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Isoperimetric inequalities in surround system and space science

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Abstract

By means of the algebraic, analysis, convex geometry, computer, and inequality theories we establish the following isoperimetric inequality in the centered 2-surround system $S^{(2)}\{P,\Gamma,l\}$:

$$\left(\frac{1}{|\Gamma|}\oint_{\Gamma}\bar{r}_P^p\right)^{1/p}\le\frac{|\Gamma|}{4\pi}\sin\frac{l\pi}{|\Gamma|}\left[\csc\frac{l\pi}{|\Gamma|}+\cot^2\frac{l\pi}{|\Gamma|}\ln\left(\tan\frac{l\pi}{|\Gamma|}+\sec\frac{l\pi}{|\Gamma|}\right)\right],$$
$$\forall p\le -2.$$

As an application of the inequality in space science, we obtain the best lower bounds of the mean $\boldsymbol{\lambda}$-gravity norm $\overline{\|\mathbf{F}_{\lambda}(\Gamma,P)\|}$ as follows:

$$\overline{\left\|\mathbf{F}_{\lambda}(\Gamma,P)\right\|}\triangleq\frac{1}{|\Gamma|}\oint_{\Gamma}\frac{1}{\|A-P\|^{\lambda}}\ge\left(\frac{2\pi}{|\Gamma|}\right)^{\lambda},\quad\forall\lambda\ge 2.$$

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Keywords: *p*-power mean; centered 2-surround system; *l*-central region; $\boldsymbol{\lambda}$-gravity function

1 Introduction

The gravity is an essential attribute of any physical matter. Therefore, the study of gravity has great theoretical significance and extensive application value.

The theory of satellite is important in space science. In [1–4], the authors systematically studied the theory of satellite and obtained some interesting results. In particular, in [2], the authors defined the centered 2-surround system, established several geometric inequalities for the centered 2-surround system under the proper hypotheses, and illustrated the background of the centered 2-surround system in space science.

It is well known that the Moon is a satellite of the Earth. In space science, we are concerned with the gravity of the Moon since the gravity may be disastrous causing *tsunami* and *tidal wave*, *etc.*

In this paper, we first define the *mean central distance* $\bar{r}_P$ of a centered 2-surround system $S^{(2)}\{P,\Gamma,l\}$. Next, we study the boundary curve of the l-central regions and the properties of the *asymptotic system* and establish several identities and inequalities involving the centered 2-surround system. Next, we prove an isoperimetric inequality in the centered 2-surround system. Finally, we demonstrate the application of our results in space science and obtain the best lower bounds of the *mean λ-gravity norm* $\overline{\|\mathbf{F}_\lambda(\Gamma,P)\|}$.

A large number of algebraic, functional analysis, differential equation, convex geometry, physics, computer, and inequality theories are used in this paper. The proofs of our results are both interesting and difficult, as well as which are depend on our previous work. Some of our proof methods can also be found in the references of this paper, such as [1–3].

2 Basic concepts and main results

We continue to use the notation of the references [1–3].

We begin by recalling some of the basic concepts and preliminary results of [1–3].

Let $\gamma : I \to \mathbb{R}^2$ be a continuous function, where $I \subset \mathbb{R}$ is an interval, and let the image

$$\Gamma \triangleq \gamma(I) = \left\{\gamma(t) \in \mathbb{R}^2 | \gamma(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, t \in I\right\}$$

of γ be a smooth curve [5], that is, the derivatives $x'(t)$ and $y'(t)$ are continuous, and the derivative of the vector $\gamma(t)$ satisfies the condition

$$\gamma'(t) \triangleq x'(t)\mathbf{i} + y'(t)\mathbf{j} \neq \mathbf{0}, \quad \forall t \in I,$$

where

$$\mathbf{0} = (0,0), \qquad \mathbf{i} = (1,0), \qquad \mathbf{j} = (0,1), \qquad \mathbb{R} \triangleq (-\infty,\infty), \qquad \mathbb{R}^2 \triangleq \mathbb{R} \times \mathbb{R}.$$

Then the length $|\Gamma|$ of the curve Γ exists:

$$|\Gamma| \triangleq \int_I \|\gamma'(t)\| \, \mathrm{d}t = \int_I \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2} \, \mathrm{d}t > 0,$$

and $|\Gamma| < \infty$ if I is a bounded interval, where the norm $\|x\mathbf{i} + y\mathbf{j}\|$ of the vector $x\mathbf{i} + y\mathbf{j} \in \mathbb{R}^2$ is defined as

$$\|x\mathbf{i} + y\mathbf{j}\| \triangleq \sqrt{x^2 + y^2}.$$

In this paper, we assume that Γ is a smooth and convex Jordan closed curve in $\mathbb{R}^2$ [1–3]. Then

$$\Gamma \triangleq \gamma(\mathbb{R}) = \left\{\gamma(t) \in \mathbb{R}^2 | \gamma(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, t \in \mathbb{R}\right\} \quad \text{and} \quad \gamma(t) \equiv \gamma\left(t + |\Gamma|\right), \quad \forall t \in \mathbb{R},$$

that is, $\gamma(t)$ is a periodic function, where the parameter t is the natural parameter, that is,

$$0 < l \leq |\Gamma| \quad \Rightarrow \quad \left|\gamma\left([t,t+l]\right)\right| \triangleq \int_t^{t+l} \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2} \, \mathrm{d}t = l, \quad \forall t \in \mathbb{R}.$$

We denote by $D(\Gamma)$ the convex region enclosed by the Jordan closed curve Γ, that is,

$$P_1, P_2 \in D(\Gamma) \quad \Rightarrow \quad \lambda P_1 + (1-\lambda)P_2 \in D(\Gamma), \quad \forall \lambda \in [0,1]$$

and

$$\left|D(\Gamma)\right| \triangleq \operatorname{Area} D(\Gamma)$$

denote the area of the region $D(\Gamma)$.

We remark here that, for the Jordan closed curve, we have the following *Jordan theorem* [3]: An arbitrary Jordan closed curve must divide a plane into two regions, and one of the regions is bounded and the another is unbounded. The bounded region is called the *interior* and the another is called the *outside* of the Jordan closed curve.

In this paper, we also assume that

$$A_- = \gamma(t_A - l), \qquad A = \gamma(t_A), \qquad A_+ = \gamma(t_A + l), \quad t_A \in \mathbb{R}. \tag{1}$$

If l is a fixed real number such that $0 < l < |\Gamma|/2$, then we say that the plane point set

$$D(\Gamma, l) \triangleq \bigcap_{A \in \Gamma} \widehat{A_-AA_+} \subset D(\Gamma) \subset \mathbb{R}^2$$

is an *l-central region* of the curve Γ, where the *angular region*

$$\widehat{A_-AA_+} \triangleq \left\{(1-\lambda)\gamma(t_A) + \lambda\gamma(t) | 0 < \lambda < \infty, t_A + l < t < t_A - l + |\Gamma|\right\}.$$

Let the l-central region $D(\Gamma, l)$ be nonempty, and let $P \in D(\Gamma, l)$ be a fixed point. We say that the set

$$S^{(2)}\{P, \Gamma, l\} \triangleq \{P, \Gamma, l\}$$

is a *centered* 2*-surround system* or *centered* 2*-satellite system*, P is a *center* and $A, A_+ \in \Gamma$ are two *satellites* of the system [1–3].

For the centered 2-surround system $S^{(2)}\{P, \Gamma, l\}$, we may think of the point P as the center of the Earth, Γ as the orbit of two satellites A, A_+. In order to avoid hitting, the satellites A, A_+ must move by the same curve velocity, that is,

$$l \triangleq \left|\gamma\left([t_A, t_A + l]\right)\right| \in \left(0, \frac{|\Gamma|}{2}\right)$$

is invariable. This is the significance of the centered 2-surround system $S^{(2)}\{P, \Gamma, l\}$ in the theory of satellites.

We remark here that, in [1, 3], the authors extended the centered 2-surround system $S^{(2)}\{P, \Gamma, l\}$ and defined the centered N-surround system, that is, if $S^{(2)}\{P, \Gamma, l_j\}$ is a centered 2-surround system, where $j = 1, 2, \ldots, N$, $N \geq 3$, then we say that the set

$$S^{(2)}\{P, \Gamma, \mathbf{l}\} \triangleq \{P, \Gamma, \mathbf{l}\}$$

is a *centered N-surround system* and P is a *center* of the system, where

$$\mathbf{l} \triangleq (l_1, l_2, \ldots, l_N) \in \mathbb{R}^N, \qquad 0 < l_j < \frac{|\Gamma|}{2}, \quad \forall j\colon 1 \leq j \leq N, \qquad \sum_{j=1}^{N} l_j = |\Gamma|,$$

and if

$$A_j \triangleq \gamma\left(t_A + \sum_{k=1}^{j} l_k\right), \quad j = 1, 2, \ldots, N,$$

then we say that $A_1, A_2, \ldots, A_N$ are N *satellites* of the system.

We remark here that, where the (2) in $S^{(2)}\{P, \Gamma, \mathbf{l}\}$ means that $P \in \mathbb{R}^2$ and $\Gamma \subset \mathbb{R}^2$. If $P \in \mathbb{R}^m$ and $\Gamma \subset \mathbb{R}^m$, $m \geq 3$ [6], then we can define [1, 3]

$$S^{(m)}\{P, \Gamma, \mathbf{l}\} \triangleq \{P, \Gamma, \mathbf{l}\}$$

as a *centered* N*-surround system* and

$$S^{(m)}\{\Gamma, \mathbf{l}\} \triangleq \{\Gamma, \mathbf{l}\}$$

as a N*-surround system without any central*.

For centered 2-surround system $S^{(2)}\{P, \Gamma, l\}$, let

$$P' \triangleq \mathrm{Projection}_{AA_+} P$$

denote the projection of the point P in the line AA_+, that is,

$$PP' \perp AA_+ \quad \text{and} \quad PP' \cap AA_+ = P'.$$

In the centered 2-surround system $S^{(2)}\{P, \Gamma, l\}$, we say that the distance

$$r_P \triangleq \mathrm{Distance}(P, AA_+) = \left\|P' - P\right\|$$

from the point P to line AA_+ is a *central distance* of the system, the distances

$$r_A \triangleq \mathrm{Distance}(A, PA_+) \quad \text{and} \quad r_{A_+} \triangleq \mathrm{Distance}(A_+, PA)$$

are the *Brocard distances* of the system [2], and the positive real number

$$\bar{r}_P \triangleq \frac{1}{\|A_+ - A\|} \int_{M \in [AA_+]} \|M - P\|$$

is the *mean central distance* of the system, which is the mean of the distance between the point P and the point M in the straight line segment

$$[AA_+] \triangleq \left\{(1 - \lambda)A + \lambda A_+ | 0 \leq \lambda \leq 1\right\}.$$

We remark here that if $l = 0$, then $\bar{r}_P = \|A - P\|$. This is another geometrical meaning of $\bar{r}_P$, which has applications in space science; see Section 5.

