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A note on the rank inequality for diagonally magic matrices

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Abstract

We prove that a positive matrix with all permutation products equal is diagonally equivalent to J, the all-1s matrix. Then we give a simple proof of the rank inequality for diagonally magic matrices (J. Inequal. Appl. 2015:318, 2015).

MSC: 15A03; 15A06

Keywords: rank; diagonally magic matrices; positive matrix

1 Introduction

We denote by $\mathbb{C}^{n\times n}$ and $\mathbb{R}^{n\times n}$ the sets of $n \times n$ complex matrices and $n \times n$ real matrices, respectively. For a positive integer n, let S_n be the set of all $n!$ permutations of $\{1, 2, \dots, n\}$. If $A = (a_{i,j}) \in \mathbb{C}^{n\times n}$ and $\sigma \in S_n$, then the sequence $a_{1,\sigma(1)}, a_{2,\sigma(2)}, \dots, a_{n,\sigma(n)}$ is called the *transversal* of A [2]. Let $A \in \mathbb{C}^{n\times n}$, $1 \le i_1 \le i_2 \le \cdots \le i_k \le n$, and $1 \le j_1 \le j_2 \le \cdots \le j_s \le n$. We denote by $A[i_1, i_2, \dots, i_k | j_1, j_2, \dots, j_s]$ the $k \times s$ submatrix of A that lies in the rows $i_1, i_2, \dots, i_k$ and columns $j_1, j_2, \dots, j_s$. Denote by $A(i_1, i_2, \dots, i_k | j_1, j_2, \dots, j_s)$ the $(n-k) \times (n-s)$ submatrix of A obtained by deleting the rows $i_1, i_2, \dots, i_k$ and columns $j_1, j_2, \dots, j_s$. A matrix $A = (a_{i,j}) \in \mathbb{C}^{n\times n}$ is called *diagonally magic* if

$$\sum_{i=1}^{n} a_{i,\sigma(i)} = \sum_{i=1}^{n} a_{i,\pi(i)}$$

for all $\sigma, \pi \in S_n$.

Obviously, the zero matrix $0_{n\times n}$ and $J = [1]_{n\times n}$, the matrix of all ones, are diagonally magic matrices. In [1], we prove that

$$B_n = \begin{pmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & \ddots & \vdots \\ (n-1)n+1 & (n-1)n+2 & \cdots & n^2 \end{pmatrix}$$

and the Henkel matrix

$$C_n = \begin{pmatrix} 1 & 2 & \cdots & n \\ 2 & 3 & \cdots & n+1 \\ \vdots & \vdots & \ddots & \vdots \\ n & n+1 & \cdots & 2n-1 \end{pmatrix}$$

are diagonally magic matrices. So, there are a lot of diagonally magic matrices. The non-negative matrices B_n and C_n have been a hot research area [3, 4].

2 Main result

The rank inequality for diagonally magic matrices can be stated as follows.

Theorem 2.1 ([1], Theorem 2.1) *Let $A \in \mathbb{C}^{n\times n}$ be a diagonally magic matrix. Then* $\operatorname{rank}(A) \leq 2$.

There are diagonally magic matrices of ranks 0, 1, 2. Indeed, $\operatorname{rank}(0_{n\times n}) = 0$, $\operatorname{rank}([1]_{n\times n}) = 1$, and $\operatorname{rank}(B_n) = \operatorname{rank}(C_n) = 2$.

The purpose of this note is to give a simple proof of Theorem 2.1. Our proof depends only on the following fact.

Theorem 2.2 *Let $C = (c_{i,j}) \in \mathbb{R}^{n\times n}$ be a positive matrix with*

$$\prod_{i=1}^{n} c_{i,\gamma(i)} = \prod_{j=1}^{n} c_{j,\tau(j)}$$

for all $\gamma, \tau \in S_n$. Then there exist positive diagonal matrices $X = \operatorname{diag}(x_1, x_2, \ldots, x_n)$ and $Y = \operatorname{diag}(y_1, y_2, \ldots, y_n)$ such that

$$C = XJY.$$

Proof Let B be a $k \times k$ submatrix of C. Then there are row and column indices $\alpha = (i_1, i_2, \ldots, i_k)$ and $\beta = (j_1, j_2, \ldots, j_k)$ such that $B = C[\alpha|\beta]$. Note that the union of a transversal of B and a transversal of $C(\alpha|\beta)$ is a transversal of C. Choose an arbitrary but fixed transversal T of the square matrix $C(\alpha|\beta)$. For any $\sigma, \pi \in S_k$, $c_{i_1,j_{\sigma(1)}}, \ldots, c_{i_k,j_{\sigma(k)}}$ and the entries in T constitute a transversal of C, whereas $c_{i_1,j_{\pi(1)}}, \ldots, c_{i_k,j_{\pi(k)}}$ and the entries in T also constitute a transversal of C. Let b be the product of the entries in T. Obviously, $b > 0$. Since

$$\prod_{i=1}^{n} c_{i,\gamma(i)} = \prod_{j=1}^{n} c_{j,\tau(j)}$$

for all $\gamma, \tau \in S_n$, we have

$$b\prod_{t=1}^{k} c_{i_t,j_{\sigma(t)}} = b\prod_{t=1}^{k} c_{i_t,j_{\pi(t)}},$$

which yields

$$\prod_{t=1}^{k} c_{i_t,j_{\sigma(t)}} = \prod_{t=1}^{k} c_{i_t,j_{\pi(t)}}.$$

Particularly, this shows that any 2×2 submatrix

$$B = \begin{pmatrix} c_{i_1,j_1} & c_{i_1,j_2} \\ c_{i_2,j_1} & c_{i_2,j_2} \end{pmatrix}$$

