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Generalized multivalued equilibrium-like problems: auxiliary principle technique and predictor-corrector methods

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Abstract

This paper is dedicated to the introduction a new class of equilibrium problems named generalized multivalued equilibrium-like problems which includes the classes of hemiequilibrium problems, equilibrium-like problems, equilibrium problems, hemivariational inequalities, and variational inequalities as special cases. By utilizing the auxiliary principle technique, some new predictor-corrector iterative algorithms for solving them are suggested and analyzed. The convergence analysis of the proposed iterative methods requires either partially relaxed monotonicity or jointly pseudomonotonicity of the bifunctions involved in generalized multivalued equilibrium-like problem. Results obtained in this paper include several new and known results as special cases.

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Keywords: generalized multivalued equilibrium-like problems; multivalued hemiequilibrium problems; auxiliary principle technique; predictor-corrector methods; convergence analysis

1 Introduction

During the last decades, the theory of variational analysis including variational inequalities (VI) have attracted a lot of attention because of its applications in optimization, nonlinear analysis, game theory, economics, and so forth; see, for example, [1] and the references therein. Because of the importance and active impact of VI in branches of sciences, engineering, and social sciences, it has been extended and generalized in many different directions. It has been used as a tool to study different aspects of optimization problems; see, for example, [1–4] and the references therein. By replacing the linear term appearing in the formulation of variational inequalities by a vector-valued term, Parida *et al.* [5] and Yang and Chen [6] independently introduced and studied a class of variational inequalities known as variational-like inequalities or pre-variational inequalities which is an important extension of the variational inequalities. Another useful and important generalization of variational inequalities is a class of variational inequalities known as hemivariational inequalities involving the nonlinear Lipschitz continuous functions. It should be pointed out that the hemivariational inequalities are connected with nonconvex and possibly nonsmooth energy functions and have important applications in structural analysis and nonconvex optimization. For more details, we refer the reader to [7–10].

On the other hand, the concept of equilibrium plays a central role in various applied sciences, such as physics (especially, mechanics), economics, finance, optimization, image reconstruction, network, ecology, sociology, chemistry, biology, engineering sciences, transportation, and other fields. The study of equilibrium problems (EP), which was introduced by Blum and Oettli [11] in 1994, in terms of their formulation, qualitative analysis, and computation has been the focus of much research in the past several decades and has given rise to the development of a variety of mathematical methodologies. Examples of mathematical formulations that have been used for equilibrium problems are: nonlinear equations, optimization problems, complementarity problems, fixed point problems, and, most recently, variational inequality problems; see, for example, [11–15] and the references therein.

In recent years, EP has received much attention by many authors due to the fact that it provides a unified model including the above mentioned problems, and various important generalizations of it have been proposed and analyzed; see, for example, [16–26] and the references therein. An important and useful generalization of EP is the multivalued equilibrium problems involving a nonlinear bifunction. It has been shown that a wide class of unrelated odd order and nonsymmetric free, moving, obstacle and equilibrium problems can be studied via the multivalued equilibrium problems.

Inspired and motivated by the research going in this interesting and fascinating area, Noor [27, 28] introduced and investigated the class of equilibrium-like (or invex equilibrium) problems as a useful and important extension of the class of equilibrium problems. It has been shown that equilibrium-like problems include equilibrium problems, variational-like inequalities, variational inequalities and their invariant forms as special cases. In the meanwhile, related to the hemivariational inequalities, Noor [29] introduced and studied the class of hemiequilibrium problems as a generalization of the class of equilibrium problems. It is shown that the hemiequilibrium problems include equilibrium problems, hemivariational inequalities and variational inequalities as special cases. It is worth mentioning that the hemiequilibrium problems and equilibrium-like problems are two quite different extensions of the classical equilibrium problems.

One of the most important and interesting problems in the theory of variational inequalities is the development of numerical methods which provide an efficient and implementable algorithm for solving variational inequalities and its generalizations. In the last decades, many efforts have been devoted to the development of efficiency and of implementable methods for solving variational inequalities and their extensions. Though, various numerical techniques are proposed for solving variational inequalities, but the nature of equilibrium problems does not allow us to use these methods in their present forms. As an example, projection technique, one of the main methods used in existence theory of variational inequalities, cannot be used in a similar way for equilibrium problems. The auxiliary principle technique helps to avoid these constraints and addresses the demand of nature of equilibrium problems in a right way. In this technique, a supporting (auxiliary) problem linked to the original one is considered. This actually is a way to define a mapping that relates the original problem wit the auxiliary problem. This technique was used by Glowinski *et al.* [30] to study the existence of a solution of mixed variational inequality and later was developed by many authors for solving various classes of variational inequalities and equilibrium problems; see, for example, [18, 30–32] and the references therein.

