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Constrained extremal solutions of multi-valued linear inclusions in Banach spaces

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Abstract
Let X and Y be Banach spaces, L be a linear manifold in $X \times Y$, or, equivalently, the graph of a multi-valued linear operator from X to Y, and let S be a prescribed hyperplane in X, *i.e.* $S = g + N$. A central problem in our general setting is to determine, for a given $y \in Y$, a vector $w \in S \cap D(L)$ such that, for some $z \in L(w)$, $\|z - y\| = \operatorname{dist}(y, L(S \cap D(L)))$, such a vector w is called the constrained extremal solution of multi-valued linear inclusions $y \in L(x)$ in Banach spaces. We establish three equivalent characterizations of constrained extremal solution of linear inclusions in Banach spaces by means of the algebraic operator parts, the metric generalized inverse of multi-valued linear operator L, and the dual mapping of the spaces. As follows from the main results in this paper, we may get the constrained extremal solution of multi-valued linear inclusions, by using the extremal solution of some interrelated multi-valued linear inclusions in the same spaces. The setting in this paper includes large classes of constrained extremal problems and optimal control problems subject to generalized boundary conditions.

MSC: 47A05

Keywords: Banach space; linear manifold; constrained extremal solution; metric generalized inverse; algebraic operator part

1 Introduction

For convenience, we first recall some related notations. Throughout this paper, X and Y denote Banach spaces and A denotes a linear manifold in the product space $X \times Y$. We may view A as a multi-valued linear operator from X to Y by taking $A(x) = \{y : \{x, y\} \in A\}$. The domain, range, and null space of A are defined, respectively, by

$$D(A) = \{x \in X : \{x, y\} \text{ for some } y \in Y\};$$
$$R(A) = \{y \in Y : \{x, y\} \text{ for some } x \in X\};$$
$$N(A) = \{x \in D(A) : \{x, \theta\} \in A\}.$$

It is well known that the quadratic control problem subject to a certain class of boundary conditions can be equivalently formulated as the problem of finding a least-squares solutions (or extremal solutions) of an appropriate linear operator equations in Hilbert spaces (or Banach spaces). When the generalized quadratic cost function and the generalized boundary conditions are involved, the problem can be reformulated as a constrained least-squares solution (or extremal solution) of multi-valued linear operators $y \in A(x)$ between Hilbert spaces (or Banach spaces) X and Y (see [1]). If X and Y are Hilbert spaces, the orthogonal operator parts, the orthogonal generalized inverse of a linear manifold A in $X \times Y$ and the least-squares solutions or the constrained least-squares solutions of multi-valued linear operators $y \in A(x)$ were investigated by Lee and Nashed [1–3]. If X and Y are Banach spaces, Lee and Nashed [4] also introduced a concept of a generalized inverse $A^{\#}$ for the linear manifold A in $X \times Y$ by means of algebraic projection and topological projection. In order to give the characterization of the set of all extremal solutions or least-extremal solutions of a linear inclusion $y \in A(x)$ in Banach space, in 2005, Wang and Liu [5] introduced the concept of the metric generalized inverse $A^{\#}$ by means of the metric projection, which is nonlinear in general. In 2012, Wang *et al.* [6] also gave the criteria for the metric generalized inverse of multi-valued linear operators in Banach space.

Let L be a linear manifold in $X \times Y$, or, equivalently, the graph of a multi-valued linear operator from X to Y and let S be a prescribed hyperplane in X, *i.e.* $S = g + N$, we denote $A := L|_N$. The problem in our general setting is to determine, for a given $y \in Y$, a vector $\omega \in S \cap D(A)$ such that, for some $z \in A|_S(\omega)$, $\|z - y\| = \operatorname{dist}(y, R(A|_S))$, such a vector w is called the constrained extremal solution of multi-valued linear inclusions $y \in A(x)$ in Banach space. The main purpose of this paper is to investigate the constrained extremal solution problem in Banach spaces in an abstract general setting. We first establish three equivalent characterizations of a constrained extremal solution of linear inclusions in Banach spaces by means of the algebraic operator parts, the metric generalized inverse of multi-valued linear operator, and the dual mapping of the spaces. It follows from the main results in this paper that we may get the constrained extremal solution of multi-valued linear inclusions, by using the extremal solution of some interrelated multi-valued linear inclusions in the same spaces, which are well investigated by using the algebraic operator parts, the metric generalized inverse of multi-valued linear operator in [5] and [6]. The setting in this paper includes large classes of constrained extremal problems and optimal control problems subject to generalized boundary conditions [7].

