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Complete convergence and complete moment convergence for arrays of rowwise ANA random variables

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Abstract

In this article, we investigate complete convergence and complete moment convergence for weighted sums of arrays of rowwise asymptotically negatively associated (ANA) random variables. The results obtained not only generalize the corresponding ones of Sung (Stat. Pap. 52:447-454, 2011), Zhou *et al.* (J. Inequal. Appl. 2011:157816, 2011), and Sung (Stat. Pap. 54:773-781, 2013) to the case of ANA random variables, but also improve them.

MSC: 60F15

Keywords: arrays of rowwise ANA random variables; complete convergence; complete moment convergence; weighted sums

1 Introduction

Recently, Sung [1] proved the following strong laws of large numbers for weighted sums of identically distributed negatively associated (NA) random variables.

Theorem A *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed NA random variables, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n) \tag{1.1}$$

for some $0 < \alpha \leq 2$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Furthermore, suppose that $EX_1 = 0$ for $1 < \alpha \leq 2$. If

$$\begin{aligned} &E|X_1|^{\alpha} < \infty \quad \text{for } \alpha > \gamma, \\ &E|X_1|^{\alpha} \log(1 + |X_1|) < \infty \quad \text{for } \alpha = \gamma, \\ &E|X_1|^{\gamma} < \infty \quad \text{for } \alpha < \gamma, \end{aligned} \tag{1.2}$$

then

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon b_n\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{1.3}$$

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Zhou *et al.* [2] partially extended Theorem A for NA random variables to the case of $\tilde{\rho}$-mixing (or ρ^*-mixing) random variables by using a different method.

Theorem B *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed $\tilde{\rho}$-mixing random variables, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^{\max\{\alpha,\gamma\}} = O(n) \tag{1.4}$$

for some $0 < \alpha \leq 2$ and $\gamma > 0$ with $\alpha \neq \gamma$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$. Assume that $EX_1 = 0$ for $1 < \alpha \leq 2$. If (1.2) *is satisfied for $\alpha \neq \gamma$, then* (1.3) *holds.*

Zhou *et al.* [2] left an open problem whether the case $\alpha = \gamma$ of Theorem A holds for $\tilde{\rho}$-mixing random variables. Sung [3] solved this problem and obtained the following result.

Theorem C *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed $\tilde{\rho}$-mixing random variables, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying* (1.4) *for some $0 < \alpha \leq 2$ and $\gamma > 0$ with $\alpha = \gamma$. Let $b_n = n^{1/\alpha}(\log n)^{1/\alpha}$. Assume that $EX_1 = 0$ for $1 < \alpha \leq 2$. If* (1.2) *is satisfied for $\alpha = \gamma$, then* (1.3) *holds.*

Inspired by these theorems, in this paper, we further investigate the limit convergence properties and obtain some much stronger conclusions, which extend and improve Theorems A, B, and C to a wider class of dependent random variables under the same conditions.

Now we introduce some definitions of dependent structures.

Definition 1.1 A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be NA if for any disjoint subsets A and B of $\{1, 2, \ldots, n\}$,

$$\operatorname{Cov}\big(f_1(X_i, i \in A), f_2(X_j, j \in B)\big) \leq 0 \tag{1.5}$$

whenever f_1 and f_2 are any real coordinatewise nondecreasing functions such that this covariance exists. An infinite family of random variables $\{X_n, n \geq 1\}$ is NA if every finite its subfamily is NA.

Definition 1.2 A sequence of random variables $\{X_n, n \geq 1\}$ is called $\tilde{\rho}$-mixing if for some integer $n \geq 1$, the mixing coefficient

$$\tilde{\rho}(s) = \sup\big\{\rho(S, T) : S, T \subset \mathrm{N}, \operatorname{dist}(S, T) \geq s\big\} \to 0 \quad \text{as } s \to \infty, \tag{1.6}$$

where

$$\rho(S, T) = \sup\left\{\frac{|EXY - EXEY|}{\sqrt{\operatorname{Var} X} \cdot \sqrt{\operatorname{Var} Y}}; X \in L_2\big(\sigma(S)\big), Y \in L_2\big(\sigma(T)\big)\right\}, \tag{1.7}$$

and $\sigma(S)$ and $\sigma(T)$ are the σ-fields generated by $\{X_i, i \in S\}$ and $\{X_i, i \in T\}$, respectively.

Definition 1.3 A sequence of random variables $\{X_n, n \geq 1\}$ is said to be ANA if

$$\rho^-(s) = \sup\bigl\{\rho^-(S,T) : S, T \subset \mathrm{N}, \mathrm{dist}(S,T) \geq s\bigr\} \to 0 \quad \text{as } s \to \infty, \tag{1.8}$$

where

$$\rho^-(S,T) = 0 \vee \left\{ \frac{\mathrm{Cov}(f(X_i, i \in S), g(X_j, j \in T))}{(\mathrm{Var} f(X_i, i \in S))^{1/2} (\mathrm{Var}\, g(X_j, j \in T))^{1/2}}, f, g \in C \right\}, \tag{1.9}$$

and C is the set of nondecreasing functions.