Motivated and inspired by the work mentioned above, the purpose of this paper is to introduce a new class of equilibrium problems named generalized multivalued equilibrium-like problems (GMELP), which includes hemiequilibrium problems, equilibrium-like problems, equilibrium problems, hemivariational inequalities, and variational-like inequalities as special cases. By using the auxiliary principle technique, we suggest and analyze some predictor-corrector methods for solving GMELP. The convergence analysis of the proposed iterative methods requires either partially relaxed monotonicity or jointly pseudomonotonicity of the bifunctions involved in GMELP. As special cases, one can obtain several new and known methods for solving variational inequalities and equilibrium problems. The results presented in this paper generalize and improve some recent results in this field.

2 Formulations, algorithms, and convergence results

Let $\mathcal{H}$ be a real Hilbert space whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let K be a nonempty closed set in $\mathcal{H}$ and let $CB(\mathcal{H})$ be the family of all nonempty, closed, and bounded subsets of $\mathcal{H}$. Suppose further that $S, T : K \to CB(\mathcal{H})$ are two multivalued operators and let $g : K \to K$ and $\eta : K \times K \to \mathcal{H}$ be two operators. For given bifunctions $F, G : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, we consider the problem of finding $u \in K$, $\nu \in T(u)$, and $\vartheta \in S(u)$ such that

$$F\big(\nu, g(v)\big) + G\big(\vartheta, \eta\big(g(v), g(u)\big)\big) \geq 0, \quad \forall v \in K, \tag{2.1}$$

which is called the *generalized multivalued equilibrium-like problem* (GMELP).

If $\eta(x,y) = x - y$, for all $x, y \in K$, then the problem (2.1) reduces to the problem of finding $u \in K$, $\nu \in T(u)$, and $\vartheta \in S(u)$ such that

$$F\big(\nu, g(v)\big) + G\big(\vartheta, g(v) - g(u)\big) \geq 0, \quad \forall v \in K, \tag{2.2}$$

which is called the *generalized multivalued hemiequilibrium problem* (GMHEP).

It should be remarked that by taking different choices of the operators S, T, η, g, and the bifunctions F and G in the above problems, one can easily obtain the problems studied in [11, 33] and the references therein.

In the sequel, we denote by GMELP(F, G, S, T, η, g, K) and GMHEP(F, G, S, T, g, K) the set of solutions of the problems (2.1) and (2.2), respectively.

Lemma 2.1 [34] *Let X be a complete metric space, $T : X \to CB(X)$ be a multivalued mapping. Then for any $\varepsilon > 0$ and for any given $x, y \in X$, $u \in T(x)$, there exists $v \in T(y)$ such that*

$$d(u,v) \leq (1+\varepsilon)M\big(T(x), T(y)\big),$$

where $M(\cdot,\cdot)$ is the Hausdorff metric on $CB(X)$ defined by

$$M(A,B) = \max\Big\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\Big\}, \quad \forall A, B \in CB(X).$$

Let F, G, S, T, η, and g be the same as in GMELP (2.1). For given $u \in K$, $\nu \in T(u)$, and $\vartheta \in S(u)$, we consider the auxiliary generalized multivalued hemiequilibrium-like problem of finding $w \in K$ such that

$$\rho F\bigl(\nu, g(v)\bigr) + \rho G\bigl(\vartheta, \eta\bigl(g(v), g(u)\bigr)\bigr) + \bigl\langle g(w) - g(u), g(v) - g(w)\bigr\rangle \geq 0, \quad \forall v \in K, \tag{2.3}$$

where $\rho > 0$ is a constant. Obviously, if $w = u$, then (w, ν, ϑ) is a solution of GMELP (2.1). This observation and Nadler's technique [34] enables us to suggest the following finite step predictor-corrector method for solving GMELP (2.1).