According to the definitions of the central distance and the l-central region, we know that r_P is a *support function* of the curve $\partial D(\Gamma, l)$, which is the boundary curve of the

l-central region $D(\Gamma, l)$, and we have that [7]

$$\left|D(\Gamma, l)\right| = \frac{1}{2}\oint_{\partial D(\Gamma, l)} r_P, \tag{2}$$

where

$$\left|D(\Gamma, l)\right| \triangleq \operatorname{Area} D(\Gamma, l)$$

is the area of the l-central region $D(\Gamma, l)$.

Let $f : \Gamma \to (0, \infty)$ be a continuous function defined on the curve Γ. Then the functional

$$M_\Gamma^{[p]}(f) \triangleq \begin{cases} (\frac{1}{|\Gamma|}\int_\Gamma f^p)^{1/p}, & p \in \mathbb{R}, p \neq 0, \\ \exp(\frac{1}{|\Gamma|}\int_\Gamma \ln f), & p = 0, \end{cases}$$

is called the *p-power mean* of the function f, where

$$M_\Gamma(f) \triangleq M_\Gamma^{[1]}(f) = \frac{1}{|\Gamma|}\int_\Gamma f$$

is the mean of the function f.

We remark here that $M_\Gamma^{[p]}(f)$ is increasing with respect to p [8–12], that is,

$$p < q \quad \Rightarrow \quad M_\Gamma^{[p]}(f) \le M_\Gamma^{[q]}(f), \tag{3}$$

where equality in (3) holds if and only if f is a constant function.

As pointed out in [13], the theory of inequalities plays an important role in all the fields of mathematics. The concept of mean is the most prominent in the theory, and the p-power mean is the crucial one. The references [8–13] studied the sharp bounds of the p-power mean.

In the convex geometry, a well-known isoperimetric inequality can be expressed as follows: If Γ is a smooth Jordan closed curve, then we have

$$\left|D(\Gamma)\right| \le \frac{|\Gamma|^2}{4\pi}. \tag{4}$$

Equality in (4) holds if and only if Γ is a circle.

In the convex geometry, a large number of isoperimetric inequalities similar to (4) was obtained [14–16]. Recently, we obtained some new isoperimetric inequalities in the surround system [1–3].

In [1], the authors obtained the following results. For the centered 2-surround system $S^{(2)}\{P, \Gamma, l\}$, we have the following isoperimetric inequalities:

$$\left(\frac{1}{|\Gamma|}\oint_\Gamma r_P^p\right)^{1/p} \le \frac{|\Gamma|}{2\pi}\cos\frac{l\pi}{|\Gamma|}, \quad \forall l\colon 0 < l \le |\Gamma|/3, \forall p \le -2 \tag{5}$$

and

$$\left(\frac{1}{|\Gamma|}\oint_\Gamma r_P^q\right)^{1/q} \ge \inf_{\frac{l\pi}{|\Gamma|} \le t < \pi}\left\{\left(\frac{2|D(\Gamma)|}{|\Gamma|}t - \frac{l}{2}\right)\csc t + \frac{l}{2}\frac{\cos t}{t}\right\}, \quad \forall q \ge 2. \tag{6}$$

Equalities in (5) and (6) hold if and only if Γ is a circle and P is the center of the circle.

In [2], the authors obtained the following results. For the centered 2-surround system $S^{(2)}\{P,\Gamma,l\}$, we have the following isoperimetric inequalities:

$$\exp\left(\frac{1}{|\Gamma|}\oint_\Gamma \ln r_A\right)+\left(\frac{1}{|\Gamma|}\oint_\Gamma r_A^{\frac{2}{3}}\right)^{\frac{3}{2}} \leq \frac{|\Gamma|}{\pi}\sin\frac{2l\pi}{|\Gamma|}, \tag{7}$$

$$\left[\frac{1}{|\Gamma|}\oint_\Gamma\left(\frac{r_A+r_{A_+}}{2}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}} \leq \frac{|\Gamma|}{2\pi}\sin\frac{2l\pi}{|\Gamma|}, \tag{8}$$

$$\left(\frac{1}{|\Gamma|}\oint_\Gamma r_P^{-2}\right)^{-\frac{1}{2}} \leq \frac{|\Gamma|}{2\pi}\cos\frac{l\pi}{|\Gamma|}, \tag{9}$$

and

$$\left(\frac{1}{|\Gamma|}\oint_\Gamma r_P^{-p}\right)^{-\frac{1}{p}} \leq \left(\frac{1}{|\Gamma|}\oint_\Gamma \left\|P'-A\right\|^q\right)^{\frac{1}{q}}\cot\frac{l\pi}{|\Gamma|}. \tag{10}$$

In (10), where $p>1$, $p^{-1}+q^{-1}=1$ and

$$0<\angle A_-AA_+\leq\pi-\arctan\left(2\sin\frac{2l\pi}{|\Gamma|}\right),\quad \forall A\in\Gamma.$$

Equalities in (7)-(10) hold if and only if Γ is a circle and P is the center of the circle.

In [3], the authors established a isoperimetric inequality in the N-surround system without any central $S^{(3)}\{\Gamma,\mathbf{l}\}$:

$$\frac{1}{|\Gamma|}\oint_\Gamma \operatorname{Area}\bigl(\Omega[P,\Gamma_N]\bigr)\leq\frac{N|\Gamma|^2}{8\pi^2}\sin\frac{2\pi}{N}, \tag{11}$$

where the N-polygon Γ_N is inscribed in Γ [17] and P is a vertex of Γ_N, and the $\Omega[P,\Gamma_N]$ is a cone surface its vertex is P and alignment is Γ_N.

Convexity and concavity are essential attributes of any real-variable function, their research and applications are important topics in mathematics and, in particular, the convex analysis [18].

In [19], the authors generalized the traditional covariance and variance of random variables, defined the ϕ-covariance, ϕ-variance, ϕ-Jensen variance, ϕ-Jensen covariance, integral variance, and γ-order variance, and studied the relationships among these variances. They also studied the monotonicity of the interval function $\mathrm{JVar}_\phi\,\varphi(X_{[a,b]})$ and proved an interesting quasi-log-concavity conjecture. They also demonstrated the applications of these results in higher education. Based on the monotonicity of the interval function $\mathrm{Var}^{[\gamma]}X_{[a,b]}$, they show that the hierarchical teaching model is normally better than the traditional teaching model under the hypothesis that

$$X_I\subset X\sim N_k(\mu,\sigma),\quad k>1.$$

In this paper, we study the best upper bounds of the p-power mean

$$M_\Gamma^{[p]}(\bar{r}_P)\triangleq\left(\frac{1}{|\Gamma|}\oint_\Gamma \bar{r}_P^p\right)^{1/p}$$

and establish a new isoperimetric inequality in the centered 2-surround system $S^{(2)}\{P,\Gamma,l\}$ as follows.

Theorem 1 (Mean central distance inequality) *Let $S^{(2)}\{P,\Gamma,l\}$ be a centered 2-surround system. If $p\in(-\infty,-2]$ and*

$$0<\angle APA_+\leq\eta,\quad \forall A\in\Gamma,$$

then we have the following isoperimetric inequality:

$$\left(\frac{1}{|\Gamma|}\oint_\Gamma \bar{r}_P^p\right)^{1/p}\leq\frac{|\Gamma|}{4\pi}\sin\frac{l\pi}{|\Gamma|}\left[\csc\frac{l\pi}{|\Gamma|}+\cot^2\frac{l\pi}{|\Gamma|}\ln\left(\tan\frac{l\pi}{|\Gamma|}+\sec\frac{l\pi}{|\Gamma|}\right)\right],\tag{12}$$

where $\eta=1.7571802619873076\ldots$ is the unique real root of the equation

$$\int_0^1\sqrt{t^2+\cot^2\frac{\eta}{2}}\,\mathrm{d}t=1,\quad \eta\in(0,\pi).\tag{13}$$

Equality in (12) *holds if and only if Γ is a circle and P is the center of the circle.*

Let $S^{(2)}\{P,\Gamma,l\}$ be a centered 2-surround system. We say that the set

$$S^{(2)}\{P,\Gamma\}\triangleq\lim_{l\to 0}S^{(2)}\{P,\Gamma,l\}=\{P,\Gamma,0\}$$

is a *centered surround system* and P is the *center* of the system.

For the centered surround system $S^{(2)}\{P,\Gamma\}$, we may think of the point P as the center of the Earth and Γ as the orbit of a satellite A (such as the Moon or an artificial Earth satellite). This is the significance of the centered surround system $S^{(2)}\{P,\Gamma\}$ in the theory of satellite.

From (15) in Section 3 we know that the centered surround system $S^{(2)}\{P,\Gamma\}$ exists for any smooth and convex Jordan closed curve Γ.

Theorem 1 implies the following interesting corollary.

Corollary 1 *Let $S^{(2)}\{P,\Gamma\}$ be a centered surround system. Then for all $p\in(-\infty,-2]$, we have the following isoperimetric inequality*:

$$\left(\frac{1}{|\Gamma|}\oint_\Gamma\|A-P\|^p\right)^{1/p}\leq\frac{|\Gamma|}{2\pi}.\tag{14}$$

Equality in (14) *holds if and only if Γ is a circle and P is the center of the circle.*

Proof Consider the 2-surround system $S^{(2)}\{P,\Gamma,l\}$. Since

$$0<\|A_+-A\|\leq l,$$

there exists $\varepsilon\in(0,|\Gamma|/2)$ such that for any $l\in(0,\varepsilon)$, we have

$$0<\angle APA_+\leq\eta,\quad \forall A\in\Gamma.$$

Letting $\varepsilon \to 0$, by Theorem 1, we get

$$\begin{aligned}
\left(\frac{1}{|\Gamma|}\oint_{\Gamma}\|A-P\|^{p}\right)^{1/p} &= \lim_{\varepsilon\to 0}\left(\frac{1}{|\Gamma|}\oint_{\Gamma}\bar{r}_{P}^{p}\right)^{1/p} \\
&= \lim_{l\to 0}\left(\frac{1}{|\Gamma|}\oint_{\Gamma}\bar{r}_{P}^{p}\right)^{1/p} \\
&\leq \lim_{l\to 0}\frac{|\Gamma|}{4\pi}\sin\frac{l\pi}{|\Gamma|}\left[\csc\frac{l\pi}{|\Gamma|}+\cot^{2}\frac{l\pi}{|\Gamma|}\ln\left(\tan\frac{l\pi}{|\Gamma|}+\sec\frac{l\pi}{|\Gamma|}\right)\right] \\
&= \lim_{l\to 0}\frac{|\Gamma|}{4\pi}\left[1+\sin^{-1}\frac{l\pi}{|\Gamma|}\cos^{2}\frac{l\pi}{|\Gamma|}\ln\left(\tan\frac{l\pi}{|\Gamma|}+\sec\frac{l\pi}{|\Gamma|}\right)\right] \\
&= \frac{|\Gamma|}{2\pi},
\end{aligned}$$

that is, inequality (14) holds.