An array of random variables $\{X_{ni}, i \geq 1, n \geq 1\}$ is said to be rowwise ANA random variables if for every $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is a sequence of ANA random variables.

It is obvious to see that $\rho^-(s) \leq \tilde{\rho}(s)$ and that a sequence of ANA random variables is NA if and only if $\rho^-(1) = 0$. So, ANA random variables include $\tilde{\rho}$-mixing and NA random variables. Consequently, the study of the limit convergence properties for ANA random variables is of much interest. Since the concept of ANA random variables was introduced by Zhang and Wang [4], many applications have been found. For example, Zhang and Wang [4] and Zhang [5, 6] obtained moment inequalities for partial sums, the central limit theorems, the complete convergence, and the strong law of large numbers, Wang and Lu [7] established some inequalities for the maximum of partial sums and weak convergence, Wang and Zhang [8] obtained the law of the iterated logarithm, Liu and Liu [9] showed moments of the maximum of normed partial sums, Yuan and Wu [10] obtained the limiting behavior of the maximum of partial sums, Budsaba *et al.* [11] investigated the complete convergence for moving-average process based on a sequence of ANA and NA random variables, Tan *et al.* [12] obtained the almost sure central limit theorem, Ko [13] obtained the Hájek-Rényi inequality and the strong law of large numbers, Zhang [14] established the complete moment convergence for moving-average process generated by ANA random variables, and so forth.

In this work, we further study the strong convergence for weighted sums of arrays of ANA random variables without assumption of identical distribution and obtain some improved results (*i.e.*, the so-called complete moment convergence, which will be introduced in Definition 1.5). As applications, the complete convergence theorems for weighted sums of arrays of identically distributed NA and $\tilde{\rho}$-mixing random variables can been obtained. The results obtained not only generalize the corresponding ones of Sung [1, 3] and Zhou *et al.* [2], but also improve them under the same conditions.

For the proofs of the main results, we need to restate some definitions used in this paper.

Definition 1.4 A sequence of random variables $\{X_n, n \geq 1\}$ converges completely to a constant λ if for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\bigl(|X_n - \lambda| > \varepsilon\bigr) < \infty.$$

This notion was first introduced by Hsu and Robbins [15].

Definition 1.5 (Chow [16]) Let $\{X_n, n \geq 1\}$ be a sequence of random variables, and $a_n > 0$, $b_n > 0$, $q > 0$. If for all $\varepsilon \geq 0$,

$$\sum_{n=1}^{\infty} a_n E\left(b_n^{-1}|X_n| - \varepsilon\right)_+^q < \infty,$$

then the sequence $\{X_n, n \geq 1\}$ is said to satisfy the complete moment convergence.

Definition 1.6 A sequence of random variables $\{X_n, n \geq 1\}$ is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P\left(|X_n| \geq x\right) \leq CP\left(|X| \geq x\right)$$

for all $x \geq 0$ and $n \geq 1$.

An array of rowwise random variables $\{X_{ni}, i \geq 1, n \geq 1\}$ is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P\left(|X_{ni}| \geq x\right) \leq CP\left(|X| \geq x\right)$$

for all $x \geq 0$, $i \geq 1$ and $n \geq 1$.

Throughout this paper, $I(A)$ is the indicator function of a set A. The symbol C denotes a positive constant, which may be different in various places, and $a_n = O(b_n)$ means that $a_n \leq Cb_n$.

2 Main results

Now, we state our main results. The proofs will be given in Section 3.

Theorem 2.1 *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise ANA random variables stochastically dominated by a random variable X, let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying* (1.1) *for some $0 < \alpha \leq 2$, and let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Furthermore, assume that $EX_{ni} = 0$ for $1 < \alpha \leq 2$. If*

$$\begin{aligned} &E|X|^{\alpha} < \infty \quad \text{for } \alpha > \gamma, \\ &E|X|^{\alpha} \log\left(1 + |X|\right) < \infty \quad \text{for } \alpha = \gamma, \\ &E|X|^{\gamma} < \infty \quad \text{for } \alpha < \gamma, \end{aligned} \tag{2.1}$$

then

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{2.2}$$

Theorem 2.2 *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise ANA random variables which is stochastically dominated by a random variable X, let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying* (1.1) *for some $0 < \alpha \leq 2$, and let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$.*

Furthermore, assume that $EX_{ni} = 0$ *for* $1 < \alpha \leq 2$. *If* (2.1) *holds, then, for* $0 < q < \alpha$,

$$\sum_{n=1}^{\infty} \frac{1}{n} E\left(\frac{1}{b_n} \max_{1\leq j\leq n} \left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| - \varepsilon\right)_+^q < \infty \quad \textit{for all } \varepsilon > 0. \tag{2.3}$$

Remark 2.1 Since ANA includes NA and $\tilde{\rho}$-mixing, Theorem 2.1 extends Theorem A for identically distributed NA random variables and Theorems B and C for identically distributed $\tilde{\rho}$-mixing random variables (by taking $X_{ni} = X_i$ in Theorem 2.1).