Algorithm 2.2 Let F, G, S, T, η, and g be the same as in GMELP (2.1). For given $u_0 \in K$, $\nu_0 \in T(u_0)$, and $\vartheta_0 \in S(u_0)$, compute the iterative sequences $\{u_n\}$, $\{\nu_n\}$, and $\{\vartheta_n\}$ by the iterative schemes

$$\begin{aligned} &\rho_1 F\bigl(\nu_{1,n}, g(v)\bigr) + \rho_1 G\bigl(\vartheta_{1,n}, \eta\bigl(g(v), g(u_{1,n})\bigr)\bigr) \\ &\quad + \bigl\langle g(u_{n+1}) - g(y_{1,n}), g(v) - g(u_{n+1})\bigr\rangle \geq 0, \quad \forall v \in K, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\rho_{i+1} F\bigl(\nu_{i+1,n}, g(v)\bigr) + \rho_{i+1} G\bigl(\vartheta_{i+1,n}, \eta\bigl(g(v), g(u_{i+1,n})\bigr)\bigr) \\ &\quad + \bigl\langle g(y_{i,n}) - g(y_{i+1,n}), g(v) - g(y_{i,n})\bigr\rangle \geq 0, \quad i = 1, 2, \ldots, q-2, \forall v \in K, \end{aligned} \tag{2.5}$$

$$\begin{aligned} &\rho_q F\bigl(\nu_n, g(v)\bigr) + \rho_q G\bigl(\vartheta_n, \eta\bigl(g(v), g(u_n)\bigr)\bigr) \\ &\quad + \bigl\langle g(y_{q-1,n}) - g(u_n), g(v) - g(y_{q-1,n})\bigr\rangle \geq 0, \quad \forall v \in K, \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\nu_{i,n} \in T(y_{i,n}) : \|\nu_{i,n+1} - \nu_{i,n}\| \leq \bigl(1 + (1+n)^{-1}\bigr) M\bigl(T(y_{i,n+1}), T(y_{i,n})\bigr), \\ &\quad i = 1, 2, \ldots, q-1, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\vartheta_{i,n} \in S(y_{i,n}) : \|\vartheta_{i,n+1} - \vartheta_{i,n}\| \leq \bigl(1 + (1+n)^{-1}\bigr) M\bigl(S(y_{i,n+1}), S(y_{i,n})\bigr), \\ &\quad i = 1, 2, \ldots, q-1, \end{aligned} \tag{2.8}$$

$$\nu_n \in T(u_n) : \|\nu_{n+1} - \nu_n\| \leq \bigl(1 + (1+n)^{-1}\bigr) M\bigl(T(u_{n+1}), T(u_n)\bigr), \tag{2.9}$$

$$\vartheta_n \in S(u_n) : \|\vartheta_{n+1} - \vartheta_n\| \leq \bigl(1 + (1+n)^{-1}\bigr) M\bigl(S(u_{n+1}), S(u_n)\bigr), \tag{2.10}$$

where $\rho_i > 0$ ($i = 1, 2, \ldots, q$) are constants and $n = 0, 1, 2, \ldots$.

If $\eta(x, y) = x - y$, for all $x, y \in K$, then Algorithm 2.2 reduces to the following predictor-corrector method.

Algorithm 2.3 For given $u_0 \in K$, $\nu_0 \in T(u_0)$ and $\vartheta_0 \in S(u_0)$, compute the iterative sequences $\{u_n\}$, $\{\nu_n\}$, and $\{\vartheta_n\}$ by the iterative schemes

$$\begin{aligned} &\rho_1 F\bigl(\nu_{1,n}, g(v)\bigr) + \rho_1 G\bigl(\vartheta_{1,n}, g(v) - g(u_{1,n})\bigr) \\ &\quad + \bigl\langle g(u_{n+1}) - g(y_{1,n}), g(v) - g(u_{n+1})\bigr\rangle \geq 0, \quad \forall v \in K, \\ &\rho_{i+1} F\bigl(\nu_{i+1,n}, g(v)\bigr) + \rho_{i+1} G\bigl(\vartheta_{i+1,n}, g(v) - g(u_{i+1,n})\bigr) \\ &\quad + \bigl\langle g(y_{i,n}) - g(y_{i+1,n}), g(v) - g(y_{i,n})\bigr\rangle \geq 0, \quad i = 1, 2, \ldots, q-2, \forall v \in K, \end{aligned}$$