According to Theorem 1, equality in (14) holds if and only if Γ is a circle and P is the center of the circle. Corollary 1 is proved. □

In Section 5, we will demonstrate the applications of Corollary 1 in space science and establish an isoperimetric inequality involving the λ-gravity of the Moon to the Earth.

3 Preliminaries

In order to prove Theorem 1, we need some preliminaries involving the centered 2-surround system.

3.1 Boundary curve of the *l*-central region

In the definition of the centered 2-surround system $S^{(2)}\{P,\Gamma,l\}$, an important assumption is that the l-central region $D(\Gamma,l)$ is nonempty, that is, the boundary curve $\partial D(\Gamma,l)$ of the l-central region $D(\Gamma,l)$ is a Jordan closed curve. Unfortunately, the l-central region $D(\Gamma,l)$ may be empty. For example, let Γ be a regular triangle of side length 3, then $D(\Gamma,4)=\emptyset$, where $\emptyset$ denote the empty set, see [2].

Since

$$\lim_{l\to 0}D(\Gamma,l)=D(\Gamma)\neq\emptyset, \tag{15}$$

there exists $\varepsilon\in(0,|\Gamma|/2)$ such that, for any $l\in(0,\varepsilon)$, we have $D(\Gamma,l)\neq\emptyset$.

On the other hand, in [2], the following statement is proved (see Lemmas 2.1 and 2.3 in [2]): Let Γ be a smooth and convex Jordan closed curve. Then $D(\Gamma,l)\neq\emptyset$ for all $l\in(0,|\Gamma|/2)$ if and only if Γ is a *central symmetric curve*.

According to this result, we know that if Γ is an ellipse, which is a central symmetric curve, then the l-central region $D(\Gamma,l)$ is nonempty. In space science, the orbit of a satellite is an ellipse, and P in $S^{(2)}\{P,\Gamma,l\}$ is one of the focuses of the ellipse [4]. Therefore, the centered 2-surround system $S^{(2)}\{P,\Gamma,l\}$ is of great application value in the theory of satellite.

Based on the definition of the l-central region $D(\Gamma,l)$, we know that the boundary curve $\partial D(\Gamma,l)$ of the l-central region $D(\Gamma,l)$ is the *envelope curve* of the family of straight line

AA_+, that is, for any point $x\mathbf{i}+y\mathbf{j} \in \partial D(\Gamma,l)$, there exists a line AA_+ such that AA_+ is tangent to $\partial D(\Gamma,l)$ at the point $x\mathbf{i}+y\mathbf{j}$. Hence, the point $x\mathbf{i}+y\mathbf{j}$ must satisfy the equation

$$\det\begin{bmatrix} x & y & 1 \\ x(t_A) & y(t_A) & 1 \\ x(t_A+l) & y(t_A+l) & 1 \end{bmatrix} = 0 \tag{16}$$

and the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(t_A+l)-y(t_A)}{x(t_A+l)-x(t_A)}. \tag{17}$$

Eliminating the parameter t_A from (16) and (17), we can obtain the equation of the boundary curve $\partial D(\Gamma,l)$; see the following Propositions 1 and 2.

Proposition 1 *Let Γ be a unit circle, that is,*

$$\Gamma \triangleq \{(x,y)|x=\cos t, y=\sin t, t\in\mathbb{R}\},$$

where t is the natural parameter. Then the equation of the boundary curve $\partial D(\Gamma,l)$ is

$$\partial D(\Gamma,l): x^2+y^2=\cos^2\frac{l}{2}, \quad \forall l: 0<l<\pi. \tag{18}$$

Proof Indeed, (16) and (17) can be rewritten as

$$\begin{cases} x\cos(t_A+\frac{l}{2})+y\sin(t_A+\frac{l}{2})=\cos\frac{l}{2}, \\ \mathrm{d}x\cos(t_A+\frac{l}{2})+\mathrm{d}y\sin(t_A+\frac{l}{2})=0. \end{cases} \tag{19}$$

Eliminating the parameter t_A from (19), we obtain that

$$(x\,\mathrm{d}y-y\,\mathrm{d}x)^2 = \left[(\mathrm{d}x)^2+(\mathrm{d}y)^2\right]\cos^2\frac{l}{2}. \tag{20}$$

Setting

$$x=\rho\cos\theta, \qquad y=\rho\sin\theta, \quad \theta\in\mathbb{R},$$

and substituting them into (20), we get

$$\rho^4 = \left[\left(\frac{\mathrm{d}\rho}{\mathrm{d}\theta}\right)^2+\rho^2\right]\cos^2\frac{l}{2} \quad\Leftrightarrow\quad \rho=\cos\frac{l}{2}\vee\rho=\cos\frac{l}{2}\sec(\theta+C).$$

Since

$$\rho=\cos\frac{l}{2}\sec(\theta+C)$$

is the equation of the family of straight line AA_+, the equation of $\partial D(\Gamma,l)$ is

$$\rho=\cos\frac{l}{2} \quad\Leftrightarrow\quad x^2+y^2=\cos^2\frac{l}{2},$$

that is, (18) holds. This ends the proof. □

We can also find the equation of $\partial D(\Gamma, l)$ if Γ is piecewise smooth.

Proposition 2 *Let $\Gamma_N \triangleq \{A_1,A_2,\ldots,A_N\}$, where $N \geq 3$, be a convex polygon* [17], *and let*

$$0 < l \leq \frac{1}{2}\min_{1\leq i\leq N}\bigl\{\|A_{i+1} - A_i\|\bigr\}.$$

Then we have

$$\bigl|D(\Gamma_N)\bigr| - \bigl|D(\Gamma_N, l)\bigr| = \frac{1}{6}l^2\sum_{i=1}^{N}\sin\angle A_{i-1}A_iA_{i+1} \leq \frac{1}{6}Nl^2\sin\frac{2\pi}{N}. \tag{21}$$

Equality in (21) *holds if and only if*

$$\angle A_{i-1}A_iA_{i+1} = \frac{N-2}{N}\pi, \quad i = 1,2,\ldots,N,$$

where $A_0 \triangleq A_N$ and $A_{N+1} \triangleq A_1$.

Proof Notice that

$$0 < l \leq \min_{1\leq i\leq N}\frac{1}{2}\bigl\{\|A_{i+1} - A_i\|\bigr\} \quad\Rightarrow\quad 0 < l < \frac{|\Gamma_N|}{2} \text{ and } D(\Gamma_N, l) \neq \phi.$$

Consider the regular region $\widehat{A_{j-1}A_jA_{j+1}}$. Let the rays A_iA_{i-1} and A_iA_{i+1} be tangent to $\partial D(\Gamma_N, l)$ at the points T_i and T_i', respectively, and let

$$\widetilde{T_iT_i'} \subset \partial D(\Gamma_N, l) \quad \text{for } i = 1,2,\ldots,N.$$

Then we have

$$\partial D(\Gamma_N, l) = \sum_{i=1}^{N}\bigl[T_{i-1}'T_i\bigr] + \sum_{i=1}^{N}\widetilde{T_iT_i'}$$

and

$$\bigl|D(\Gamma_N)\bigr| - \bigl|D(\Gamma_N, l)\bigr| = \sum_{i=1}^{N}\bigl|D\bigl([A_iT_i] + \bigl[A_iT_i'\bigr] + \widetilde{T_iT_i'}\bigr)\bigr|. \tag{22}$$

For any

$$A \in A_iA_{i+1}, \qquad B \in A_iA_{i-1} \quad \text{and} \quad \angle A_{i-1}A_iA_{i+1} = 2\alpha \in (0,\pi),$$

let the corresponding coordinates of A and B be

$$A\left(\left(\frac{l}{2} - t\right)\cos\alpha, \left(\frac{l}{2} - t\right)\sin\alpha\right) \quad \text{and} \quad B\left(\left(\frac{l}{2} + t\right)\cos\alpha, -\left(\frac{l}{2} + t\right)\sin\alpha\right),$$

respectively, where

$$-\frac{l}{2} \leq t \leq \frac{l}{2};$$

see Figure 1.

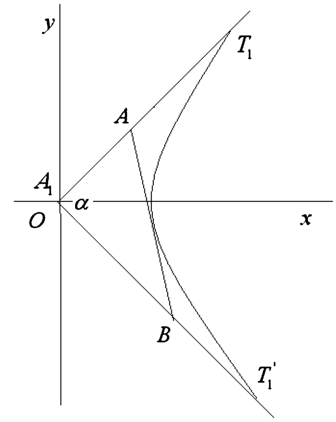

Figure 1 The graph of the curve $\widetilde{T_1T_1'}$.