In this paper, X, Y, and Z denote Banach spaces. The following are standard notations (see [1–3, 7]), but for convenience, we recall them again. For $A, B \subset X \times Y$, $C \subset Z \times X$,

$$AC = \Big\{\{z, y\} : \{z, x\} \in C, \{x, y\} \in A\Big\};$$

$$\alpha A = \Big\{\{x, \alpha y\} : \{x, y\} \in A\Big\}, \quad \alpha \in R^1;$$

$$A \dotplus B = \{a + b : a \in A, b \in B\};$$

$$A + B = \Big\{\{x, y + z\} : \{x, y\}, \{x, z\}\Big\}.$$

The main mathematics tools in this investigation are the algebraic operator part and metric generalized inverse of linear manifold in Banach spaces, we recall and describe them in Section 2 (see [5] and [6]). The other mathematics method is the generalized orthogonal decomposition theorem in Banach space, which is given by one of the authors in another paper (see Lemma 2.4 in Section 2).

2 Preliminaries and basic notions

Let X and Y be Banach spaces, and A be a linear manifold (or linear subspace) in the product space $X \times Y$. The inverse relation A^{-1}, which is the graph inverse of A, always exists and is given by

$$A^{-1} = \{\{y,x\} : \{x,y\} \in A\}.$$

If S is a set in X, then

$$A(S) = \{y : \{x,y\} \in A, \text{ for some } x \in S\}.$$

It is possible that $A(S)$ is empty. The restriction of A to S will be denoted by

$$A|_S := \{\{x,y\} : \{x,y\} \in A \text{ and } x \in S\}.$$

If $T : X \to Y$ is a single-valued operator from X to Y with domain $D(T)$, then the gragh of T, denote $Gr(T)$, defined by

$$Gr(T) = \{(x, T(x)) : x \in D(T)\}.$$

For a multi-valued linear operator A from X into Y, we may introduce a single-valued operator from $D(A)$ into Y, denoted $A_{S,P}$, which is defined as follows.

Definition 2.1 [2] Let X and Y be Banach spaces, A be a linear manifold in the product space $X \times Y$ and P be an algebraic projection from Y onto $A(\theta)$. Then the composite relation

$$A_{S,P} := Gr(I - P)A,$$

where I is the identity operator on Y, is called an algebraic operator part of A.

In this case, for any $x \in D(A)$, we may have $A(x) = A_{S,P}(x) + A(\theta)$ and express the variational set $A(x)$ as a variable $A_{S,P}(x)$ plus the fixed set $A(\theta)$. Since $A(\theta)$ is a fixed subspace of Y, then there forever is an algebraic operator part of A.

Next, we introduce the concept of a constrained extremal solution of multi-valued linear inclusions in Banach spaces.

Definition 2.2 Let X, Y be Banach spaces, A be a linear manifold in the product space $X \times Y$, S be a set in X and $y \in Y$. Then $u \in X$ is called a constrained extremal solution of the linear inclusion $y \in A(x)$ with respect to S if

(i) $u \in D(A) \cap S$;

(ii) there exists $z \in A|_S(u)$ such that

$$\|y - z\| = \operatorname{dist}(y, R(A|_S)),$$

where $\operatorname{dist}(y, R(A|_S)) = \inf_{z \in R(A|_S)} \|y - z\|$.

If $S = X$, the constrained extremal solution of the linear inclusion $y \in A(x)$ with respect to S is just the extremal solution or the extremal solution, which was defined in [5].

Now we recall some notions and results in [5], which were used in this paper on many occasions.

A subset G in a Banach space X is said to be proximal if every element $x \in X$ has at least one element of best approximation in G, *i.e.*

$$\mathcal{P}_G(x) = \left\{x_0 \in G : \|x - x_0\| = \inf_{y \in G} \|x - y\|\right\} \neq \emptyset.$$

G is said to be a semi-Chebyshev set, if every element $x \in X$ has at most one element of best approximation in G, *i.e.* $x \in X$, $x_1, x_2 \in \mathcal{P}_G(x)$ implies $x_1 = x_2$. G is said a Chebyshev set if it is simultaneously a proximal and a semi-Chebyshev set (see [8]). When G is a Chebyshev set, we denote $\mathcal{P}_G(x) = \{\pi_G(x)\}$, where π_G is called the metric projector from X onto G.