Remark 2.2 Under the conditions of Theorem 2.2, we can obtain that

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} \frac{1}{n} E\left(\frac{1}{b_n} \max_{1\leq j\leq n} \left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| - \varepsilon\right)_+^q \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} P\left(\frac{1}{b_n} \max_{1\leq j\leq n} \left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| - \varepsilon > t^{1/q}\right) dt \\
&\geq C \int_0^1 \sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1\leq j\leq n} \left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > b_n\varepsilon\right) dt \\
&= C \sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1\leq j\leq n} \left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > b_n\varepsilon\right) \quad \text{for all } \varepsilon > 0.
\end{aligned} \tag{2.4}$$

Hence, from (2.4) we get that the complete moment convergence implies the complete convergence. Compared with the results of Sung [1, 3] and Zhou *et al.* [2], it is worth pointing out that our main result is much stronger under the same conditions. So, Theorem 2.2 is an extension and improvement of the corresponding ones of Sung [1, 3] and Zhou *et al.* [2].

3 Proofs

To prove the main results, we need the following lemmas.

Lemma 3.1 (Wang and Lu [7]) *Let* $\{X_n, n \geq 1\}$ *be a sequence of ANA random variables. If* $\{f_n, n \geq 1\}$ *is a sequence of real nondecreasing* (*or nonincreasing*) *functions, then* $\{f_n(X_n), n \geq 1\}$ *is still a sequence of ANA random variables.*

From Wang and Lu's [7] Rosenthal-type inequality for ANA random variables we obtain the following result.

Lemma 3.2 (Wang and Lu [7]) *For a positive integer* $\mathrm{N} \geq 1$ *and* $0 \leq s < \frac{1}{12}$, *let* $\{X_n, n \geq 1\}$ *be a sequence of ANA random variables with* $\rho^-(\mathrm{N}) \leq s$, $EX_n = 0$, *and* $E|X_n|^2 < \infty$. *Then for all* $n \geq 1$, *there exists a positive constant* $C = C(2, \mathrm{N}, s)$ *such that*

$$E\left(\max_{1\leq j\leq n} \left|\sum_{i=1}^{j} X_i\right|^2\right) \leq C \sum_{i=1}^{n} EX_i^2. \tag{3.1}$$

Lemma 3.3 (Adler and Rosalsky [17]; Adler *et al.* [18]) *Suppose that* $\{X_{ni}, i \geq 1, n \geq 1\}$ *is an array of random variables stochastically dominated by a random variable* X. *Then, for* *all* $q > 0$ *and* $x > 0$,

$$E|X_{ni}|^q I\big(|X_{ni}| \le x\big) \le C\big(E|X|^q I\big(|X| \le x\big) + x^q P\big(|X| > x\big)\big), \tag{3.2}$$

$$E|X_{ni}|^q I\big(|X_{ni}| > x\big) \le CE|X|^q I\big(|X| > x\big). \tag{3.3}$$

Lemma 3.4 (Wu *et al.* [19]) *Let* $\{a_{ni}, i \ge 1, n \ge 1\}$ *be an array of real constants satisfying* (1.1) *for some* $\alpha > 0$, *and* X *be a random variable. Let* $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ *for some* $\gamma > 0$. *If* $p > \max\{\alpha, \gamma\}$, *then*

$$\sum_{n=1}^{\infty} \frac{1}{nb_n^p} \sum_{i=1}^{n} E|a_{ni}X|^p I\big(|a_{ni}X| \le b_n\big) \le \begin{cases} CE|X|^\alpha & \text{for } \alpha > \gamma, \\ CE|X|^\alpha \log(1 + |X|) & \text{for } \alpha = \gamma, \\ CE|X|^\gamma & \text{for } \alpha < \gamma. \end{cases} \tag{3.4}$$