Then the equation of the curve $\widetilde{T_iT_i'}$ is determined by (16) and (17). Hence,

$$\widetilde{T_iT_i'}\colon \frac{y + t\sin\alpha}{x - l/2\cos\alpha} = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{l}{2t}\tan\alpha, \tag{23}$$

where

$$\left(\frac{l}{2}\cos\alpha, -t\sin\alpha\right)$$

is the midpoint of $[AB]$. Eliminating the parameter t_A from (23), we get

$$\frac{2y}{l\sin\alpha} - \frac{\mathrm{d}\frac{2x}{l\cos\alpha}}{\mathrm{d}\frac{2y}{l\sin\alpha}} = \frac{(\frac{2x}{l\cos\alpha} - 1)\,\mathrm{d}\frac{2y}{l\sin\alpha}}{\mathrm{d}\frac{2x}{l\cos\alpha}}. \tag{24}$$

Set

$$x^* = \frac{2x}{l\cos\alpha}, \qquad y^* = \frac{2y}{l\sin\alpha} \quad \text{and} \quad u = \frac{\mathrm{d}x^*}{\mathrm{d}y^*}.$$

Then (24) can be rewritten as

$$y^* - u = u^{-1}\bigl(x^* - 1\bigr) \quad \text{and} \quad \mathrm{d}x^* = u\,\mathrm{d}y^*. \tag{25}$$

From (25) we get

$$\mathrm{d}y^* - \mathrm{d}u = \frac{u\,\mathrm{d}x^* - (x^* - 1)\,\mathrm{d}u}{u^2} = \mathrm{d}y^* - \frac{(x^* - 1)\,\mathrm{d}u}{u^2}$$

and

$$x^* = 1 + u^2, \qquad y^* = 2u. \tag{26}$$

Eliminating the parameter u in (26), we see that the curve $\widetilde{T_iT_i'}$ is a parabola whose equation is

$$\widetilde{T_iT_i'}\colon x^* = 1 + \left(\frac{y^*}{2}\right)^2 \quad \Leftrightarrow \quad \frac{2x}{l\cos\alpha} = 1 + \left(\frac{y}{l\sin\alpha}\right)^2, \quad -l\sin\alpha \le y \le l\sin\alpha. \tag{27}$$

Consequently,

$$\begin{aligned}
\left|D\bigl([A_iT_i]+\bigl[A_iT_i'\bigr]+\widetilde{T_iT_i'}\bigr)\right| &= 2\int_0^{l\sin\alpha}\left\{\frac{l\cos\alpha}{2}\left[1+\left(\frac{y}{l\sin\alpha}\right)^2\right]-y\cot\alpha\right\}\mathrm{d}y \\
&= l^2\cos\alpha\sin\alpha\int_0^{l\sin\alpha}\left(\frac{y}{l\sin\alpha}-1\right)^2\mathrm{d}\left(\frac{y}{l\sin\alpha}-1\right) \\
&= l^2\cos\alpha\sin\alpha\,\frac{(\frac{y}{l\sin\alpha}-1)^3}{3}\bigg|_0^{l\sin\alpha} \\
&= \frac{l^2}{6}\sin 2\alpha \\
&= \frac{l^2}{6}\sin\angle A_{i-1}A_iA_{i+1}.
\end{aligned}$$

From (22), the formula

$$\sum_{i=1}^{N}\angle A_{i-1}A_iA_{i+1}=(N-2)\pi,\quad 0<\angle A_{i-1}A_iA_{i+1}<\pi, i=1,2,\ldots,N,$$

and the Jensen inequality [19, 20]

$$\frac{1}{N}\sum_{i=1}^{N}\sin\angle A_{i-1}A_iA_{i+1}\le\sin\frac{1}{N}\sum_{i=1}^{N}\angle A_{i-1}A_iA_{i+1}=\sin\left(\pi-\frac{2\pi}{N}\right)=\sin\frac{2\pi}{N}$$

we get

$$\begin{aligned}
\left|D(\Gamma_N)\right|-\left|D(\Gamma_N,l)\right| &= \sum_{i=1}^{N}\left|D\bigl([A_iT_i]+\bigl[A_iT_i'\bigr]+\widetilde{T_iT_i'}\bigr)\right| \\
&= \sum_{i=1}^{N}\frac{l^2}{6}\sin\angle A_{i-1}A_iA_{i+1} \\
&= \frac{l^2}{6}\sum_{i=1}^{N}\sin\angle A_{i-1}A_iA_{i+1} \\
&\le \frac{1}{6}Nl^2\sin\frac{2\pi}{N},
\end{aligned}$$

that is, (21) holds.

Based on this proof, we see that the equality in (21) holds if and only if

$$\angle A_{i-1}A_iA_{i+1}=\frac{N-2}{N}\pi,\quad i=1,2,\ldots,N.$$

The proof is completed. □

For example, if $N=4$, Γ_4 is a square of side length 2 and $l=1$, then, by Proposition 2, we have

$$D(\Gamma_N,l)\neq\phi\quad\text{and}\quad\left|D(\Gamma_N)\right|-\left|D(\Gamma_N,l)\right|=\frac{1}{6}Nl^2\sin\frac{2\pi}{N}=\frac{2}{3}.$$

3.2 Asymptotic system

In the theory of surround system, one of the important concepts is the *asymptotic system*.

Definition 1 (see [1–3]) Let $S^{(2)}\{P,\Gamma,l\}$ be a centered 2-surround system. Suppose that:

(i) $\{N_n\}_{n=1}^{\infty}$ and $\{k_n\}_{n=1}^{\infty}$ are two positive integer sequences, and

$$N_n \geq 3, \qquad 1 \leq k_n < \frac{N_n}{2}, \qquad \lim_{n\to\infty} N_n = +\infty, \qquad \lim_{n\to\infty} \frac{k_n}{N_n} = \frac{l}{|\Gamma|};$$

(ii) $\Gamma_{N_n} \triangleq \{A_1, A_2, \ldots, A_{N_n}\}$ is the inscribed N_n-sided polygonal of the closed curve Γ, and

$$\|A_2 - A_1\| = \|A_3 - A_2\| = \cdots = \|A_{N_n} - A_{N_n-1}\| = \|A_1 - A_{N_n}\|.$$

Then the set

$$S^{(2)}\left\{P, \Gamma_{N_n}, \frac{k_n}{N_n}|\Gamma_{N_n}|\right\} \triangleq \left\{P, \Gamma_{N_n}, \frac{k_n}{N_n}|\Gamma_{N_n}|\right\}$$

is called an *asymptotic system* of the system $S^{(2)}\{P,\Gamma,l\}$.

The asymptotic system has the properties as follows.

Lemma 1 (see Lemma 2.4 in [2]) *If $S^{(2)}\{P,\Gamma,l\}$ is a centered 2-surround system, then we have*

$$\lim_{n\to\infty} S^{(2)}\left\{P, \Gamma_{N_n}, \frac{k_n}{N_n}|\Gamma_{N_n}|\right\} = S^{(2)}\{P,\Gamma,l\}. \tag{28}$$

Lemma 2 (see Lemma 2.5 in [2]) *If $S^{(2)}\{P,\Gamma_{N_n},\frac{k_n}{N_n}|\Gamma_{N_n}|\}$ is an asymptotic system of $S^{(2)}\{P,\Gamma,l\}$, then there exists a sequence*

$$\{A_{i_n}^{(n)}\}_{n=0}^{\infty} \subseteq \{A_i^{(n)}\}_{n=0}^{\infty}, \quad i_n \in \{1,2,\ldots,N_n\},$$

such that

$$\lim_{n\to\infty} A_{i_n}^{(n)} = A, \qquad \lim_{n\to\infty} A_{i_n+k_n}^{(n)} = A_+ \quad \textit{and} \quad \lim_{n\to\infty} A_{i_n-k_n}^{(n)} = A_-. \tag{29}$$

Lemma 3 (see Lemma 2.6 in [2]) *Let the image $\Gamma = \gamma([a,b])$ of a continuous function $\gamma : [a,b] \to \mathbb{R}^m$ be a smooth curve, and let $f : \Gamma \to \mathbb{R}$ be a Riemann-integrable function over Γ. Suppose that Γ is partitioned by means of $N+1$ points*

$$A_0, A_1, \ldots, A_{i-1}, A_i, \ldots, A_N, \quad N \geq 3,$$

such that

$$A_i = \gamma(t_i), \quad i = 0,1,2,\ldots,N, a = t_0 < t_1 < \cdots < t_N = b,$$

$$\lim_{N\to\infty} (t_{i+1} - t_i) = 0, \quad i = 0,1,2,\ldots,N-1,$$

and

$$\|A_1 - A_0\| = \|A_2 - A_1\| = \cdots = \|A_i - A_{i-1}\| = \cdots = \|A_N - A_{N-1}\| = |\Gamma_N|/N.$$

Then we have

$$\frac{1}{|\Gamma|}\int_\Gamma f = \lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1} f(A_i). \tag{30}$$

3.3 Associated identities and inequalities

In order to prove Theorem 1, we need to establish several identities and inequalities involving the centered 2-surround system as follows.

Lemma 4 *Let $S^{(2)}\{P,\Gamma,l\}$ be a centered 2-surround system. Then we have the following identity*:

$$\oint_\Gamma \angle A_- P A_+ = 2l\pi. \tag{31}$$

Proof This proof is similar to that of Lemma 2.13 in [2].

We need the following definition:

$$A_i = A_j \quad \Leftrightarrow \quad i = j \pmod{N_n}.$$

Consider the asymptotic system $S^{(2)}\{P, \Gamma_{N_n}, \frac{k_n}{N_n}|\Gamma_{N_n}|\}$. By Lemmas 1 and 2 we have that $P \in D(\Gamma_{N_n})$ if n is sufficiently large. By Lemmas 1, 2, and 3 and by the identity

$$\sum_{i=1}^{N_n} \angle A_i P A_{i+1} = 2\pi$$

we get

$$\begin{aligned}
\frac{1}{|\Gamma|}\oint_\Gamma \angle A_- P A_+ &= \lim_{n\to\infty}\frac{1}{N_n}\sum_{i=1}^{N_n}\angle A_i P A_{i+k_n} \\
&= \lim_{n\to\infty}\frac{1}{N_n}\sum_{i=1}^{N_n}\sum_{j=0}^{k_n-1}\angle A_{i+j} P A_{i+1+j} \\
&= \lim_{n\to\infty}\frac{1}{N_n}\sum_{j=0}^{k_n-1}\sum_{i=1}^{N_n}\angle A_{i+j} P A_{i+1+j} \\
&= \lim_{n\to\infty}\frac{1}{N_n}\sum_{j=0}^{k_n-1} 2\pi \\
&= \lim_{n\to\infty}\frac{2k_n\pi}{N_n} \\
&= \frac{2l\pi}{|\Gamma|}.
\end{aligned}$$

The proof of Lemma 4 is completed. □

Lemma 5 (see Lemma 2.7 in [2]) *Let $S^{(2)}\{P,\Gamma,l\}$ be a centered 2-surround system. Then we have the following inequality*:

$$\sqrt{\frac{1}{|\Gamma|}\oint_{\Gamma}\|A_{+}-A\|^{2}}\leq\frac{|\Gamma|}{\pi}\sin\frac{l\pi}{|\Gamma|}. \tag{32}$$

Equality in (32) *holds if Γ is a circle in $\mathbb{R}^2$*.