It is well known that if X is a reflexive Banach space and $G \subset X$ is a convex closed set, then G is a proximal set, while if X is a strictly convex Banach space and G is a convex closed set, then G is a semi-Chebyshev set (see [8]).

We may use some properties of the metric projector, now we recall them.

Proposition 2.3 [8] *Let X be a Banach space and G a Chebyshev subspace of X. Then*
- (i) $\pi_G(x) = x$ *if* $x \in G$;
- (ii) $\pi_G^2(x) = \pi_G(x)$ *for any* $x \in X$;
- (iii) $\pi_G(\lambda x) = \lambda \pi_G(x)$ *for any* $x \in X$ *and* $\lambda \in R^1$;
- (iv) $\pi_G(x + y) = \pi_G(x) + y$ *for any* $x \in X$, $y \in G$.

Proof See Theorem 4.1 in [8]. □

We also use the dual mapping of Banach space, let us recall it.

Let X be a Banach space, the set-valued mapping $F_X : X \to 2^{X^*}$, defined by

$$F_X(x) = \left\{x^* \in X^* : \langle x^*, x\rangle = \left\|x^*\right\|^2 = \|x\|^2\right\}$$

for $x \in X$, is called the dual mapping of X, where $\langle x^*, x\rangle$ denotes the value of the functional $x^* \in X^*$ on $x \in X$ (see [9]).

In Banach space, there is no the concept of the orthogonal property just as in Hilbert space. By using the dual mapping of the Banach space X and the Chebyshev property of subspace G, we can extend the Riesz orthogonal decomposition theorem from Hilbert space into Banach space.

Lemma 2.4 [10] (Generalized orthogonal decomposition theorem) *Let G be a Chebyshev subspace. Then for any $x \in X$, we have a unique decomposition*

$$x = \pi_G(x) + x_1, \quad x_1 \in F_X^{-1}\left(G^\perp\right),$$

where $G^\perp = \{x^* \in X^* : \langle x^*, x\rangle = 0, \forall x \in G\}$, $F_X^{-1}(G^\perp) = \{x : F_X(x) \cap G^\perp \neq \phi\}$, π_G *is the metric projector. In other words, we have*

$$X = G \dotplus F_X^{-1}\left(G^\perp\right).$$

Proof See Lemma 2.2 in [5], also see [10] and [11]. □

Remark 2.5 In the above lemma, if $X = H$ is a Hilbert space, $X^* = X$, for any closed subspace G, it must be a Chebyshev space, the metric projector π_G is just the orthogonal projector P_G, and $F_X = I_X$, the identity of X, then $X = G \dotplus F_X^{-1}(G^\perp)$ become

$$H = G \oplus G^\perp.$$

which is just Riesz orthogonal decomposition theorem.

Next, we recall the concept of the metric generalized inverse $A^\#$, which is a single-valued operator of a multi-valued linear operator A from X into Y. By means of the metric generalized inverse $A^\#$, we can express the constrained extremal solution of multi-valued linear inclusions $y \in A(x)$ in Banach space.

Definition 2.6 [5] Let X and Y be Banach spaces, $A \subset X \times Y$ be a linear manifold, $N(A)$ and $R(A)$ be Chebyshev subspaces in X and Y, respectively, $\pi_{N(A)} : X \to N(A)$ and $\pi_{R(A)} : Y \to R(A)$ be the metric projectors. The metric generalized inverse $A^\#$ of A is defined by

$$A^\# = \{\{y, (I_{D(A)} - \pi_{N(A)})(g)\} : y \in Y \text{ and } \{g, \pi_{R(A)}(y)\} \in M\}.$$

Remark 2.7 If both X and Y are Hilbert spaces, the metric generalized inverse $A^\#$ of A is just the orthogonal generalized inverse (see [1–3, 7]).

Remark 2.8 If X and Y are Banach spaces, $T : X \to Y$ is a linear operator, and $N(T)$, $R(T)$ are Chebyshev subspaces in X and Y, respectively, then $T^\#$ is just the Moore-Penrose metric generalized inverse of T, denoted by T^M.