Proof of Theorem 2.1 Without loss of generality, assume that $a_{ni} \ge 0$ (otherwise, we shall use a_{ni}^+ and a_{ni}^- instead of a_{ni}, and note that $a_{ni} = a_{ni}^+ - a_{ni}^-$). For fixed $n \ge 1$, define

$$\begin{aligned}
&Y_{ni} = -b_n I(a_{ni}X_{ni} < -b_n) + a_{ni}X_{ni} I\big(|a_{ni}X_{ni}| \le b_n\big) + b_n I(a_{ni}X_{ni} > b_n), \quad i \ge 1; \\
&Z_{ni} = a_{ni}X_{ni} - Y_{ni} = (a_{ni}X_{ni} + b_n)I(a_{ni}X_{ni} < -b_n) + (a_{ni}X_{ni} - b_n)I(a_{ni}X_{ni} > b_n); \\
&A = \bigcap_{i=1}^{n} (Y_{ni} = a_{ni}X_{ni}), \qquad B = \bar{A} = \bigcup_{i=1}^{n} (Y_{ni} \ne a_{ni}X_{ni}) = \bigcup_{i=1}^{n} \big(|a_{ni}X_{ni}| > b_n\big); \\
&E_{ni} = \left(\max_{1 \le j \le n} \left| \sum_{i=1}^{j} a_{ni}X_{ni} \right| > \varepsilon b_n \right).
\end{aligned}$$

It is easy to check that for all $\varepsilon > 0$,

$$E_{ni} = E_{ni}A \cup E_{ni}B \subset \left(\max_{1 \le j \le n} \left| \sum_{i=1}^{j} Y_{ni} \right| > \varepsilon b_n \right) \cup \left(\bigcup_{i=1}^{n} \big(|a_{ni}X_{ni}| > b_n\big) \right),$$

which implies that

$$\begin{aligned}
P(E_{ni}) &\le P\left(\max_{1 \le j \le n} \left| \sum_{i=1}^{j} Y_{ni} \right| > \varepsilon b_n \right) + P\left(\bigcup_{i=1}^{n} \big(|a_{ni}X_{ni}| > b_n\big) \right) \\
&\le P\left(\max_{1 \le j \le n} \left| \sum_{i=1}^{j} (Y_{ni} - EY_{ni}) \right| > \varepsilon b_n - \max_{1 \le j \le n} \left| \sum_{i=1}^{j} EY_{ni} \right| \right) \\
&\quad + \sum_{i=1}^{n} P\big(|a_{ni}X_{ni}| > b_n\big).
\end{aligned} \tag{3.5}$$

First, we shall show that

$$\frac{1}{b_n} \max_{1 \le j \le n} \left| \sum_{i=1}^{j} EY_{ni} \right| \to 0 \quad \text{as } n \to \infty. \tag{3.6}$$

For $0 < \alpha \leq 1$, it follows from (3.2) of Lemma 3.3, the Markov inequality, and $E|X|^\alpha < \infty$ that

$$\begin{aligned}
\frac{1}{b_n}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} EY_{ni}\right| &\leq C\frac{1}{b_n}\sum_{i=1}^{n}|EY_{ni}| \\
&\leq C\frac{1}{b_n}\sum_{i=1}^{n}E|a_{ni}X_{ni}|I\big(|a_{ni}X_{ni}|\leq b_n\big) + C\sum_{i=1}^{n}P\big(|a_{ni}X_{ni}|>b_n\big) \\
&\leq C\frac{1}{b_n}\sum_{i=1}^{n}\big(E|a_{ni}X|I\big(|a_{ni}X|\leq b_n\big) + b_nP\big(|a_{ni}X|>b_n\big)\big) \\
&\quad + C\sum_{i=1}^{n}P\big(|a_{ni}X|>b_n\big) \\
&\leq C\frac{1}{b_n^\alpha}\sum_{i=1}^{n}a_{ni}^\alpha E|X|^\alpha I\big(|a_{ni}X|\leq b_n\big) + C\frac{1}{b_n^\alpha}\sum_{i=1}^{n}a_{ni}^\alpha E|X|^\alpha \\
&\leq C(\log n)^{-\alpha/\gamma}E|X|^\alpha \to 0 \quad \text{as } n\to\infty.
\end{aligned} \tag{3.7}$$

From the definition of $Z_{ni} = a_{ni}X_{ni} - Y_{ni}$ we know that when $a_{ni}X_{ni} > b_n$, $0 < Z_{ni} = a_{ni}X_{ni} - b_n < a_{ni}X_{ni}$, and when $a_{ni}X_{ni} < -b_n$, $a_{ni}X_{ni} < Z_{ni} = a_{ni}X_{ni} + b_n < 0$. Hence, $|Z_{ni}| < |a_{ni}X_{ni}|I(|a_{ni}X_{ni}| > b_n)$.