Lemma 6 *Let $0<\theta<\pi/2$. Then the inequality*

$$\int_0^1\sqrt{t^2+\cot^2\theta}\,\mathrm{d}t\geq 1 \tag{33}$$

holds if and only if $0<\theta\leq\eta/2$, where $\eta=1.7571802619873076\ldots$ is the unique real root of equation (13).

Proof Using the formula

$$\int\sqrt{t^2+a^2}\,\mathrm{d}t=\frac{1}{2}\Bigl[t\sqrt{t^2+a^2}+a^2\ln\bigl(t+\sqrt{t^2+a^2}\bigr)\Bigr]+C, \tag{34}$$

we get

$$\begin{aligned}\int_0^1\sqrt{t^2+\cot^2\theta}\,\mathrm{d}t&=\frac{1}{2}\Bigl[\sqrt{1+\cot^2\theta}+\cot^2\theta\ln\bigl(1+\sqrt{1+\cot^2\theta}\bigr)\Bigr]-\frac{1}{2}\cot^2\theta\ln\cot\theta\\&=\frac{1}{2}\bigl[\csc\theta+\cot^2\theta\ln(1+\csc\theta)\bigr]-\frac{1}{2}\cot^2\theta\ln\cot\theta\\&=\frac{1}{2}\left(\csc\theta+\cot^2\theta\ln\frac{1+\csc\theta}{\cot\theta}\right)\\&=\frac{1}{2}\bigl[\csc\theta+\cot^2\theta\ln(\tan\theta+\sec\theta)\bigr],\end{aligned}$$

that is,

$$\int_0^1\sqrt{t^2+\cot^2\theta}\,\mathrm{d}t=\frac{1}{2}\bigl[\csc\theta+\cot^2\theta\ln(\tan\theta+\sec\theta)\bigr]. \tag{35}$$

Consider the auxiliary function

$$\begin{aligned}&\varphi_*:\left(0,\frac{\pi}{2}\right)\to\mathbb{R},\\&\varphi_*(\theta)=\int_0^1\sqrt{t^2+\cot^2\theta}\,\mathrm{d}t-1=\frac{1}{2}\bigl[\csc\theta+\cot^2\theta\ln(\tan\theta+\sec\theta)\bigr]-1.\end{aligned}$$

The graph of the function φ_* is depicted in Figures 2 and 3.

Inequality (33) can be rewritten as

$$\varphi_*(\theta)\geq 0,\quad \theta\in\left(0,\frac{\pi}{2}\right).$$

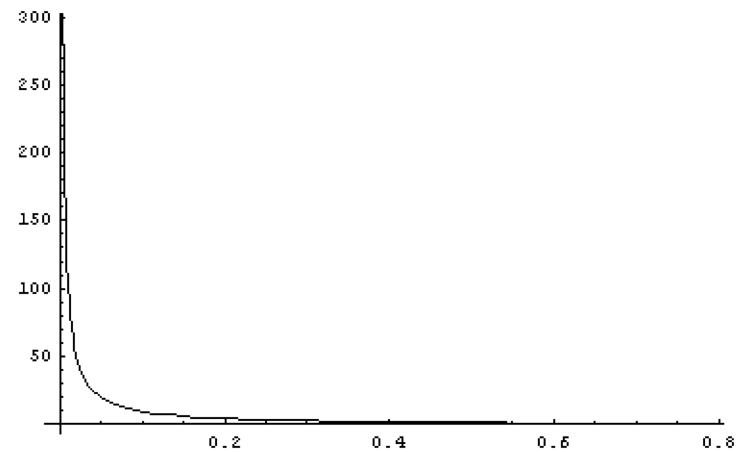

Figure 2 The graph of the function φ_*, where $t \in (0, \frac{\pi}{4}]$.

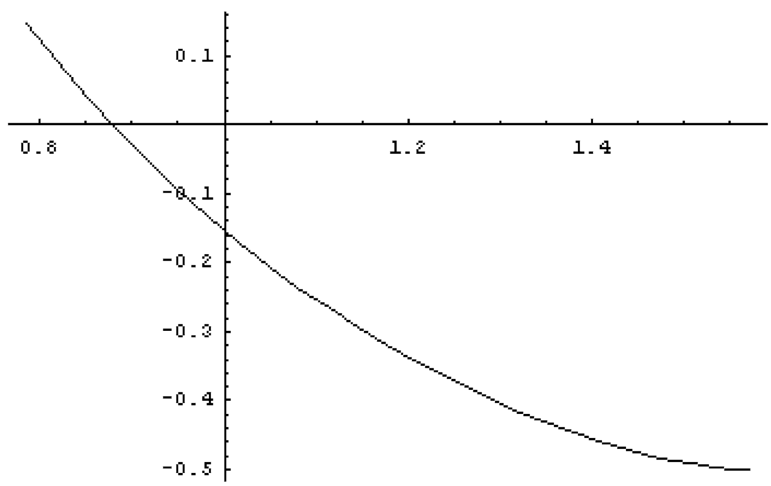

Figure 3 The graph of the function φ_*, where $t \in [\frac{\pi}{4}, \frac{\pi}{2})$.

By means of the Mathematica software we know that the equation

$$\frac{\mathrm{d}\varphi_*}{\mathrm{d}\theta} = 0, \quad \theta \in (0, \pi/2),$$

has no real roots and $\mathrm{d}\varphi_*/\mathrm{d}\theta < 0$. Hence, the function φ_* is decreasing. The solution of the inequality

$$\varphi_*(\theta) \geq 0, \quad \theta \in \left(0, \frac{\pi}{2}\right),$$

is

$$0 < \theta \leq 0.8785901309936538\ldots = \frac{\eta}{2},$$

where $\eta = 1.7571802619873076\ldots$ is the unique real root of equation (13). This ends the proof. □

Lemma 7 *Let $S^{(2)}\{P, \Gamma, l\}$ be a centered 2-surround system. Then the inequality*

$$\bar{r}_P \leq \frac{\|A_+ - A\|}{2} \int_0^1 \sqrt{t^2 + \cot^2 \frac{\angle APA_+}{2}}\,\mathrm{d}t, \quad \forall A \in \Gamma, \tag{36}$$

holds if and only if

$$0 < \angle APA_+ \leq \eta, \quad \forall A \in \Gamma, \tag{37}$$

where $\eta = 1.7571802619873076\ldots$ is the unique real root of equation (13). *Equality in* (36) *holds if and only if P' is the midpoint of the closed straight line segment $[AA_+]$.*

Proof The relevant calculations in the proof are dependent on the Mathematica software since these calculations are very complex.

Let

$$A = x_A\mathbf{i}, \qquad A_+ = x_{A_+}\mathbf{i}, \qquad M = x\mathbf{i}, \quad x_A \leq x \leq x_{A_+}, \qquad P = r\mathbf{j}, \qquad P' = 0\mathbf{i} + 0\mathbf{j},$$

where $r \triangleq r_P > 0$, and let

$$\angle APA_+ = 2\theta, \qquad \alpha \triangleq \angle APP' = -\arctan\frac{x_A}{r}, \qquad \beta \triangleq \angle A_+PP' = \arctan\frac{x_{A_+}}{r}.$$

Then

$$\alpha, \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \qquad \alpha + \beta = 2\theta, \quad \theta \in \left(0, \frac{\pi}{2}\right), \qquad \|M - P\| = \sqrt{x^2 + r^2},$$
$$\|A_+ - A\| = x_{A_+} - x_A, \qquad r = \frac{\|A_+ - A\|}{\tan\alpha + \tan\beta} = \frac{\|A_+ - A\|\cos\alpha\cos\beta}{\sin 2\theta},$$

and

$$\begin{aligned}
\bar{r}_P &= \frac{1}{\|A_+ - A\|}\int_{[AA_+]} \|M - P\| \\
&= \frac{1}{\|A_+ - A\|}\int_{x_A}^{x_{A_+}} \sqrt{x^2 + r^2}\,\mathrm{d}x \\
&= \frac{1}{\|A_+ - A\|}\int_{-r\tan\alpha}^{r\tan\beta} \sqrt{x^2 + r^2}\,\mathrm{d}x \\
&= \frac{1}{\|A_+ - A\|}\int_{\frac{\|A_+ - A\|\sin\alpha\cos\beta}{\sin 2\theta}}^{\frac{\|A_+ - A\|\cos\alpha\sin\beta}{\sin 2\theta}} \sqrt{x^2 + \left(\frac{\|A_+ - A\|\cos\alpha\cos\beta}{\sin 2\theta}\right)^2}\,\mathrm{d}x \\
&= \frac{1}{\|A_+ - A\|}\left(\frac{\|A_+ - A\|\cos\alpha\cos\beta}{\sin 2\theta}\right)^2 \int_{-\tan\alpha}^{\tan\beta} \sqrt{t^2 + 1}\,\mathrm{d}t \\
&= \frac{\|A_+ - A\|}{\sin^2 2\theta(1 + \tan^2\alpha)(1 + \tan^2\beta)} \int_{-\tan\alpha}^{\tan\beta} \sqrt{t^2 + 1}\,\mathrm{d}t,
\end{aligned}$$

where

$$x = \frac{\|A_+ - A\|\cos\alpha\cos\beta}{\sin 2\theta} t.$$

Hence,

$$\bar{r}_P = \frac{\|A_+ - A\|}{\sin^2 2\theta(1+\tan^2\alpha)(1+\tan^2\beta)} \int_{-\tan\alpha}^{\tan\beta} \sqrt{t^2+1}\,\mathrm{d}t. \tag{38}$$

By (38) we see that inequality (36) can be rewritten as

$$\frac{1}{(1+\tan^2\alpha)(1+\tan^2\beta)} \int_{-\tan\alpha}^{\tan\beta} \sqrt{t^2+1}\,\mathrm{d}t \le \frac{\sin^2 2\theta}{2} \int_0^1 \sqrt{t^2+\cot^2\theta}\,\mathrm{d}t. \tag{39}$$

Equality in (39) holds if $\alpha = \beta$.

By the symmetry we can further assume that $\beta \ge \alpha$.