For convenience we list them as the following propositions the results in [5].

Proposition 2.9 [5] *Let X and Y be Banach spaces, $A \subset X \times Y$ be a linear manifold, $N(A)$ and $R(A)$ be Chebyshev subspaces in X and Y, respectively. Then, for any $y \in Y \backslash R(A)$, the following statements are equivalent*:

(i) *$u \in D(A)$ is the extremal solution of linear inclusion $y \in A(x)$*;
(ii) *$u \in D(A)$ and $\pi_{R(A)}(y) \in A(u)$*;
(iii) *$u \in D(A)$ and $y \in A(u) \dotplus F_Y^{-1}(R(A)^\perp)$.*

Proof See Theorem 5.1 in [5]. □

Proposition 2.10 [5] *Let X and Y be Banach spaces, $A \subset X \times Y$ be a linear manifold, $N(A)$ and $R(A)$ be Chebyshev subspaces in X and Y, respectively. Then we have the following*:

(i) *Let K be any algebraic operator part of A^{-1}, then the coset*

$$K(\pi_{R(A)}(y)) + N(A)$$

is the set of all extremal solutions of $y \in A(x)$.

(ii) *Let $A^\#$ be the metric generalized inverse of A, then the coset*

$$A^\#(y) + N(A)$$

is the set of all extremal solutions of $y \in A(x)$.

(iii) *$u = A^{\#}(y)$ is the unique least extremal solution of $y \in A(x)$.*

Proof See Theorem 5.2 in [5], also see [12]. □

3 Main theorems

In this section, we consider the constrained extremal problems for a linear inclusions restricted to a hyperplane in Banach space. By using Proposition 2.3, we establish several equivalent characterizations of the constrained extremal solution of linear inclusions in Banach spaces by means of the algebraic operator parts, the metric generalized inverse of multi-valued linear operator, and the dual mapping of the spaces. It follows from these results that we may get the constrained extremal solution of multi-valued linear inclusions by using the extremal solution of some interrelated multi-valued linear inclusions in the same space, which are well investigated in [5] and [6]. These characterizations involve algebraic operator parts, the metric generalized inverse, and the dual mapping of the spaces.

Theorem 3.1 *Let X and Y be Banach spaces, $L \subset X \times Y$ be a multi-valued linear operator from X to Y, $N \subset X$ be a subspace, P be an algebraic projector from Y onto the subspace $L(\theta)$, $L_{S,P}$ be any fixed algebraic operator part of L with respect to the projector P. Let*

$$S = g + N \quad \textit{and} \quad A := L|_N,$$

where $g \in D(L)$. Suppose that $N(A)$ and $R(A)$ are Chebyshev subspaces in X and Y, respectively. Then the following are equivalent:

(1) *$w \in D(L) \cap S$ is a constrained extremal solution of the linear inclusion $y \in L(x)$ with respect to S*;

(2) *$k := g - w \in D(L) \cap N$ is an extremal solution of the linear inclusion $L_{S,P}(g) - y \in A(x)$*;

(3) *$w \in D(L) \cap S$ and*

$$L_{S,P}(w) - y \in L(\theta) + F_Y^{-1}\bigl(R(A)^{\perp}\bigr); \tag{3.1}$$

(4) *$g \in D(L)$ such that*

$$L_{S,P}(g) - y \in R(A) \dotplus F_Y^{-1}\bigl(R(A)^{\perp}\bigr). \tag{3.2}$$

Proof (1) ⇔ (2) From Definition 2.2, $w \in D(L) \cap S$ is a constrained extremal solution of the linear inclusion $y \in L(x)$ if and only if there exists $z \in Y$ such that

$$z \in L|_S(w) \tag{3.3}$$

and

$$\|y - z\| = \operatorname{dist}\bigl(y, R(L|_S)\bigr). \tag{3.4}$$

We will characterize (3.3) and (3.4).