For $1 < \alpha \leq 2$, it follows from $EX_{ni} = 0$, (3.3) of Lemma 3.3, and $E|X|^\alpha < \infty$ again that

$$\begin{aligned}
\frac{1}{b_n}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} EY_{ni}\right| &= \frac{1}{b_n}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} EZ_{ni}\right| \\
&\leq C\frac{1}{b_n}\sum_{i=1}^{n}E|Z_{ni}| \\
&\leq C\frac{1}{b_n}\sum_{i=1}^{n}E|a_{ni}X_{ni}|I\big(|a_{ni}X_{ni}|>b_n\big) \\
&\leq C\frac{1}{b_n}\sum_{i=1}^{n}E|a_{ni}X|I\big(|a_{ni}X|>b_n\big) \\
&\leq C\frac{1}{b_n^\alpha}\sum_{i=1}^{n}a_{ni}^\alpha E|X|^\alpha I\big(|a_{ni}X|>b_n\big) \\
&\leq C(\log n)^{-\alpha/\gamma}E|X|^\alpha \to 0 \quad \text{as } n\to\infty.
\end{aligned} \tag{3.8}$$

By (3.7) and (3.8) we immediately obtain (3.6). Hence, for n large enough,

$$P(E_n) \leq P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(Y_{ni} - EY_{ni})\right| > \frac{\varepsilon b_n}{2}\right) + \sum_{i=1}^{n}P\big(|a_{ni}X_{ni}|>b_n\big). \tag{3.9}$$

To prove (2.2), it suffices to show that

$$I \triangleq \sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(Y_{ni} - EY_{ni})\right| > \frac{\varepsilon b_n}{2}\right) < \infty, \tag{3.10}$$

$$J \triangleq \sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}P\big(|a_{ni}X_{ni}|>b_n\big)<\infty. \tag{3.11}$$

By Lemma 3.1 it obviously follows that $\{Y_{ni}-EY_{ni}, i\geq 1, n\geq 1\}$ is still an array of rowwise ANA random variables. Hence, it follows from the Markov inequality and Lemma 3.2 that

$$\begin{aligned}
I &\leq C\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{b_n^2}E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\right|^2\right)\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{b_n^2}\sum_{i=1}^{n}E|Y_{ni}-EY_{ni}|^2\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{b_n^2}\sum_{i=1}^{n}EY_{ni}^2\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{b_n^2}\sum_{i=1}^{n}E|a_{ni}X_{ni}|^2I\big(|a_{ni}X_{ni}|\leq b_n\big)+C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}P\big(|a_{ni}X_{ni}|>b_n\big)\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{b_n^2}\sum_{i=1}^{n}E|a_{ni}X|^2I\big(|a_{ni}X|\leq b_n\big)\\
&\quad+C\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{b_n^\alpha}\sum_{i=1}^{n}E|a_{ni}X|^\alpha I\big(|a_{ni}X|>b_n\big)\\
&\triangleq I_1+I_2.
\end{aligned} \tag{3.12}$$

From Lemma 3.4 (for $p=2$) and (2.1) we obtain that $I_1<\infty$. Hence, it follows from (3.2) of Lemma 3.3 and from (1.1) that

$$\begin{aligned}
I_2 &= C\sum_{n=1}^{\infty}\frac{1}{n^2}(\log n)^{-\alpha/\gamma}\sum_{i=1}^{n}E|a_{ni}X|^\alpha I\big(|a_{ni}X|^\alpha>n(\log n)^{\alpha/\gamma}\big)\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{n^2}(\log n)^{-\alpha/\gamma}\sum_{i=1}^{n}E|a_{ni}X|^\alpha I\left(|X|^\alpha>\frac{n(\log n)^{\alpha/\gamma}}{\sum_{i=1}^{n}|a_{ni}|^\alpha}\right)\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{n^2}(\log n)^{-\alpha/\gamma}\sum_{i=1}^{n}E|a_{ni}X|^\alpha I\big(|X|>(\log n)^{1/\gamma}\big)\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{n}(\log n)^{-\alpha/\gamma}E|X|^\alpha I\big(|X|>(\log n)^{1/\gamma}\big)\\
&= C\sum_{n=1}^{\infty}\frac{1}{n}(\log n)^{-\alpha/\gamma}\sum_{k=n}^{\infty}E|X|^\alpha I\big((\log k)^{1/\gamma}<|X|<\big(\log(k+1)\big)^{1/\gamma}\big)\\
&= C\sum_{k=1}^{\infty}E|X|^\alpha I\big((\log k)^{1/\gamma}<|X|<\big(\log(k+1)\big)^{1/\gamma}\big)\sum_{n=1}^{k}\frac{1}{n}(\log n)^{-\alpha/\gamma}.
\end{aligned}$$

Note that

$$\sum_{n=1}^{k}\frac{1}{n}(\log n)^{-\alpha/\gamma}=\begin{cases}C & \text{for } \alpha>\gamma,\\ C\log\log k & \text{for } \alpha=\gamma,\\ C(\log k)^{1-\alpha/\gamma} & \text{for } \alpha<\gamma.\end{cases}$$

Therefore, we obtain that

$$I_2 \le \begin{cases} CE|X|^\alpha & \text{for } \alpha > \gamma, \\ CE|X|^\alpha \log(1+|X|) & \text{for } \alpha = \gamma, \\ CE|X|^\gamma & \text{for } \alpha < \gamma. \end{cases}$$

Under the conditions of Theorem 2.1 it follows that $I_2 < \infty$.