For any fixed θ, set

$$\frac{\beta-\alpha}{2} = \omega.$$

Then

$$\alpha = \alpha(\omega) = \theta - \omega, \qquad \frac{\mathrm{d}\alpha}{\mathrm{d}\omega} = -1, \qquad \beta = \beta(\omega) = \theta + \omega, \qquad \frac{\mathrm{d}\beta}{\mathrm{d}\omega} = 1, \tag{40}$$

and

$$\omega \in \left[0, \frac{\pi}{2} - \theta\right) \quad \Leftrightarrow \quad \cos\omega \in (\sin\theta, 1]. \tag{41}$$

Now we define two auxiliary functions:

$$\varphi : \left[0, \frac{\pi}{2} - \theta\right) \to \mathbb{R}, \qquad \varphi(\omega) \triangleq \frac{1}{(1+\tan^2\alpha)(1+\tan^2\beta)} \int_{-\tan\alpha}^{\tan\beta} \sqrt{t^2+1}\,\mathrm{d}t$$

and

$$f : \left[0, \frac{\pi}{2} - \theta\right) \to \mathbb{R}, \qquad f(\omega) \triangleq \int_{-\tan\alpha}^{\tan\beta} \sqrt{t^2+1}\,\mathrm{d}t - \left(1+\tan^2\alpha\right)\left(1+\tan^2\beta\right)\varphi(0).$$

Since equality in (39) holds if $\alpha = \beta \Leftrightarrow \omega = 0$, by (35) we see that

$$\varphi(0) = \frac{\sin^2 2\theta}{2} \int_0^1 \sqrt{t^2+\cot^2\theta}\,\mathrm{d}t = \frac{\sin^2 2\theta}{4}\left[\csc\theta + \cot^2\theta \ln(\tan\theta + \sec\theta)\right]. \tag{42}$$

By (42) we know that inequality (39) can be rewritten as

$$f(\omega) \le 0, \quad \forall \omega \in \left[0, \frac{\pi}{2} - \theta\right). \tag{43}$$

Next, we prove that if (37) holds, then (43) holds, that is, (36) holds.

By means of the Mathematica software and (40) we can get

$$\frac{\cos^2\alpha + \cos^2\beta + \cos\alpha\cos\beta}{\cos\alpha + \cos\beta} = \frac{4\cos^2\theta - 1}{2\cos\theta}\cos\omega + \frac{\sin^2\theta}{2\cos\theta\cos\omega}$$

and

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}\omega} &= \sqrt{\tan^2\beta+1}\sec^2\beta\frac{\mathrm{d}\beta}{\mathrm{d}\omega} - \sqrt{\tan^2\alpha+1}\left(-\sec^2\alpha\right)\frac{\mathrm{d}\alpha}{\mathrm{d}\omega} \\
&\quad - \left[2\tan\alpha\sec^2\alpha\left(1+\tan^2\beta\right)\varphi(0)\frac{\mathrm{d}\alpha}{\mathrm{d}\omega} + 2\tan\beta\sec^2\beta\left(1+\tan^2\alpha\right)\varphi(0)\frac{\mathrm{d}\beta}{\mathrm{d}\omega}\right] \\
&= \sec^3\beta - \sec^3\alpha - 2\sec^2\alpha\sec^2\beta(\tan\beta-\tan\alpha)\varphi(0) \\
&= (\tan\beta-\tan\alpha)\left[(\tan\beta+\tan\alpha)\frac{\sec^3\beta-\sec^3\alpha}{\sec^2\beta-\sec^2\alpha} - 2\sec^2\alpha\sec^2\beta\varphi(0)\right] \\
&= \frac{\sin 2\omega}{\cos\beta\cos\alpha}\left[\frac{\sin 2\theta}{\cos\beta\cos\alpha}\frac{\sec^2\beta+\sec^2\alpha+\sec\beta\sec\alpha}{\sec\beta+\sec\alpha} - 2\sec^2\alpha\sec^2\beta\varphi(0)\right] \\
&= \sin 2\omega\sec^3\alpha\sec^3\beta\left[\sin 2\theta\frac{\cos^2\beta+\cos^2\alpha+\cos\beta\cos\alpha}{\cos\beta+\cos\alpha} - 2\varphi(0)\right] \\
&= \sin 2\omega\sec^3\alpha\sec^3\beta\left[\sin 2\theta\left(\frac{4\cos^2\theta-1}{2\cos\theta}\cos\omega + \frac{\sin^2\theta}{2\cos\theta\cos\omega}\right) - 2\varphi(0)\right] \\
&= \sin 2\omega\sec^3\alpha\sec^3\beta\left\{\sin\theta\left[\left(4\cos^2\theta-1\right)\cos\omega + \frac{\sin^2\theta}{\cos\omega}\right] - 2\varphi(0)\right\} \\
&= \sin 2\omega\sec^3\alpha\sec^3\beta g(\omega),
\end{aligned}$$

that is,

$$\frac{\mathrm{d}f}{\mathrm{d}\omega} = \sin 2\omega\sec^3\alpha\sec^3\beta g(\omega), \tag{44}$$

where

$$g:\left[0,\frac{\pi}{2}-\theta\right)\to\mathbb{R}, \qquad g(\omega)\triangleq\sin\theta\left[\left(4\cos^2\theta-1\right)\cos\omega+\frac{\sin^2\theta}{\cos\omega}\right]-2\varphi(0).$$

Since

$$g_*:(\sin\theta,1]\to\mathbb{R}, \qquad g_*(\xi)\triangleq\sin\theta\left[\left(4\cos^2\theta-1\right)\xi+\frac{\sin^2\theta}{\xi}\right]-2\varphi(0)$$

is a convex function, that is,

$$\frac{\mathrm{d}^2 g_*}{\mathrm{d}\xi^2} = \frac{2\sin^3\theta}{\xi^3} > 0, \quad \forall\xi\in(\sin\theta,1]$$

and

$$g(\omega) = g_*(\cos\xi),$$

we have

$$g_*(\xi)\le\max\left\{g_*(\sin\theta),g_*(1)\right\} \quad\Leftrightarrow\quad g(\omega)\le\max\left\{g(0),g\left(\frac{\pi}{2}-\theta\right)\right\}. \tag{45}$$

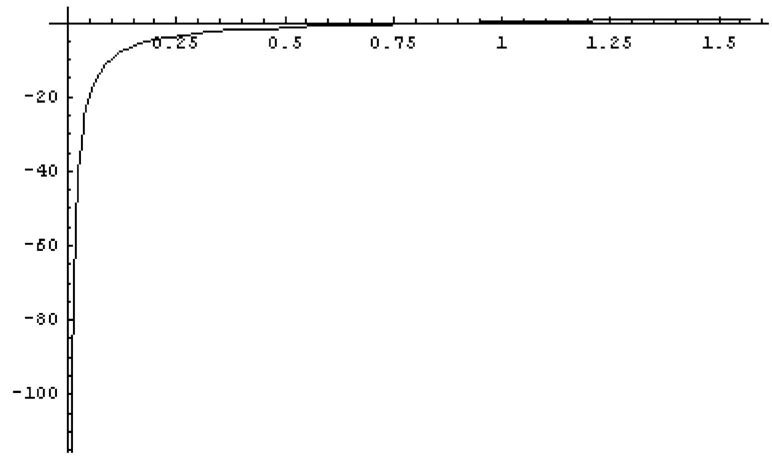

Figure 4 The graph of the function g^*, where $\theta \in (0, \frac{\pi}{2})$.

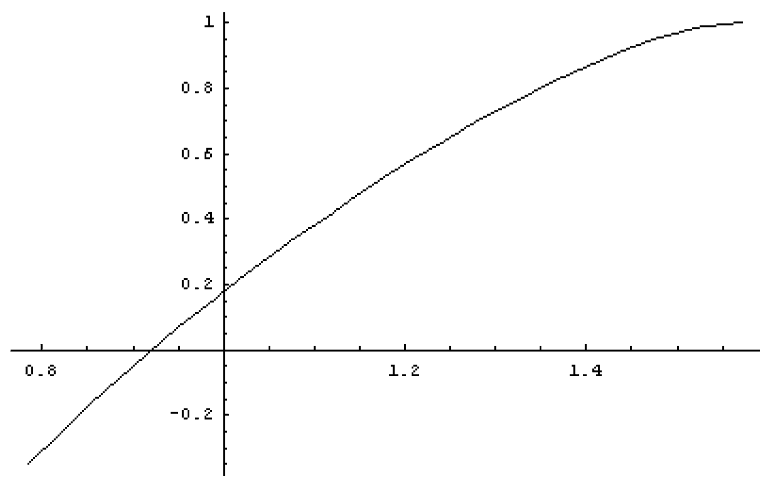

Figure 5 The graph of the function g^*, where $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$.

Now we prove that

$$g(0) \le 0 \quad \text{and} \quad g\left(\frac{\pi}{2} - \theta\right) \le 0. \tag{46}$$

Consider the auxiliary function:

$$g^* : \left(0, \frac{\pi}{2}\right) \to \mathbb{R}, \qquad g^*(\theta) = \csc\theta - 2\cot^2\theta \ln(\tan\theta + \sec\theta).$$

The graph of the function g^* is depicted in Figures 4 and 5.

