\le \frac{1}{4}\left[\left(\Phi - \frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\big({}_aI_q^\alpha f(s)\big)(b)\right) + \left(\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\big({}_aI_q^\alpha f(s)\big)(b) - \phi\right)\right]^2$$
$$\le \frac{1}{4}(\Phi - \phi)^2. \tag{3.16}$$

A similar argument gives

$$\big({}_aI_q^{2\alpha}\big(g(s)-g(r)\big)^2\big)(b) \le \frac{1}{4}(\Psi - \psi)^2. \tag{3.17}$$

Using inequality (3.13) via (3.14) and the estimations (3.16) and (3.17), we get

$$\left|\frac{1}{2}\big({}_aI_q^{2\alpha}\big(f(s)-f(r)\big)\big(g(s)-g(r)\big)\big)(b)\right| \le \frac{(b-a)^\alpha}{4}(\Phi-\phi)(\Psi-\psi).$$

Therefore, inequality (3.12) holds, as desired. □

Remark 3.10 If $\alpha = 1$ and $q \to 1$, then inequality (3.12) is reduced to the classical Grüss integral inequality as

$$\left|\frac{1}{b-a}\int_a^b f(s)g(s)\,ds - \left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\left(\frac{1}{b-a}\int_a^b g(s)\,ds\right)\right|$$
$$\le \frac{1}{4}(\Phi-\phi)(\Psi-\psi).$$

See also [22, 23].

Next, we are going to prove the fractional q-Grüss-Čebyšev integral inequality on the interval $[a,b]$.

Theorem 3.11 *Let $f,g:[a,b]\to\mathbb{R}$ be L_1-, L_2-Lipschitzian continuous functions, so that*

$$\big|f(s)-f(r)\big| \le L_1|s-r|, \qquad \big|g(s)-g(r)\big| \le L_2|s-r|, \tag{3.18}$$

for all $s,r \in [a,b]$, $0<q<1$, $L_1, L_2 > 0$, and $\alpha > 0$. Then we have the inequality

$$\left|\frac{(b-a)^\alpha}{\Gamma_q(\alpha+1)}\big({}_aI_q^\alpha f(s)g(s)\big)(b) - \big({}_aI_q^\alpha f(s)\big)(b)\big({}_aI_q^\alpha g(s)\big)(b)\right|$$
$$\le \frac{L_1L_2(b-a)^{2\alpha+2}}{\Gamma_q(\alpha+2)\Gamma_q(\alpha+3)}\big((1+q)[\alpha+1]_q - [\alpha+2]_q\big). \tag{3.19}$$

Proof Recall the fractional q-Korkine equality as

$$\frac{(b-a)^\alpha}{\Gamma_q(\alpha+1)}\big({}_aI_q^\alpha f(s)g(s)\big)(b) - \big({}_aI_q^\alpha f(s)\big)(b)\big({}_aI_q^\alpha g(s)\big)(b)$$
$$= \frac{1}{2}\big({}_aI_q^{2\alpha}\big(f(s)-f(r)\big)\big(g(s)-g(r)\big)\big)(b). \tag{3.20}$$

It follows from (3.18) that

$$\big|\big(f(s)-f(r)\big)\big(g(s)-g(r)\big)\big| \le L_1L_2(s-r)^2, \tag{3.21}$$

for all $s, r \in [a,b]$. Taking the double fractional q-integration of order α with respect to $s, r \in [a,b]$, we get