By (3.3) of Lemma 3.3 and the proof of $I_2 < \infty$,

$$J \le \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{b_n^\alpha} \sum_{i=1}^{n} E|a_{ni}X|^\alpha I\bigl(|a_{ni}X| > b_n\bigr) < \infty. \tag{3.13}$$

The proof of Theorem 2.1 is completed. □

Proof of Theorem 2.2 For all $\varepsilon > 0$, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{n} E\left(\frac{1}{b_n} \max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| - \varepsilon\right)_+^q \\ &\quad = \sum_{n=1}^{\infty} \frac{1}{n} \int_0^\infty P\left(\frac{1}{b_n} \max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| - \varepsilon > t^{1/q}\right) dt \\ &\quad = \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 P\left(\frac{1}{b_n} \max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon + t^{1/q}\right) dt \\ &\qquad + \sum_{n=1}^{\infty} \frac{1}{n} \int_1^\infty P\left(\frac{1}{b_n} \max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon + t^{1/q}\right) dt \\ &\quad \le \sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right) \\ &\qquad + \sum_{n=1}^{\infty} \frac{1}{n} \int_1^\infty P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > b_n t^{1/q}\right) dt \\ &\quad \triangleq K_1 + K_2. \end{aligned} \tag{3.14}$$

To prove (2.3), it suffices to prove $K_1 < \infty$ and $K_2 < \infty$. By Theorem 2.1 we obtain that $K_1 < \infty$. By applying a similar notation and the methods of Theorem 2.1, for fixed $n \ge 1$, $i \ge 1$, and all $t \ge 1$, define

$$\begin{aligned} Y'_{ni} &= -b_n t^{1/q} I\bigl(a_{ni}X_{ni} < -b_n t^{1/q}\bigr) + a_{ni}X_{ni} I\bigl(|a_{ni}X_{ni}| \le b_n t^{1/q}\bigr) + b_n t^{1/q} I\bigl(a_{ni}X_{ni} > b_n t^{1/q}\bigr); \\ Z'_{ni} &= a_{ni}X_{ni} - Y'_{ni} \\ &= \bigl(a_{ni}X_{ni} + b_n t^{1/q}\bigr) I\bigl(a_{ni}X_{ni} < -b_n t^{1/q}\bigr) + \bigl(a_{ni}X_{ni} - b_n t^{1/q}\bigr) I\bigl(a_{ni}X_{ni} > b_n t^{1/q}\bigr); \\ A' &= \bigcap_{i=1}^{n} \bigl(Y'_{ni} = a_{ni}X_{ni}\bigr), \qquad B' = \bar{A}' = \bigcup_{i=1}^{n} \bigl(Y'_{ni} \ne a_{ni}X_{ni}\bigr) = \bigcup_{i=1}^{n} \bigl(|a_{ni}X_{ni}| > b_n t^{1/q}\bigr); \\ E'_{ni} &= \left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > b_n t^{1/q}\right). \end{aligned}$$

It is easy to check that for all $\varepsilon > 0$,

$$\begin{aligned} P(E'_{ni}) &\le P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} Y'_{ni}\right| > b_n t^{1/q}\right) + P\left(\bigcup_{i=1}^{n}\left(|a_{ni}X_{ni}| > b_n t^{1/q}\right)\right) \\ &\le P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}\left(Y'_{ni} - EY'_{ni}\right)\right| > b_n t^{1/q} - \max_{1\le j\le n}\left|\sum_{i=1}^{j} EY'_{ni}\right|\right) \\ &\quad + \sum_{i=1}^{n} P\left(|a_{ni}X_{ni}| > b_n t^{1/q}\right). \end{aligned} \tag{3.15}$$

First, we shall show that

$$\max_{t\ge1}\frac{1}{b_n t^{1/q}}\max_{1\le j\le n}\left|\sum_{i=1}^{j} EY'_{ni}\right| \to 0 \quad \text{as } n\to\infty. \tag{3.16}$$

Similarly to the proofs of (3.7) and (3.8), for $0 < \alpha \le 1$, it follows from (3.2) of Lemma 3.3, the Markov inequality, and $E|X|^{\alpha} < \infty$ that