$$
\begin{aligned}
&\rho_q F\big(\nu_n, g(\nu)\big) + \rho_q G\big(\vartheta_n, g(\nu) - g(u_n)\big) \\
&\quad + \big\langle g(y_{q-1,n}) - g(u_n), g(\nu) - g(y_{q-1,n})\big\rangle \geq 0, \quad \forall \nu \in K, \\
&\nu_{i,n} \in T(y_{i,n}) : \|\nu_{i,n+1} - \nu_{i,n}\| \leq \big(1 + (1+n)^{-1}\big) M\big(T(y_{i,n+1}), T(y_{i,n})\big), \quad i = 1, 2, \ldots, q-1, \\
&\vartheta_{i,n} \in S(y_{i,n}) : \|\vartheta_{i,n+1} - \vartheta_{i,n}\| \leq \big(1 + (1+n)^{-1}\big) M\big(S(y_{i,n+1}), S(y_{i,n})\big), \quad i = 1, 2, \ldots, q-1, \\
&\nu_n \in T(u_n) : \|\nu_{n+1} - \nu_n\| \leq \big(1 + (1+n)^{-1}\big) M\big(T(u_{n+1}), T(u_n)\big), \\
&\vartheta_n \in S(u_n) : \|\vartheta_{n+1} - \vartheta_n\| \leq \big(1 + (1+n)^{-1}\big) M\big(S(u_{n+1}), S(u_n)\big),
\end{aligned}
$$

where $\rho_i > 0$ ($i = 1, 2, \ldots, q$) are constants and $n = 0, 1, 2, \ldots$.

In order to study the convergence analysis of the iterative sequences generated by Algorithm 2.2, we need the following definitions.

Definition 2.1 Let $T : K \to CB(\mathcal{H})$ be a multivalued operator and $g : K \to K$ be an operator. The bifunction $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is said to be

(a) *g-monotone* with respect to T, if

$$
F\big(w_1, g(u_2)\big) + F\big(w_2, g(u_1)\big) \leq 0, \quad \forall u_1, u_2 \in K, w_1 \in T(u_1), w_2 \in T(u_2);
$$

(b) *partially α-relaxed g-monotone* with respect to T, if there exists a constant $\alpha > 0$ such that

$$
\begin{aligned}
&F\big(w_1, g(u_2)\big) + F\big(w_2, g(z)\big) \leq \alpha \big\|g(z) - g(u_1)\big\|^2, \\
&\quad \forall u_1, u_2, z \in K, w_1 \in T(u_1), w_2 \in T(u_2).
\end{aligned}
$$

It should be pointed out that if $z = u_1$, then the partially α-relaxed g-monotonicity of the bifunction F with respect to T reduces to g-monotonicity with respect to T. Meanwhile, if $g \equiv I$, then parts (a) and (b) of Definition 2.1 reduce to the definition of monotonicity and partially α-relaxed monotonicity of the bifunction F with respect to T, respectively.

Definition 2.2 Let $S : K \to CB(\mathcal{H})$ be a multivalued operator and let $g : K \to K$ and $\eta : K \times K \to \mathcal{H}$ be two operators. The bifunction $G : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is said to be

(a) *g-η-monotone* with respect to S, if

$$
\begin{aligned}
&G\big(w_1, \eta\big(g(u_2), g(u_1)\big)\big) + G\big(w_2, \eta\big(g(u_1), g(u_2)\big)\big) \leq 0, \\
&\quad \forall u_1, u_2 \in K, w_1 \in S(u_1), w_2 \in S(u_2);
\end{aligned}
$$

(b) *partially β-relaxed g-η-monotone* with respect to S, if there exists a constant $\beta > 0$ such that

$$
\begin{aligned}
&G\big(w_1, \eta\big(g(u_2), g(u_1)\big)\big) + G\big(w_2, \eta\big(g(z), g(u_2)\big)\big) \leq \beta \big\|g(z) - g(u_1)\big\|^2, \\
&\quad \forall u_1, u_2, z \in K, w_1 \in S(u_1), w_2 \in S(u_2).
\end{aligned}
$$

It should be remarked that if $z = u_1$, then the partially β-relaxed g-η-monotonicity of the bifunction G with respect to S reduces to g-monotonicity of the bifunction G with respect to S. Furthermore, for the case when $\eta(x,y) = x - y$, for all $x, y \in K$, then parts (a) and (b) of Definition 2.2 reduce to the definition of g-monotonicity and partially β-relaxed g-monotonicity of the bifunction G with respect to S, respectively.