$$\begin{aligned}
&\left({}_aI_q^{2\alpha}\left|\big(f(s)-f(r)\big)\big(g(s)-g(r)\big)\right|\right)(b) \\
&= \frac{1}{\Gamma_q^2(\alpha)}\int_a^b\int_a^b {}_a\big(b-{}_a\Phi_q(s)\big)_q^{(\alpha-1)}{}_a\big(b-{}_a\Phi_q(r)\big)_q^{(\alpha-1)} \\
&\quad\times\left|\big(f(s)-f(r)\big)\big(g(s)-g(r)\big)\right|{}_a\,d_qs_a\,d_qr \\
&\le \frac{L_1L_2}{\Gamma_q^2(\alpha)}\int_a^b\int_a^b {}_a\big(b-{}_a\Phi_q(s)\big)_q^{(\alpha-1)}{}_a\big(b-{}_a\Phi_q(r)\big)_q^{(\alpha-1)}(s-r)^2{}_a\,d_qs_a\,d_qr \\
&= \frac{L_1L_2}{\Gamma_q^2(\alpha)}\int_a^b\int_a^b {}_a\big(b-{}_a\Phi_q(s)\big)_q^{(\alpha-1)}{}_a\big(b-{}_a\Phi_q(r)\big)_q^{(\alpha-1)}s^2{}_a\,d_qs_a\,d_qr \\
&\quad-\frac{2L_1L_2}{\Gamma_q^2(\alpha)}\int_a^b\int_a^b {}_a\big(b-{}_a\Phi_q(s)\big)_q^{(\alpha-1)}{}_a\big(b-{}_a\Phi_q(r)\big)_q^{(\alpha-1)}sr_a\,d_qs_a\,d_qr \\
&\quad+\frac{L_1L_2}{\Gamma_q^2(\alpha)}\int_a^b\int_a^b {}_a\big(b-{}_a\Phi_q(s)\big)_q^{(\alpha-1)}{}_a\big(b-{}_a\Phi_q(r)\big)_q^{(\alpha-1)}r^2{}_a\,d_qs_a\,d_qr \\
&= 2L_1L_2\left(\frac{(b-a)^\alpha}{\Gamma_q(\alpha+1)}\big({}_aI_q^\alpha s^2\big)(b)-\Big(\big({}_aI_q^\alpha s\big)(b)\Big)^2\right).
\end{aligned} \tag{3.22}$$

From Corollary 2.7(ii), with $t=b$, we have

$$\big({}_aI_q^\alpha s^2\big)(b)=\frac{(b-a)^\alpha}{\Gamma_q(\alpha+3)}\big((1+q)(b-a)^2+2a(b-a)[\alpha+2]_q+a^2[\alpha+1]_q[\alpha+2]_q\big).$$

By direct computation, we have

$$\begin{aligned}
&\frac{(b-a)^\alpha}{\Gamma_q(\alpha+1)}\big({}_aI_q^\alpha s^2\big)(b)-\Big(\big({}_aI_q^\alpha s\big)(b)\Big)^2 \\
&= \frac{(b-a)^{2\alpha}}{\Gamma_q(\alpha+1)\Gamma_q(\alpha+3)}\big((1+q)(b-a)^2+2a(b-a)[\alpha+2]_q+a^2[\alpha+1]_q[\alpha+2]_q\big) \\
&\quad-\frac{(b-a)^{2\alpha}}{\Gamma_q^2(\alpha+2)}\big(b+\big([\alpha+1]_q-1\big)a\big)^2 \\
&= \frac{(b-a)^{2\alpha+2}}{\Gamma_q(\alpha+2)\Gamma_q(\alpha+3)}\big((1+q)[\alpha+1]_q-[\alpha+2]_q\big).
\end{aligned} \tag{3.23}$$

Thus, from (3.22) and (3.23), we have

$$\begin{aligned}
&\left({}_aI_q^{2\alpha}\left|\big(f(s)-f(r)\big)\big(g(s)-g(r)\big)\right|\right)(b) \\
&\le \frac{2L_1L_2(b-a)^{2\alpha+2}}{\Gamma_q(\alpha+2)\Gamma_q(\alpha+3)}\big((1+q)[\alpha+1]_q-[\alpha+2]_q\big).
\end{aligned} \tag{3.24}$$

By applying (3.24) to (3.20), we get the desired inequality in (3.19). □

Remark 3.12 If $\alpha = 1$ and $q \to 1$, then inequality (3.19) is reduced to the classical Grüss-Čebyšev integral inequality as

$$\left|\frac{1}{b-a}\int_a^b f(s)g(s)\,ds - \left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\left(\frac{1}{b-a}\int_a^b g(s)\,ds\right)\right| \leq \frac{L_1L_2}{12}(b-a)^2.$$

See also [22, 23].

For the final result, we establish the fractional q-Pólya-Szegö integral inequality on the interval $[a,b]$.