$$\begin{aligned} \max_{t\ge1}\frac{1}{b_n t^{1/q}}\max_{1\le j\le n}\left|\sum_{i=1}^{j} EY'_{ni}\right| &\le C\max_{t\ge1}\frac{1}{b_n t^{1/q}}\sum_{i=1}^{n}\left|EY'_{ni}\right| \\ &\le C\max_{t\ge1}\frac{1}{b_n t^{1/q}}\sum_{i=1}^{n} E|a_{ni}X_{ni}|I\left(|a_{ni}X_{ni}| \le b_n t^{1/q}\right) \\ &\quad + C\max_{t\ge1}\sum_{i=1}^{n} P\left(|a_{ni}X_{ni}| > b_n t^{1/q}\right) \\ &\le C\max_{t\ge1}\frac{1}{b_n t^{1/q}}\sum_{i=1}^{n} E|a_{ni}X|I\left(|a_{ni}X| \le b_n t^{1/q}\right) \\ &\quad + C\max_{t\ge1}\sum_{i=1}^{n} P\left(|a_{ni}X| > b_n t^{1/q}\right) \\ &\le C\max_{t\ge1}\frac{1}{b_n^{\alpha} t^{\alpha/q}}\sum_{i=1}^{n} a_{ni}^{\alpha}E|X|^{\alpha}I\left(|a_{ni}X| \le b_n t^{1/q}\right) \\ &\quad + C\max_{t\ge1}\frac{1}{b_n^{\alpha} t^{\alpha/q}}\sum_{i=1}^{n} a_{ni}^{\alpha}E|X|^{\alpha} \\ &\le C(\log n)^{-\alpha/\gamma}E|X|^{\alpha} \to 0 \quad \text{as } n\to\infty. \end{aligned} \tag{3.17}$$

Noting that $|Z'_{ni}| < |a_{ni}X_{ni}|I(|a_{ni}X_{ni}| > b_n t^{1/q})$, for $1 < \alpha \le 2$, it follows from $EX_n = 0$, (3.3) of Lemma 3.3, and $E|X|^{\alpha} < \infty$ again that

$$\begin{aligned} \max_{t\ge1}\frac{1}{b_n t^{1/q}}\max_{1\le j\le n}\left|\sum_{i=1}^{j} EY'_{ni}\right| &= \max_{t\ge1}\frac{1}{b_n t^{1/q}}\max_{1\le j\le n}\left|\sum_{i=1}^{j} EZ'_{ni}\right| \\ &\le C\max_{t\ge1}\frac{1}{b_n t^{1/q}}\sum_{i=1}^{n} E\left|Z'_{ni}\right| \end{aligned}$$

$$
\begin{aligned}
&\le C \max_{t\ge 1} \frac{1}{b_n t^{1/q}} \sum_{i=1}^{n} E|a_{ni}X_{ni}| I\bigl(|a_{ni}X_{ni}| > b_n t^{1/q}\bigr) \\
&\le C \max_{t\ge 1} \frac{1}{b_n t^{1/q}} \sum_{i=1}^{n} E|a_{ni}X| I\bigl(|a_{ni}X| > b_n t^{1/q}\bigr) \\
&\le C \max_{t\ge 1} \frac{1}{b_n^{\alpha} t^{\alpha/q}} \sum_{i=1}^{n} a_{ni}^{\alpha} E|X|^{\alpha} I\bigl(|a_{ni}X| > b_n t^{1/q}\bigr) \\
&\le C(\log n)^{-\alpha/\gamma} E|X|^{\alpha} \to 0 \quad \text{as } n \to \infty. \qquad (3.18)
\end{aligned}
$$

To prove $K_2 < \infty$, it suffices to show that

$$
K_{21} \triangleq \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} P\left(\max_{1\le j\le n} \left| \sum_{i=1}^{j} \bigl(Y'_{ni} - EY'_{ni}\bigr) \right| > \frac{b_n t^{1/q}}{2} \right) dt < \infty, \qquad (3.19)
$$

$$
K_{22} \triangleq \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \sum_{i=1}^{n} P\bigl(|a_{ni}X_{ni}| > b_n t^{1/q}\bigr)\, dt < \infty. \qquad (3.20)
$$

Hence, it follows from the Markov inequality and Lemma 3.2 that

$$
\begin{aligned}
K_{21} &\le C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \frac{1}{b_n^2 t^{2/q}} E\left(\max_{1\le j\le n} \left| \sum_{i=1}^{j} \bigl(Y'_{ni} - EY'_{ni}\bigr) \right|^2 \right) dt \\
&\le C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \frac{1}{b_n^2 t^{2/q}} \sum_{i=1}^{n} E\bigl|Y'_{ni} - EY'_{ni}\bigr|^2\, dt \\
&\le C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \sum_{i=1}^{n} P\bigl(|a_{ni}X_{ni}| > b_n t^{1/q}\bigr)\, dt \\
&\quad + C \sum_{n=1}^{\infty} \frac{1}{n b_n^2} \int_1^{\infty} \frac{1}{t^{2/q}} \sum_{i=1}^{n} E|a_{ni}X_{ni}|^2 I\bigl(|a_{ni}X_{ni}| \le b_n t^{1/q}\bigr)\, dt \\
&\le C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \sum_{i=1}^{n} P\bigl(|a_{ni}X| > b_n t^{1/q}\bigr)\, dt \\
&\quad + C \sum_{n=1}^{\infty} \frac{1}{n b_n^2} \int_1^{\infty} \frac{1}{t^{2/q}} \sum_{i=1}^{n} E|a_{ni}X|^2 I\bigl(|a_{ni}X| \le b_n\bigr)\, dt \\
&\quad + C \sum_{n=1}^{\infty} \frac{1}{n b_n^2} \int_1^{\infty} \frac{1}{t^{2/q}} \sum_{i=1}^{n} E|a_{ni}X|^2 I\bigl(b_n < |a_{ni}X| \le b_n t^{1/q}\bigr)\, dt. \qquad (3.21)
\end{aligned}
$$