By (42) we get

$$\begin{aligned} g(0) &= \sin\theta\big[\big(4\cos^2\theta - 1\big) + \sin^2\theta\big] - 2\varphi(0) \\ &= 3\sin\theta\cos^2\theta - \frac{\sin^2 2\theta}{2}\big[\csc\theta + \cot^2\theta \ln(\tan\theta + \sec\theta)\big] \end{aligned}$$

$$\begin{aligned}
&= 3\sin\theta\cos^2\theta - 2\sin^2\theta\cos^2\theta\bigl[\csc\theta + \cot^2\theta\ln(\tan\theta + \sec\theta)\bigr] \\
&= \sin^2\theta\cos^2\theta\bigl[\csc\theta - 2\cot^2\theta\ln(\tan\theta + \sec\theta)\bigr] \\
&= \sin^2\theta\cos^2\theta g^*(\theta),
\end{aligned}$$

that is,

$$g(0) = \sin^2\theta\cos^2\theta g^*(\theta). \tag{47}$$

By means of the Mathematica software we get

$$\begin{aligned}
\frac{\mathrm{d}g^*}{\mathrm{d}\theta} &= -\cot\theta\csc\theta + 4\cot\theta\csc^2\theta\ln(\tan\theta + \sec\theta) \\
&\quad - \frac{2\cot^2\theta(\sec^2\theta + \sec\theta\tan\theta)}{\sec\theta + \tan\theta}.
\end{aligned}$$

The equation $\mathrm{d}g^*/\mathrm{d}\theta = 0$ has no real roots in the interval $(0, \pi/2)$, and $\mathrm{d}g^*/\mathrm{d}\theta > 0$. Hence,

$$g^*(\theta) \le 0 \quad \Leftrightarrow \quad 0 < \theta \le \theta_0 \quad \text{and} \quad g^*(\theta) \ge 0 \quad \Leftrightarrow \quad \theta_0 \le \theta < \frac{\pi}{2},$$

where $\theta_0 \triangleq 0.9212996176628999\ldots$ is the root of the equation

$$g^*(\theta) = 0, \quad \theta \in \left(0, \frac{\pi}{2}\right).$$

Since (37) holds, and

$$0 < \theta \le \frac{\eta}{2} = 0.8785901309936538\ldots < 0.9212996176628999\ldots = \theta_0,$$

by (47) we get

$$g^*(\theta) \le 0 \quad \Leftrightarrow \quad g(0) = \sin^2\theta\cos^2\theta g^*(\theta) \le 0. \tag{48}$$

By (42) and Lemma 6 we get

$$\begin{aligned}
g\left(\frac{\pi}{2} - \theta\right) &= \sin\theta\left[\left(4\cos^2\theta - 1\right)\cos\omega + \frac{\sin^2\theta}{\cos\omega}\right]_{\omega=\frac{\pi}{2}-\theta} - 2\varphi(0) \\
&= \sin\theta\bigl[\left(4\cos^2\theta - 1\right)\sin\theta + \sin\theta\bigr] - 2\frac{\sin^2 2\theta}{2}\int_0^1 \sqrt{t^2 + \cot^2\theta}\,\mathrm{d}t \\
&= 4\sin^2\theta\cos^2\theta - \sin^2 2\theta\int_0^1 \sqrt{t^2 + \cot^2\theta}\,\mathrm{d}t \\
&= \sin^2 2\theta - \sin^2 2\theta\int_0^1 \sqrt{t^2 + \cot^2\theta}\,\mathrm{d}t \\
&= \sin^2 2\theta\left(1 - \int_0^1 \sqrt{t^2 + \cot^2\theta}\,\mathrm{d}t\right) \\
&\le 0,
\end{aligned}$$

that is,

$$g\left(\frac{\pi}{2}-\theta\right) \leq 0. \tag{49}$$

Combining (48) and (49), we get (46). Hence, (46) is proved.

From (45) and (46) we get

$$g(\omega) \leq \max\left\{g(0), g\left(\frac{\pi}{2}-\theta\right)\right\} \leq 0, \quad \forall \omega \in \left[0, \frac{\pi}{2}-\theta\right). \tag{50}$$

From (44) and (50) we get

$$\frac{\mathrm{d}f}{\mathrm{d}\omega} = \sin 2\omega \sec^3 \alpha \sec^3 \beta g(\omega) \leq 0, \quad \forall \omega \in \left[0, \frac{\pi}{2}-\theta\right).$$

Therefore,

$$f(\omega) \leq f(0) = 0, \quad \forall \omega \in \left[0, \frac{\pi}{2}-\theta\right),$$

which is just inequality (43). Hence, (43) holds, and (36) is proved.

Next, we prove that if inequality (43) holds (*i.e.*, (36) holds), then (37) holds.

Indeed, inequality (43) is equivalent to inequality (39). Since

$$\begin{aligned}
&\frac{1}{(1+\tan^2\alpha)(1+\tan^2\beta)} \int_{-\tan\alpha}^{\tan\beta} \sqrt{t^2+1}\,\mathrm{d}t \\
&\quad = \cos^2\alpha \cos^2\beta \int_{-\tan\alpha}^{\tan\beta} \sqrt{t^2+1}\,\mathrm{d}t \\
&\quad = \int_{-\tan\alpha}^{\tan\beta} \sqrt{(\cos\alpha\cos\beta t)^2 + (\cos\alpha\cos\beta)^2}\,\mathrm{d}(\cos\alpha\cos\beta t) \\
&\quad = \int_{-\cos\beta\sin\alpha}^{\cos\alpha\sin\beta} \sqrt{t^2 + (\cos\alpha\cos\beta)^2}\,\mathrm{d}t,
\end{aligned}$$

we can rewrite inequality (39) as

$$\frac{2}{\sin^2 2\theta} \int_{-\cos\beta\sin\alpha}^{\cos\alpha\sin\beta} \sqrt{t^2 + (\cos\alpha\cos\beta)^2}\,\mathrm{d}t \leq \int_0^1 \sqrt{t^2+\cot^2\theta}\,\mathrm{d}t. \tag{51}$$

Set

$$\beta \to \frac{\pi}{2} \quad \Leftrightarrow \quad \alpha = 2\theta - \beta \to 2\theta - \frac{\pi}{2}$$

in (51). Then

$$\begin{aligned}
&\frac{2}{\sin^2 2\theta} \int_{-\cos\beta\sin\alpha}^{\cos\alpha\sin\beta} \sqrt{t^2 + (\cos\alpha\cos\beta)^2}\,\mathrm{d}t \leq \int_0^1 \sqrt{t^2+\cot^2\theta}\,\mathrm{d}t \\
&\quad \Leftrightarrow \quad \frac{2}{\sin^2 2\theta} \int_0^{\sin 2\theta} t\,\mathrm{d}t \leq \int_0^1 \sqrt{t^2+\cot^2\theta}\,\mathrm{d}t
\end{aligned}$$

$$\Leftrightarrow \quad \frac{1}{\sin^2 2\theta} t^2 |_0^{\sin 2\theta} \leq \int_0^1 \sqrt{t^2 + \cot^2 \theta}\, \mathrm{d}t$$
$$\Leftrightarrow \quad \int_0^1 \sqrt{t^2 + \cot^2 \theta}\, \mathrm{d}t \geq 1,$$

that is, inequality (33) holds. By Lemma 6 we have

$$0 < \angle APA_+ = 2\theta \leq \eta, \quad \forall A \in \Gamma;$$

hence, (37) holds.

Based on this proof, we know that equality in (36) holds if and only if P' is the midpoint of the closed straight line segment $[AA_+]$. This completes the proof of Lemma 7. □

Lemma 8 (see [19, 20]) *Let $E \subset \mathbb{R}^m$ be a bounded and closed region* (*or curve*), *and let the functions $f : E \to \mathbb{R}$ and $\phi : f(E) \to \mathbb{R}$ be integrable, where $f(E)$ is an interval. If $\phi : f(E) \to \mathbb{R}$ is a convex function, then we have the following Jensen inequality*:

$$\frac{\int_E \phi(f)}{\int_E} \geq \phi\left(\frac{\int_E f}{\int_E}\right). \tag{52}$$

4 Proof of Theorem 1

Proof Consider the auxiliary function

$$\psi : \left(0, \frac{\pi}{2}\right) \to \mathbb{R}, \qquad \psi(\theta) = \left(\int_0^1 \sqrt{t^2 + \cot^2 \theta}\, \mathrm{d}t\right)^{-1}.$$

By means of the Mathematica software and (35) we get

$$\psi(\theta) = 2\left[\csc\theta + \cot^2\theta \ln(\tan\theta + \sec\theta)\right]^{-1}$$

and

$$\frac{\mathrm{d}\psi}{\mathrm{d}\theta} = -2\frac{-\cot\theta\csc\theta - 2\cot\theta\csc^2\theta\ln(\sec\theta + \tan\theta) + \frac{\cot^2\theta(\sec^2\theta + \sec\theta\tan\theta)}{\sec\theta + \tan\theta}}{[\csc\theta + \cot^2\theta\ln(\sec\theta + \tan\theta)]^2}.$$

The graph of the function $\mathrm{d}\psi/\mathrm{d}\theta$ is depicted in Figure 6.

The function $\mathrm{d}\psi/\mathrm{d}\theta$ has unique extremum point $\theta_1 = 1.2313051084629325\ldots$ in the interval $(0, \pi/2)$.

The increasing and decreasing intervals of the function $\mathrm{d}\psi/\mathrm{d}\theta$ are $(0, \theta_1]$ and $[\theta_1, \pi/2)$, respectively. Hence, the convex and concave intervals of the function ψ are $(0, \theta_1]$ and $[\theta_1, \pi/2)$, respectively. Since

$$0 < \frac{\eta}{2} = 0.8785901309936538\ldots < 1.2313051084629325\ldots = \theta_1,$$

we see that the function

$$\psi : \left(0, \frac{\eta}{2}\right] \to \mathbb{R}$$

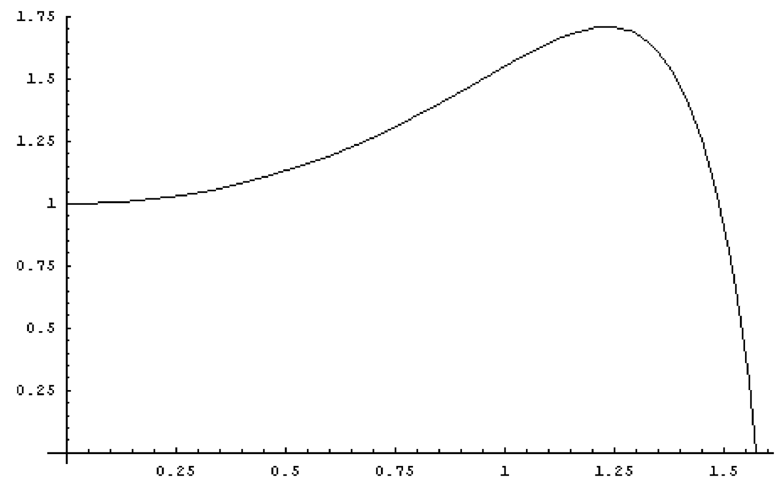

Figure 6 The graph of the function $d\psi/d\theta$, where $\theta \in (0, \frac{\pi}{2})$.

is a convex function.