Now, by definition, (3.3) holds if and only if

$$w \in D(L) \cap S \quad \text{and} \quad \{w, z\} \in L.$$

Equivalently, we see that (3.3) holds if

$$w = g - k \in D(L) \quad \text{for some } k \in N \quad \text{and} \quad z = L_{S,P}(w) + s \quad \text{for some } s \in L(\theta).$$

Since $g \in D(L)$ and $w \in D(L)$, $k = g - w \in D(L) \cap N$. Hence, we see that (3.3) holds if and only if $w = g - k$ for some $k \in D(L) \cap N$ such that

$$z = L_{S,P}(g) - L_{S,P}(k) + s \quad \text{for some } s \in L(\theta). \tag{3.5}$$

Next, we characterize (3.4). Note first that

$$\|y - z\| = \bigl\|L_{S,P}(g) - y - L_{S,P}(k) + s\bigr\| \tag{3.6}$$

and

$$\begin{aligned} \operatorname{dist}\bigl(y, R(L|_S)\bigr) &= \inf\bigl\{\|y - z\| : z \in L(u), w \in D(L) \cap S\bigr\} \\ &= \inf\bigl\{\bigl\|L_{S,P}(g) - y - L_{S,P}(k) + s\bigr\| : k \in D(A), s \in L(\theta)\bigr\} \\ &= \operatorname{dist}\bigl(L_{S,P}(g) - y, R(A)\bigr). \end{aligned} \tag{3.7}$$

Note that, from $s \in L(\theta)$ and $k = g - w \in D(L) \cap N$, we have $L(k) = L|_N(k) = A(k)$ and

$$-s + L_{S,P}(k) \in L(\theta) + L_{S,P}(k) = L|_N(k) = A(k). \tag{3.8}$$

From (3.6) and (3.7), we see that (3.4) holds if and only if (3.8) holds and

$$\bigl\|L_{S,P}(g) - y - \bigl(L_{S,P}(k) - s\bigr)\bigr\| = \operatorname{dist}\bigl(L_{S,P}(g) - y, R(A)\bigr). \tag{3.9}$$

Hence, it follows that (3.4) holds if and only if k is an extremal solution,

$$L_{S,P}(g) - y \in A(x).$$

Consequently, it follows from (3.3) and (3.4) that $w \in D(L) \cap S$ is a constrained extremal solution of the linear inclusion $y \in L(x)$ with respect to S if and only if $w = g - k$ for some $k \in D(L) \cap N$ such that k is an extremal solution of $L_{S,P}(g) - y \in A(x)$. This proves that (i) and (ii) are equivalent.

(2) $\Rightarrow$ (3). Assume that $k := g - w \in D(L) \cap N$ is an extremal solution of the linear inclusion $-y + L_{S,P}(g) \in A(x)$. From (i) $\Leftrightarrow$ (ii) in Proposition 2.9, we have $w = g - k$ for some $k \in D(L) \cap N$ and

$$-y + L_{S,P}(g) \in A(k) \dotplus F_Y^{-1}\bigl(R(A)^\perp\bigr). \tag{3.10}$$

Let us write

$$-y + L_{S,P}(g) = x + z,$$

where $x \in A(k)$, $z \in F_Y^{-1}(R(A)^{\perp})$. Note that $A = L|_N$, it follows that $k \in D(L) \cap N$ and $\{k, x\} \in L$. Thus we obtain

$$x = L_{S,P}(k) + s \quad \text{for some } s \in L(\theta).$$

Consequently, we have $w \in D(L) \cap S$ and

$$\begin{aligned} -y + L_{S,P}(w) &= -y + L_{S,P}(g) - L_{S,P}(k) \\ &= x + z - L_{S,P}(k) \\ &= s + z \\ &\in L(\theta) \dotplus F_Y^{-1}\bigl(R(A)^{\perp}\bigr). \end{aligned}$$

It follows that (2) $\Rightarrow$ (3).

Next, we intend to prove that (3) $\Rightarrow$ (2). Assume (3) is true, we want to show that there exists $k \in D(L) \cap N$ such that

$$L_{S,P}(g) - y \in A(k) \dotplus F_Y^{-1}\bigl(R(A)^{\perp}\bigr).$$

From (i) $\Leftrightarrow$ (iii) in Proposition 2.9, we have showed that (2) holds. Indeed, from (3), we write

$$L_{S,P}(w) - y = s + z \quad \text{where } s \in L(\theta) \text{ and } z \in F_Y^{-1}\bigl(R(A)^{\perp}\bigr).$$

Note that $w \in D(L) \cap S$ and $w = g - k$ for some $k \in D(L) \cap N = D(A)$. Then

$$\begin{aligned} L_{S,P}(g) - y &= L_{S,P}(w) + L_{S,P}(k) - y \\ &= s + L_{S,P}(k) + z \in A(k) \dotplus F_Y^{-1}\bigl(R(A)^{\perp}\bigr) \end{aligned}$$

since $s + L_{S,P}(k) \in L(\theta) + L_{S,P}(k) = A(\theta) + A_{S,P}(k) = A(k)$. Thus (3) $\Rightarrow$ (2). It follows that (2) $\Leftrightarrow$ (3).