Definition 2.3 A multivalued operator $T : \mathcal{H} \to CB(\mathcal{H})$ is said to be M-Lipschitz continuous with constant δ, if there exists a constant $\delta > 0$ such that

$$M\bigl(T(u), T(v)\bigr) \leq \delta \|u - v\|, \quad \forall u, v \in \mathcal{H},$$

where $M(\cdot,\cdot)$ is the Hausdorff metric on $CB(\mathcal{H})$.

The next proposition plays a crucial role in the study of convergence analysis of the iterative sequences generated by Algorithm 2.2.

Proposition 2.4 *Let F, G, S, T, η, and g be the same as in* GMELP (2.1) *and let $\hat{u} \in K$, $\hat{\nu} \in T(u)$, and $\hat{\vartheta} \in G(\hat{u})$ be the solution of* GMELP (2.1). *Suppose further that $\{u_n\}$ and $\{y_{i,n}\}$ $(i = 1, 2, \ldots, q - 1)$ are the sequences generated by Algorithm* 2.2. *If F is partially α-relaxed g-η-monotone with respect to T, and G is partially β-relaxed g-η-monotone with respect to S, then*

$$\bigl\|g(\hat{u}) - g(u_{n+1})\bigr\|^2 \leq \bigl\|g(\hat{u}) - g(y_{1,n})\bigr\|^2 - \bigl(1 - 2(\alpha + \beta)\rho_1\bigr)\bigl\|g(u_{n+1}) - g(y_{1,n})\bigr\|^2, \tag{2.11}$$

$$\bigl\|g(\hat{u}) - g(y_{i,n})\bigr\|^2 \leq \bigl\|g(\hat{u}) - g(y_{i+1,n})\bigr\|^2 - \bigl(1 - 2(\alpha + \beta)\rho_{i+1}\bigr)\bigl\|g(y_{i,n}) - g(y_{i+1,n})\bigr\|^2, \tag{2.12}$$

$$\bigl\|g(\hat{u}) - g(y_{q-1,n})\bigr\|^2 \leq \bigl\|g(\hat{u}) - g(u_n)\bigr\|^2 - \bigl(1 - 2(\alpha + \beta)\rho_q\bigr)\bigl\|g(y_{q-1,n}) - g(u_n)\bigr\|^2, \tag{2.13}$$

for all $n \geq 0$, where $i = 1, 2, \ldots, q - 2$.

Proof Since $\hat{u} \in K$, $\hat{\nu} \in T(u)$, and $\hat{\vartheta} \in S(u)$ are the solution of GMELP (2.1), we have

$$F\bigl(\hat{\nu}, g(v)\bigr) + G\bigl(\hat{\vartheta}, \eta\bigl(g(v), g(\hat{u})\bigr)\bigr) \geq 0, \quad \forall v \in K. \tag{2.14}$$

Taking $v = u_{n+1}$ in (2.14) and $v = u$ in (2.4), we get

$$F\bigl(\hat{\nu}, g(u_{n+1})\bigr) + G\bigl(\hat{\vartheta}, \eta\bigl(g(u_{n+1}), g(\hat{u})\bigr)\bigr) \geq 0 \tag{2.15}$$

and

$$\begin{aligned}
&\rho_1 F\bigl(\nu_{1,n}, g(\hat{u})\bigr) + \rho_1 G\bigl(\vartheta_{1,n}, \eta\bigl(g(\hat{u}), g(u_{1,n})\bigr)\bigr) \\
&\quad + \bigl\langle g(u_{n+1}) - g(y_{1,n}), g(\hat{u}) - g(u_{n+1})\bigr\rangle \geq 0.
\end{aligned} \tag{2.16}$$

By combining (2.15) and (2.16) and taking into account of the facts that the bifunction F is partially α-relaxed g-monotone with respect to T, and the bifunction G is partially β-relaxed strongly g-η-monotone with respect to G, it follows that

$$\begin{aligned}
&\bigl\langle g(u_{n+1}) - g(y_{1,n}), g(\hat{u}) - g(u_{n+1})\bigr\rangle \\
&\quad \geq -\rho_1 F\bigl(\nu_{1,n}, g(\hat{u})\bigr) - \rho_1 G\bigl(\vartheta_{1,n}, \eta\bigl(g(\hat{u}), g(u_{1,n})\bigr)\bigr)
\end{aligned}$$

$$\begin{aligned}
&\geq -\rho_1\big(F\big(\nu_{1,n}, g(\hat{u})\big) + F\big(\hat{\nu}, g(u_{n+1})\big) \\
&\quad + G\big(\vartheta_{1,n}, \eta\big(g(\hat{u}), g(u_{1,n})\big)\big) + G\big(\hat{\vartheta}, \eta\big(g(u_{n+1}), g(\hat{u})\big)\big)\big) \\
&\geq -\rho_1\alpha \big\| g(u_{n+1}) - g(y_{1,n})\big\|^2 - \rho_1\beta \big\| g(u_{n+1}) - g(y_{1,n})\big\|^2 \\
&= -\rho_1(\alpha+\beta)\big\| g(u_{n+1}) - g(y_{1,n})\big\|^2. \qquad (2.17)
\end{aligned}$$