of C satisfies

$$c_{i_1,j_1} c_{i_2,j_2} = c_{i_1,j_2} c_{i_2,j_1}. \tag{1}$$

For any $x_1 > 0$, let

$$y_j = \frac{c_{1,j}}{x_1} \tag{2}$$

for $j = 1, 2, \ldots, n$ and

$$x_i = \frac{c_{i,1}}{c_{1,1}} x_1 \tag{3}$$

for $i = 2, 3, \ldots, n$. According to (1), (2), and (3), we have

$$c_{i,j} = \frac{c_{i,1} c_{1,j}}{c_{1,1}} = x_i y_j$$

for all $i, j = 1, 2, \ldots, n$. Let $X = \operatorname{diag}(x_1, x_2, \ldots, x_n)$ and $Y = \operatorname{diag}(y_1, y_2, \ldots, y_n)$. Obviously, X and Y are positive diagonal matrices, and we have

$$C = XJY.$$

This completes the proof. □

We are now ready to present our proof of Theorem 2.1.

Proof of Theorem 2.1 First, let A be real. Let A be a diagonally magic matrix. Then the elementwise exponential $C = \exp(A) := (c_{i,j}) \in \mathbb{R}^{n \times n}$ is a positive matrix with all permutation products equal. Hence, by Theorem 2.2 it is diagonally equivalent to J, the all-1s matrix, that is,

$$c_{i,j} = x_i y_j, \quad i, j = 1, 2, \ldots, n,$$

for suitable positive vectors $x = (x_1, x_2, \ldots, x_n)^T$ and $y = (y_1, y_2, \ldots, y_n)^T$. Hence, if $q = (\log(x_1), \log(x_2), \ldots, \log(x_n))^T$ and $r = (\log(y_1), \log(y_2), \ldots, \log(y_n))^T$, then

$$a_{i,j} = \log(x_i) + \log(y_j), \quad i, j = 1, 2, \ldots, n.$$

Hence,

$$A = e_n \cdot q^T + r \cdot e_n^T,$$

where $e_n = (\underbrace{1, 1, \ldots, 1}_{n})^T$. Thus, A is the sum of two matrices of rank 1 and, hence, at most of rank 2.

Now let A be complex, so that

$$A = B + iC \quad (B, C \text{ real}).$$

Since A is a diagonally magic matrix, so are B and C. Hence, both B and C are of the form

$$B = e_n \cdot q_1^T + r_1 \cdot e_n^T \quad \text{with } q_1, r_1 \text{ real},$$
$$C = e_n \cdot q_2^T + r_2 \cdot e_n^T \quad \text{with } q_2, r_2 \text{ real},$$

and hence

$$A = e_n \cdot (q_1 + iq_2)^T + (r_1 + ir_2) \cdot e_n^T. \tag{4}$$

The matrix A has rank at most 2. This completes the proof. □

According to (4), we obtain that a diagonally magic matrix A can be presented in the form

$$A = e_n \cdot x^T + y \cdot e_n^T = \begin{pmatrix} x_1 + y_1 & x_1 + y_2 & \cdots & x_1 + y_n \\ x_2 + y_1 & x_2 + y_2 & \cdots & x_2 + y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n + y_1 & x_n + y_2 & \cdots & x_n + y_n \end{pmatrix}. \tag{5}$$

If $A = (a_{i,j}) \in \mathbb{C}^{n \times n}$ is a diagonally magic matrix, for any $x_1 \in \mathbb{C}$, let

$$y_j = a_{1,j} - x_1$$

for $j = 1, 2, \ldots, n$ and

$$x_i = a_{i,1} - a_{1,1} + x_1$$

for $i = 2, 3, \ldots, n$. By (5) we have

$$a_{i,j} = x_i + y_j$$

for all $i, j = 1, 2, \ldots, n$. For example,

$$B_n = \begin{pmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & \ddots & \vdots \\ (n-1)n+1 & (n-1)n+2 & \cdots & n^2 \end{pmatrix}$$

$$= \begin{pmatrix} 0+1 & 0+2 & \cdots & 0+n \\ n+1 & n+2 & \cdots & n+n \\ \vdots & \vdots & \ddots & \vdots \\ (n-1)n+1 & (n-1)n+2 & \cdots & (n-1)n+n \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} 1 & 2 & \cdots & n \\ 2 & 3 & \cdots & n+1 \\ \vdots & \vdots & \ddots & \vdots \\ n & n+1 & \cdots & 2n-1 \end{pmatrix} = \begin{pmatrix} 1+0 & 1+1 & \cdots & 1+(n-1) \\ 2+0 & 2+1 & \cdots & 2+(n-1) \\ \vdots & \vdots & \ddots & \vdots \\ n+0 & n+1 & \cdots & n+(n-1) \end{pmatrix}.$$

By (5) we can get the characteristic polynomial, the eigenvalues, and the eigenvectors of A. In fact, the characteristic polynomial of A is

$$p_A(\lambda) = \lambda^{n-2}\left(\lambda^2 - \lambda\left(\sum_{i=1}^{n}(x_i + y_i)\right) + \sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j - n\sum_{i=1}^{n}(x_i y_i)\right). \tag{6}$$

From (6) we can see that *the algebraic multiplicity of the eigenvalue* 0 of the diagonally magic matrix A is *at least* $n-2$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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