Theorem 3.13 *Let $f,g:[a,b]\to\mathbb{R}$ be two positive integrable functions satisfying*

$$0<\phi\leq f(s)\leq\Phi,\qquad 0<\psi\leq g(s)\leq\Psi,\quad \text{for all } s\in[a,b], \phi,\Phi,\psi,\Psi\in\mathbb{R}^+. \tag{3.25}$$

Then for $0<q<1$ and $\alpha>0$, we have the inequality

$$\frac{({}_aI_q^\alpha(f^2(s)))(b)({}_aI_q^\alpha(g^2(s)))(b)}{(({}_aI_q^\alpha(f(s)g(s))(b))^2} \leq \frac{1}{4}\left(\sqrt{\frac{\phi\psi}{\Phi\Psi}}+\sqrt{\frac{\Phi\Psi}{\phi\psi}}\right)^2. \tag{3.26}$$

Proof From (3.25), for $s\in[a,b]$, we have

$$\frac{\phi}{\Psi}\leq\frac{f(s)}{g(s)}\leq\frac{\Phi}{\psi},$$

which yields

$$\left(\frac{\Phi}{\psi}-\frac{f(s)}{g(s)}\right)\geq 0 \tag{3.27}$$

and

$$\left(\frac{f(s)}{g(s)}-\frac{\phi}{\Psi}\right)\geq 0. \tag{3.28}$$

Multiplying (3.27) and (3.28), we obtain

$$\left(\frac{\Phi}{\psi}-\frac{f(s)}{g(s)}\right)\left(\frac{f(s)}{g(s)}-\frac{\phi}{\Psi}\right)\geq 0,$$

or

$$\left(\frac{\Phi}{\psi}+\frac{\phi}{\Psi}\right)\frac{f(s)}{g(s)}\geq\frac{f^2(s)}{g^2(s)}+\frac{\phi\Psi}{\psi\Phi}. \tag{3.29}$$

Inequality (3.29) can be written as

$$(\phi\psi+\Phi\Psi)f(s)g(s)\geq\psi\Psi f^2(s)+\phi\Phi g^2(s). \tag{3.30}$$

Multiplying both sides of (3.30) by ${}_0(b - {}_0\Phi_q(s))_q^{(\alpha-1)}/\Gamma_q(\alpha)$ and integrating with respect to s from a to b, we get

$$(\phi\psi + \Phi\Psi)\big({}_aI_q^\alpha\big(f(s)g(s)\big)\big)(b) \geq \psi\Psi\big({}_aI_q^\alpha\big(f^2(s)\big)\big)(b) + \phi\Phi\big({}_aI_q^\alpha\big(g^2(s)\big)\big)(b).$$

Applying the AM-GM inequality, $A + B \geq 2\sqrt{AB}$, $A, B \in \mathbb{R}^+$, we have

$$(\phi\psi + \Phi\Psi)\big({}_aI_q^\alpha\big(f(s)g(s)\big)\big)(b) \geq 2\sqrt{\phi\psi\Phi\Psi\big({}_aI_q^\alpha\big(f^2(s)\big)\big)(b)\big({}_aI_q^\alpha\big(g^2(s)\big)\big)(b)},$$

which leads to

$$\phi\psi\Phi\Psi\big({}_aI_q^\alpha\big(f^2(s)\big)\big)(b)\big({}_aI_q^\alpha\big(g^2(s)\big)\big)(b) \leq \frac{1}{4}\big((\phi\psi + \Phi\Psi)\big({}_aI_q^\alpha\big(f(s)g(s)\big)(b)\big)^2.$$

Therefore, inequality (3.26) is proved. □

Remark 3.14 If $\alpha = 1$ and $q \to 1$, then inequality (3.26) is reduced to the classical Pólya-Szegö integral inequality as

$$\frac{\int_a^b f^2(s)\,ds \int_a^b g^2(s)\,ds}{(\int_a^b f(s)g(s)\,ds)^2} \leq \frac{1}{4}\left(\sqrt{\frac{\phi\psi}{\Phi\Psi}} + \sqrt{\frac{\Phi\Psi}{\phi\psi}}\right)^2.$$

See also [24].