Since

$$0 < \angle APA_+ \leq \eta \quad \Leftrightarrow \quad 0 < \frac{\angle APA_+}{2} \leq \frac{\eta}{2}, \quad \forall A \in \Gamma,$$

by Lemma 7 we have

$$\bar{r}_P \leq \frac{\|A_+ - A\|}{2} \int_0^1 \sqrt{t^2 + \cot^2 \frac{\angle APA_+}{2}}\, \mathrm{d}t,$$

that is,

$$2\psi\left(\frac{\angle APA_+}{2}\right) \leq \frac{\|A_+ - A\|}{\bar{r}_P}. \tag{53}$$

Form (53) we get

$$\frac{2}{|\Gamma|} \oint_\Gamma \psi\left(\frac{\angle APA_+}{2}\right) \leq \frac{1}{|\Gamma|} \oint_\Gamma \frac{\|A_+ - A\|}{\bar{r}_P}. \tag{54}$$

Since

$$\angle APA_+ \in (0, \eta], \qquad \oint_\Gamma = |\Gamma|,$$

by Lemmas 8 and 4 we get

$$\frac{1}{|\Gamma|} \oint_\Gamma \psi\left(\frac{\angle APA_+}{2}\right) \geq \psi\left(\frac{1}{|\Gamma|} \oint_\Gamma \frac{\angle APA_+}{2}\right) = \left(\int_0^1 \sqrt{t^2 + \cot^2 \frac{l\pi}{|\Gamma|}}\, \mathrm{d}t\right)^{-1}. \tag{55}$$

By the Cauchy inequality [2]

$$\frac{1}{|\Gamma|} \oint_\Gamma \frac{\|A_+ - A\|}{\bar{r}_P} \leq \sqrt{\frac{1}{|\Gamma|} \oint_\Gamma \|A_+ - A\|^2} \sqrt{\frac{1}{|\Gamma|} \oint_\Gamma \bar{r}_P^{-2}}$$

and Lemma 5 we get

$$\frac{1}{|\Gamma|}\oint_{\Gamma}\frac{\|A_{+}-A\|}{\bar{r}_P}\le\frac{|\Gamma|}{\pi}\sin\frac{l\pi}{|\Gamma|}\sqrt{\frac{1}{|\Gamma|}\oint_{\Gamma}\bar{r}_P^{-2}}.\tag{56}$$

By (54), (55), and (56) we get

$$2\left(\int_0^1\sqrt{t^2+\cot^2\frac{l\pi}{|\Gamma|}}\,\mathrm{d}t\right)^{-1}\le\frac{|\Gamma|}{\pi}\sin\frac{l\pi}{|\Gamma|}\sqrt{\frac{1}{|\Gamma|}\oint_{\Gamma}\bar{r}_P^{-2}}.\tag{57}$$

By (57), the power mean inequality (3), and (35), for $p\in(-\infty,-2]$, we have

$$\begin{aligned}\left(\frac{1}{|\Gamma|}\oint_{\Gamma}\bar{r}_P^{p}\right)^{\frac{1}{p}}&\le\left(\frac{1}{|\Gamma|}\oint_{\Gamma}\bar{r}_P^{-2}\right)^{-\frac{1}{2}}\\&\le\frac{|\Gamma|}{2\pi}\sin\frac{l\pi}{|\Gamma|}\int_0^1\sqrt{t^2+\cot^2\frac{l\pi}{|\Gamma|}}\,\mathrm{d}t\\&=\frac{|\Gamma|}{4\pi}\sin\frac{l\pi}{|\Gamma|}\left[\csc\frac{l\pi}{|\Gamma|}+\cot^2\frac{l\pi}{|\Gamma|}\ln\left(\tan\frac{l\pi}{|\Gamma|}+\sec\frac{l\pi}{|\Gamma|}\right)\right].\end{aligned}$$

This proves inequality (12).

Based on this proof, we know that the equality in (12) holds if and only if Γ is a circle and P is the center of the circle. This completes the proof of Theorem 1. □

5 Applications in space science

Corollary 1 is of great significance in space science.

Let $S^{(2)}\{P,\Gamma\}$ be a centered surround system. We may regard P as the Earth (or an atomic nucleus, *etc.*) with mass M, A as the Moon (or an electron of the atom, *etc.*) with mass m, which is a satellite of the Earth, and Γ as the orbit of the Moon. According to the law of universal gravitation, the gravity of the Moon A to the Earth P is

$$\mathbf{F}(A,P)=\frac{GmM(A-P)}{\|A-P\|^3},\tag{58}$$

and the norm $\|\mathbf{F}(A,P)\|$ of the gravity $\mathbf{F}(A,P)$ between the Moon A and the Earth P is

$$\left\|\mathbf{F}(A,P)\right\|=\frac{GmM}{\|A-P\|^2},\tag{59}$$

where G is the gravitational constant of the solar system. Without loss of generality, we may assume that $GmM=1$.

When the Moon A traverses one cycle along its orbit Γ, the mean of the norm $\|\mathbf{F}(\Gamma,P)\|$ of the gravity $\mathbf{F}(A,P)$ between the Moon A and the Earth P is

$$\overline{\left\|\mathbf{F}(\Gamma,P)\right\|}\triangleq M_{\Gamma}\left(\left\|\mathbf{F}(\Gamma,P)\right\|\right)=\frac{1}{|\Gamma|}\oint_{\Gamma}\frac{1}{\|A-P\|^2}.\tag{60}$$

In [10], the authors defined the λ-*gravity* as follows:

$$\mathbf{F}_{\lambda}(A,P)=\frac{GmM(A-P)}{\|A-P\|^{\lambda+1}}=\frac{A-P}{\|A-P\|^{\lambda+1}},\quad\lambda>0,\tag{61}$$

where

$$\mathbf{F}_2(A,P) = \mathbf{F}(A,P).$$

In the solar system, the gravity of the physical matter X to another physical matter P is $\mathbf{F}(A,P)$, whereas for another galaxy in the universe, the gravity may be $\mathbf{F}_\lambda(A,P)$, where $\lambda \in (0,2) \cup (2,+\infty)$. For example, in the black hole of the universe, we conjecture that the gravity is $\mathbf{F}_\lambda(A,P)$ with $\lambda \in (0,2)$, P can be regarded as an atomic nucleus of an atom, A can be regarded as an electron of the atom, and Γ can be regarded as the orbit of the electron.

We define as

$$\mathbf{F}_\lambda(A,P) \triangleq \frac{A-P}{\|A-P\|^{\lambda+1}}, \tag{62}$$

$$\left\|\mathbf{F}_\lambda(A,P)\right\| \triangleq \frac{1}{\|A-P\|^{\lambda}}, \tag{63}$$

and

$$\overline{\left\|\mathbf{F}_\lambda(\Gamma,P)\right\|} \triangleq \frac{1}{|\Gamma|}\oint_\Gamma \frac{1}{\|A-P\|^{\lambda}} \tag{64}$$

the λ-*gravity function*, λ-*gravity norm*, and *mean* λ-*gravity norm* between the Moon A and the Earth P, respectively, where $\lambda \in (0,\infty)$.

In [21], the authors defined the planet system $\mathrm{PS}\{P,m,\mathrm{B}(g,r)\}_{\mathbb{E}}^{n}$ in an Euclidean space $\mathbb{E}$ and the λ-gravity function

$$\mathbf{F}_\lambda : \mathbb{E}^n \to \mathbb{E}, \qquad \mathbf{F}_\lambda(P) \triangleq \sum_{i=1}^{n} \frac{m_i p_i}{\|p_i\|^{\lambda+1}}$$

in the planet system, and obtained some interesting results. For example, in the planet system $\mathrm{PS}\{P,m,\mathrm{B}(g,r)\}_{\mathbb{E}}^{n}$, if $\lambda > \mu > 2$ and $\|g\| \geq \sqrt{2}$, then we have the following inequality:

$$\frac{\mathrm{Var}_\lambda^*(P)}{\mathrm{Var}_\mu^*(P)} \geq \frac{\mu}{\lambda}\left[\frac{\|\mathbf{F}_2(P)\|}{\|\mathbf{F}_0(P)\|}\right]^{\lambda-\mu}, \tag{65}$$

where

$$\mathrm{Var}_\lambda^*(P) \triangleq \frac{8}{\lambda(\lambda-2)}\left[\left(\frac{\|\mathbf{F}_\lambda(P)\|}{\|\mathbf{F}_0(P)\|}\right)^2 - \left(\frac{\|\mathbf{F}_2(P)\|}{\|\mathbf{F}_0(P)\|}\right)^\lambda\right],$$

and the coefficient μ/λ in (65) is the best constant.

In this section, we will establish a new isoperimetric inequality involving the λ-gravity.

Corollary 1 implies the following interesting result, which is significant in space science.

Proposition 3 (λ-gravity inequality) *Let $S^{(2)}\{P,\Gamma\}$ be a centered surround system. Then we have the following isoperimetric inequality*:

$$\overline{\left\|\mathbf{F}_\lambda(\Gamma,P)\right\|} \triangleq \frac{1}{|\Gamma|}\oint_\Gamma \frac{1}{\|A-P\|^{\lambda}} \geq \left(\frac{2\pi}{|\Gamma|}\right)^\lambda, \quad \forall \lambda \geq 2. \tag{66}$$

Equality in (66) *holds if and only if Γ is a circle and P is the center of the circle.*

In [10], the authors obtained the following interesting inequality:

$$\sum_{j=1}^{m} \mu_j \overline{\left\| \mathbf{F}_{\alpha_j}(A,P) \right\|}^{\frac{\gamma}{\alpha_j}} \geq \overline{\left\| \mathbf{F}_{\lambda}(\Gamma,P) \right\|}^{\frac{\gamma}{\lambda}}, \tag{67}$$

where

$$\alpha, \mu \in (0,\infty)^m, \qquad \sum_{j=1}^{m} \mu_j = 1, \quad m \geq 2, \qquad \gamma \in (0,\infty), \qquad 0 < \lambda \leq \left(\sum_{j=1}^{m} \frac{\mu_j}{\alpha_j} \right)^{-1}.$$

According to Proposition 3 and (67), we know that in the centered surround system $S^{(2)}\{P,\Gamma\}$, if

$$\alpha, \mu \in (0,\infty)^m, \qquad \sum_{j=1}^{m} \mu_j = 1, \quad m \geq 2, \qquad \gamma \in (0,\infty), \qquad \left(\sum_{j=1}^{m} \frac{\mu_j}{\alpha_j} \right)^{-1} \geq 2,$$

then we have the following isoperimetric inequality:

$$\sum_{j=1}^{m} \mu_j \overline{\left\| \mathbf{F}_{\alpha_j}(A,P) \right\|}^{\frac{\gamma}{\alpha_j}} \geq \left(\frac{2\pi}{|\Gamma|} \right)^{\gamma}. \tag{68}$$

Equality in (68) holds if and only if Γ is a circle and P is the center of the circle.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in this paper. All authors read and approved the final manuscript.

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