For $0 < q < \alpha$ and $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n)$, similarly as in the proof of $I_2 < \infty$, we obtain that

$$
\begin{aligned}
K_{22} &\le \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \sum_{i=1}^{n} P\bigl(|a_{ni}X| > b_n t^{1/q}\bigr)\, dt \\
&\le C \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} \sum_{i=1}^{n} P\bigl(|a_{ni}X| > b_n t^{1/q}\bigr)\, dt \\
&\le C \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} \sum_{i=1}^{n} P\left(\frac{|a_{ni}X|^q}{b_n^q} > t \right) dt
\end{aligned}
$$

$$\begin{aligned}
&\leq C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{E|a_{ni}X|^q}{b_n^q}I\bigl(|a_{ni}X|>b_n\bigr)\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{nb_n^{\alpha}}\sum_{i=1}^{n}E|a_{ni}X|^{\alpha}I\bigl(|a_{ni}X|>b_n\bigr)\\
&<\infty \quad \text{(see the proof of } I_2<\infty).
\end{aligned}$$

For $0<q<\alpha\leq 2$, it follows from Lemma 3.4 and (2.1) that

$$\begin{aligned}
\nabla_1 &\triangleq \sum_{n=1}^{\infty}\frac{1}{nb_n^2}\int_1^{\infty}\frac{1}{t^{2/q}}\sum_{i=1}^{n}E|a_{ni}X|^2 I\bigl(|a_{ni}X|\leq b_n\bigr)\,dt\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{b_n^2}\sum_{i=1}^{n}E|a_{ni}X|^2 I\bigl(|a_{ni}X|\leq b_n\bigr)<\infty.
\end{aligned}$$

Taking $t=x^q$, by the Markov inequality from (3.2) of Lemma 3.3 it follows that

$$\begin{aligned}
\nabla_2 &\triangleq \sum_{n=1}^{\infty}\frac{1}{nb_n^2}\int_1^{\infty}\frac{1}{t^{2/q}}\sum_{i=1}^{n}E|a_{ni}X|^2 I\bigl(b_n<|a_{ni}X|\leq b_n t^{1/q}\bigr)\,dt\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{nb_n^2}\int_1^{\infty}x^{q-3}\sum_{i=1}^{n}E|a_{ni}X|^2 I\bigl(b_n<|a_{ni}X|\leq b_n x\bigr)\,dx\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{nb_n^2}\sum_{m=1}^{\infty}\int_m^{m+1}x^{q-3}\sum_{i=1}^{n}E|a_{ni}X|^2 I\bigl(b_n<|a_{ni}X|\leq b_n x\bigr)\,dx\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{nb_n^2}\sum_{m=1}^{\infty}m^{q-3}\sum_{i=1}^{n}E|a_{ni}X|^2 I\bigl(b_n<|a_{ni}X|\leq b_n(m+1)\bigr)\\
&= C\sum_{n=1}^{\infty}\frac{1}{nb_n^2}\sum_{i=1}^{n}\sum_{m=1}^{\infty}\sum_{s=1}^{m}m^{q-3}E|a_{ni}X|^2 I\bigl(b_n s<|a_{ni}X|\leq b_n(s+1)\bigr)\\
&= C\sum_{n=1}^{\infty}\frac{1}{nb_n^2}\sum_{i=1}^{n}\sum_{s=1}^{\infty}E|a_{ni}X|^2 I\bigl(b_n s<|a_{ni}X|\leq b_n(s+1)\bigr)\sum_{m=s}^{\infty}m^{q-2}\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{nb_n^2}\sum_{i=1}^{n}\sum_{s=1}^{\infty}E|a_{ni}X|^2 I\bigl(b_n s<|a_{ni}X|\leq b_n(s+1)\bigr)s^{q-2}\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{nb_n^q}\sum_{i=1}^{n}E|a_{ni}X|^q I\bigl(|a_{ni}X|>b_n\bigr)\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{nb_n^{\alpha}}\sum_{i=1}^{n}E|a_{ni}X|^{\alpha} I\bigl(|a_{ni}X|>b_n\bigr)\\
&<\infty \quad \text{(see the proof of } I_2<\infty).
\end{aligned}$$

The proof of Theorem 2.2 is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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