4 Conclusion

In this work, some important integral inequalities involving the new q-shifting operator ${}_a\Phi_q(m) = qm + (1 - q)a$, introduced in [1], are established in the context of fractional quantum calculus. The derived results constitute contributions to the theory of integral inequalities and fractional calculus and can be specialized to yield numerous interesting fractional integral inequalities including some known results. Furthermore, they are expected to lead to some applications in fractional boundary value problems.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this article. They read and approved the final manuscript.

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References

1. Tariboon, J, Ntouyas, SK, Agarwal, P: New concepts of fractional quantum calculus and applications to impulsive fractional q-difference equations. Adv. Differ. Equ. **2015**, 18 (2015)
2. Kac, V, Cheung, P: Quantum Calculus. Springer, New York (2002)
3. Jackson, FH: q-Difference equations. Am. J. Math. **32**, 305-314 (1910)
4. Al-Salam, WA: Some fractional q-integrals and q-derivatives. Proc. Edinb. Math. Soc. **15**(2), 135-140 (1966/1967)
5. Agarwal, RP: Certain fractional q-integrals and q-derivatives. Proc. Camb. Philos. Soc. **66**, 365-370 (1969)
6. Ernst, T: The history of q-calculus and a new method. UUDM Report 2000:16, Department of Mathematics, Uppsala University (2000)

7. Ferreira, R: Nontrivial solutions for fractional q-difference boundary value problems. Electron. J. Qual. Theory Differ. Equ. **2010**, 70 (2010)
8. Annaby, MH, Mansour, ZS: q-Fractional Calculus and Equations. Lecture Notes in Mathematics, vol. 2056. Springer, Berlin (2012)
9. Bangerezako, G: Variational q-calculus. J. Math. Anal. Appl. **289**, 650-665 (2004)
10. Ismail, MEH, Simeonov, P: q-Difference operators for orthogonal polynomials. J. Comput. Appl. Math. **233**, 749-761 (2009)
11. Yu, C, Wang, J: Existence of solutions for nonlinear second-order q-difference equations with first-order q-derivatives. Adv. Differ. Equ. **2013**, 124 (2013)
12. Ahmad, B, Ntouyas, SK: Boundary value problems for q-difference inclusions. Abstr. Appl. Anal. **2011**, 292860 (2011)
13. Ahmad, B: Boundary-value problems for nonlinear third-order q-difference equations. Electron. J. Differ. Equ. **2011**, 94 (2011)
14. Graef, JR, Kong, L: Positive solutions for a class of higher-order boundary value problems with fractional q-derivatives. Appl. Math. Comput. **218**, 9682-9689 (2012)
15. Ahmad, B, Ntouyas, SK, Purnaras, IK: Existence results for nonlocal boundary value problems of nonlinear fractional q-difference equations. Adv. Differ. Equ. **2012**, 140 (2012)
16. Tariboon, J, Ntouyas, SK: Quantum calculus on finite intervals and applications to impulsive difference equations. Adv. Differ. Equ. **2013**, 282 (2013)
17. Tariboon, J, Ntouyas, SK: Quantum integral inequalities on finite intervals. J. Inequal. Appl. **2014**, 121 (2014)
18. Noor, MA, Noor, KI, Awan, MU: Some quantum estimates for Hermite-Hadamard inequalities. Appl. Math. Comput. **251**, 675-679 (2015)
19. Noor, MA, Noor, KI, Awan, MU: Some quantum integral inequalities via preinvex functions. Appl. Math. Comput. **269**, 242-251 (2015)
20. Taf, S, Brahim, K, Riahi, L: Some results for Hadamard-type inequalities in quantum calculus. Matematiche **LXIX**(2), 243-258 (2014)
21. Anastassiou, GA: Intelligent Mathematics: Computational Analysis. Springer, New York (2011)
22. Cerone, P, Dragomir, SS: Mathematical Inequalities. CRC Press, New York (2011)
23. Pachpatte, BG: Analytic Inequalities. Atlantis Press, Paris (2012)
24. Pólya, G, Szegö, G: Aufgaben und Lehrsatze aus der Analysis, Band 1. Die Grundlehren der mathematischen Wissenschaften, vol. 19. Springer, Berlin (1925)