(2) $\Leftrightarrow$ (4) Assume (2) is true, *i.e.* $k := g - w \in D(L) \cap N$ is an extremal solution of the linear inclusion $L_{S,P}(g) - y \in A(x)$. By (i) $\Leftrightarrow$ (iii) in Proposition 2.9, we have $L_{S,P}(g) - y \in A(k) \dotplus F_Y^{-1}(R(A)^{\perp}) \subset R(A) \dotplus F_Y^{-1}(R(A)^{\perp})$. This proves that (2) $\Rightarrow$ (4). Conversely, we assume (4) is true, *i.e.* $L_{S,P}(g) - y \in R(A) \dotplus F_Y^{-1}(R(A)^{\perp})$, then there exist $y_1 \in R(A)$ and $y_2 \in F_Y^{-1}(R(A)^{\perp})$ such that $L_{S,P}(g) - y = y_1 + y_2$. By the definition, from $y_1 \in R(A)$, we have a $k \in D(A) = D(L) \cap N$ such that $y_1 \in A(k)$, hence

$$L_{S,P}(g) - y = y_1 + y_2 \in A(k) \dotplus F_Y^{-1}\bigl(R(A)^{\perp}\bigr).$$

Again, by (i) $\Leftrightarrow$ (iii) in Proposition 2.9, $k \in D(L) \cap N$ is an extremal solution of the linear inclusion $L_{S,P}(g) - y \in A(x)$. Thus (4) $\Rightarrow$ (2). □

Theorem 3.2 *Let the assumptions of Theorem* 3.1 *hold. Then, for any $y \in Y$, the set of all constrained extremal solution of the linear inclusion $y \in L(x)$ with respect to S, denoted by Ω_y, is not empty and is given by*

$$\Omega_y = \{g - A^{\#}[L_{S,P}(g) - y]\} \dotplus N(A), \tag{3.11}$$

where $A^{\#}$ is the metric generalized inverse of the multi-valued linear operator A, $A = L|_N$, $S = g + N$.

Proof (i) Since $R(A)$ is a Chebyshev subspace in Y, by Lemma 2.4, we have

$$Y = R(A) \dotplus F_Y^{-1}(R(A)^{\perp}).$$

For any $y \in Y$, we must have $L_{S,P}(g) - y \in R(A) \dotplus F_Y^{-1}(R(A)^{\perp})$. From (1) $\Leftrightarrow$ (4) in Theorem 3.1, we see that $\Omega_y \neq \emptyset$.

(ii) For any $y \in Y$, we see that $L_{S,P}(g) - y \in R(A) \dotplus F_Y^{-1}(R(A)^{\perp})$, hence $\Omega_y \neq \emptyset$ by (i).

For any $w \in \Omega_y$, *i.e.* $w \in D(L) \cap S$ is a constrained extremal solution of the linear inclusion $y \in L(x)$ with respect to S. By (1) $\Leftrightarrow$ (2) in Theorem 3.1, we have $w \in \Omega_y$ if and only if $k := g - w \in D(L) \cap N$ is an extremal solution of the linear inclusion $L_{S,P}(g) - y \in A(x)$. By (ii) in Proposition 2.10, we see that $w \in \Omega_y \Leftrightarrow$

$$\begin{aligned} w &\in \{g - k : k \in D(A) \cap N \text{ is an extremal solution of } L_{S,P}(g) - y \in A(x)\} \\ &= \{g - k : k \in A^{\#}[L_{S,P}(g) - y] + N(A)\} \\ &= \{g - A^{\#}[L_{S,P}(g) - y]\} \dotplus N(A), \end{aligned}$$

where $A^{\#}$ is the metric generalized inverse of the multi-valued linear operator A. Hence, it follows that

$$\Omega_y = \{g - A^{\#}[L_{S,P}(g) - y]\} \dotplus N(A). \qquad \square$$

Corollary 3.3 *Let X, Y, and Z be reflexive strictly convex Banach spaces. Let $A \subset X \times Z$ be a linear relation and $L \subset X \times Y$ be a single-valued linear operator. Assume that $N(A)$ is closed in X. Let $z \in R(A) \dotplus F_Z^{-1}(R(A)^{\perp})$ and $y \in Y$ be given. Define*

$$S := A^{\#}(z) + N(A), \qquad T := L|_{N(A)}.$$

Assume that $A^{\#}(z) \in D(L)$. Then we have the following:

(I) *The following statements are equivalent*:

(i) *$w \in D(L) \cap S$ is a constrained extremal solution of the linear operator equation $L(x) = y$ with respect to S*;

(ii) *$k := A^{\#}(z) - w \in D(L) \cap N(A)$ is an extremal solution of the linear operator equation*

$$L(A^{\#}(z)) - y = T(x);$$

(iii) *$w \in D(L) \cap S$ and*

$$L(w) - y \in F_Y^{-1}(R(T)^{\perp}).$$

(II) *$L(x) = y$ has a constrained extremal solution with respect to S if and only if*

$$L\big(A^{\#}(z)\big) - y \in R(T) \dotplus F_Y^{-1}\big(R(T)^{\perp}\big). \tag{3.12}$$

In particular, if $R(T)$ is closed in Y, then $L(x) = y$ always has a constrained extremal solution with respect to S.

(III) *Assume that* (3.12) *and that $N(T) = N(A) \cap N(L)$ is closed in X. Then the set of all constrained extremal solution of $L(x) = y$ with respect to S is given by*

$$\Omega_y = \big\{A^{\#}(z) - T^M\big[L\big(A^{\#}(z)\big) - y\big]\big\} \dotplus N(A) \cap N(L).$$

The main results, (i)-(iii) in Theorem 3.1 and (i)-(ii) in Theorem 3.2 in [7], will be especial cases of Theorem 3.1 and Theorem 3.2. We express them as the following corollary.

Corollary 3.4 [7] *Let H_1 and H_2 be Hilbert spaces. Let $L \subset H_1 \times H_2$ and $N \subset H_1$ be linear manifolds. Let $L_{S,P}$ be an arbitrary, but fixed algebraic operator part of L corresponding to an algebraic projector P of H_1 onto $L(\theta)$. Let*

$$S := g \dotplus N \quad \textit{and} \quad M := L|_N,$$

where $g \in D(L)$. Then we have:

(I) *for fixed $h \in H_2$, the following statements are equivalent*:

- (i) *w is a restricted least-squares solution* (*LSS*) *of the linear inclusion $y \in L(x)$ with respect to S.*
- (ii) *$k := g - w$ is an LSS of*

  $$L_{S,P}(g) - h \in M(x).$$

- (iii) *$w \in S \cap D(L)$ and*

  $$L_{S,P}(g) - h \in L(\theta) + N\big(M^*\big),$$

  where $M^* := \{(x, y) : (-y, x) \in M^{\perp}\}$ is the adjoint subspace of the linear manifold $M \subset H_1 \times H_2$, and $M^{\perp}$ is the orthogonal complement of M in Hilbert space $H_1 \times H_2$.
- (iv) *$g \in D(L)$ such that*

  $$L_{S,P}(g) - h \in R(M) \dotplus N\big(M^*\big).$$

  In particular, if $R(M)$ is closed, then a restricted LSS exists for each $h \in H_2$.

(II) *The set of all restricted LSS of the linear inclusion $y \in L(x)$ with respect to S, denoted by Ω_y, is not empty and is given by*

$$\Omega_y = \big\{g - M^{\#}\big[L_{S,P}(g) - y\big]\big\} \dotplus N(M).$$

Proof In Theorem 3.1, take $X = H_1$ and $Y = H_2$, $A = M = L|_N$, since H_1, H_2, and $H_1 \times H_2$ are Hilbert spaces, $F_Y = I$ the identity operator of H_2, and $N(M^*) = R(M)^{\perp} = R(A)^{\perp}$ (see [13]). (I) in Corollary 3.4 follows from Theorem 3.1, and (II) in Corollary 3.4 follows from Theorem 3.2. □

Remark 3.5 In [7], the authors gave an application of Theorem 3.1, *i.e.* Corollary 3.4, to concrete cases of singular optimal control problems involving ordinary differential equations with general boundary conditions where both the control space and the state space are Hilbert space $L_2^m = L_2([a,b],\mathbb{C}^m)$ and $L_2^n = L_2([a,b],\mathbb{C}^n)$, but for the same problem with the control space $L_p^m = L_p([a,b],\mathbb{C}^m)$ and the state space $L_p^n = L_p([a,b],\mathbb{C}^n)$ $(1 < p < \infty)$, we cannot apply Theorem 3.1 in [7], while we can apply Theorem 3.1, and Theorem 3.2, in this paper.

Remark 3.6 In Theorem 3.1, the three equivalent characterizations of a constrained extremal solution of the linear inclusion $y \in L(x)$ with respect to S are expressed in terms of algebraic operator parts and the generalized orthogonal complement of $R(A)$. In characterization (2) of Theorem 3.1, the constrained extremal solution is equivalent to an unconstrained, but modified extremal solution. Characterization (3) is a generalized form of the normal equation; we call

$$L_{S,P}(w) - y \in L(\theta) + F_Y^{-1}\bigl(R(A)^{\perp}\bigr)$$

the 'normal inclusion' for the given inclusion $y \in L(x)$. In the case of a single-valued operator with domain H_1 ($X = H_1$, $Y = H_2$ are Hilbert spaces with $(H_2)^* = H_2$), $g = \theta$, $N = H_1$, $A = L$, $R(L)$ is closed in H_2. Then $F_Y^{-1}(R(A)^{\perp}) = R(A)^{\perp} = N(A^*)$ by the Banach closed range theorem (see the theorem in [13]), $L(\theta) = \theta$, hence the 'normal inclusion' reduces to

$$A(w) - y \in N\bigl(A^*\bigr), \quad \textit{i.e.}\ A^*\bigl[A(w) - y\bigr] = \theta,$$

which gives the normal equation.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ZW and BW conceived and designed the study. ZW wrote and edited the manuscript. BW and YW examined all the steps of the proofs in this research and gave some advice. All authors read and approved the final manuscript.

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References
1. Lee, SJ, Nashed, MZ: Least-squares solutions of multi-valued linear operator equations in Hilbert spaces. J. Approx. Theory **38**, 380-391 (1983)
2. Lee, SJ: Boundary conditions for linear manifold. J. Math. Anal. Appl. **73**, 366-380 (1980)
3. Lee, SJ, Nashed, MZ: Generalized inverses for linear manifolds and applications to boundary value problems in Banach spaces. C. R. Math. Acad. Sci. Soc. R. Can. **4**, 347-352 (1982)
4. Lee, SJ, Nashed, MZ: Algebraic and topological selections of multi-valued linear relations. Ann. Sc. Norm. Super. Pisa, Cl. Sci. **17**, 111-126 (1990)
5. Wang, YW, Liu, J: Metric generalized inverse for linear manifolds and extremal solutions of linear inclusions in Banach spaces. J. Math. Anal. Appl. **302**, 360-371 (2005)
6. Wang, YW, Zhang, J, Cui, YA: Criteria for the single-valued metric generalized inverse multi-valued linear operators in Banach spaces. Acta Math. Sin. Engl. Ser. **28**(3), 637-644 (2012)
7. Lee, SJ, Nashed, MZ: Constrained least-squares solutions of linear inclusions and singular control problems in Hilbert spaces. Appl. Math. Optim. **19**, 225-242 (1989)

8. Singer, I: The Theory of Best Approximation and Functional Analysis. Regional Conf. Ser. in Appl. Math. SIAM, Philadelphia (1974)
9. Barbu, V, Precpuan, T: The Convexity and Optimization in Banach Spaces. Acad. Rep. Soc., Bucuresti (1978)
10. Wang, H, Wang, YW: Metric generalized inverses of linear operators in Banach spaces. Chin. Ann. Math., Ser. B **24**(4), 509-520 (2003)
11. Hudzik, H, Wang, YW, Sha, RL: Orthogonally complemented subspaces in Banach spaces. Numer. Funct. Anal. Optim. **29**, 779-790 (2008)
12. Wang, YW: Operator Generalized Inverses in Banach Spaces and Applications. Science Press, Beijing (2005)
13. Yosida, K: Functional Analysis. Springer, New York (1978)