On the other hand, letting $x = g(\hat{u}) - g(u_{n+1})$ and $y = g(u_{n+1}) - g(y_{1,n})$ and by utilizing the well-known property of the inner product, we have

$$\begin{aligned}
&2\big\langle g(u_{n+1}) - g(y_{1,n}), g(\hat{u}) - g(u_{n+1})\big\rangle \\
&\quad = \big\| g(\hat{u}) - g(y_{1,n})\big\|^2 - \big\| g(\hat{u}) - g(u_{n+1})\big\|^2 - \big\| g(u_{n+1}) - g(y_{1,n})\big\|^2. \qquad (2.18)
\end{aligned}$$

Applying (2.17) and (2.18), it follows that

$$\big\| g(\hat{u}) - g(u_{n+1})\big\|^2 \leq \big\| g(\hat{u}) - g(y_{1,n})\big\|^2 - \big(1 - 2(\alpha+\beta)\rho_1\big)\big\| g(u_{n+1}) - g(y_{1,n})\big\|^2,$$

which is the required result (2.11).

Taking $\nu = y_{i,n}$ $(i = 1, 2, \ldots, q-2)$ in (2.14) and $\nu = \hat{u}$ in (2.5), for each $i = 1, 2, \ldots, q-2$, we have

$$F\big(\hat{\nu}, g(y_{i,n})\big) + G\big(\hat{\vartheta}, \eta\big(g(y_{i,n}), g(\hat{u})\big)\big) \geq 0 \qquad (2.19)$$

and

$$\begin{aligned}
&\rho_{i+1}F\big(\nu_{i+1,n}, g(\hat{u})\big) + \rho_{i+1}G\big(\vartheta_{i+1,n}, \eta\big(g(\hat{u}), g(u_{i+1,n})\big)\big) \\
&\quad + \big\langle g(y_{i,n}) - g(y_{i+1,n}), g(\hat{u}) - g(y_{i,n})\big\rangle \geq 0. \qquad (2.20)
\end{aligned}$$

Letting $x = g(\hat{u}) - g(y_{i,n})$ and $y = g(y_{i,n}) - g(y_{i+1,n})$ for each $i = 1, 2, \ldots, q-2$, and by using the well-known property of the inner product, one has

$$\begin{aligned}
&2\big\langle g(y_{i,n}) - g(y_{i+1,n}), g(\hat{u}) - g(y_{i,n})\big\rangle \\
&\quad = \big\| g(\hat{u}) - g(y_{i+1,n})\big\|^2 - \big\| g(\hat{u}) - g(y_{i,n})\big\|^2 - \big\| g(y_{i,n}) - g(y_{i+1,n})\big\|^2. \qquad (2.21)
\end{aligned}$$

In a similar fashion to the preceding analysis, employing (2.19)-(2.21) and considering the facts that the bifunction F is partially α-relaxed g-η-monotone with respect to T, and the bifunction G is partially β-relaxed g-η-monotone with respect to S, for each $i = 1, 2, \ldots, q-2$, one can deduce that

$$\big\| g(\hat{u}) - g(y_{i,n})\big\|^2 \leq \big\| g(\hat{u}) - g(y_{i+1,n})\big\|^2 - \big(1 - 2(\alpha+\beta)\rho_{i+1}\big)\big\| g(y_{i,n}) - g(y_{i+1,n})\big\|^2,$$

which is the required result (2.12).

Taking $\nu = y_{q-1,n}$ in (2.14) and $\nu = \hat{u}$ in (2.6), we have

$$F\big(\hat{\nu}, g(y_{q-1,n})\big) + G\big(\hat{\vartheta}, \eta\big(g(y_{q-1,n}), g(\hat{u})\big)\big) \geq 0 \qquad (2.22)$$

and

$$\begin{aligned}&\rho_q F\big(\nu_n, g(\hat{u})\big) + \rho_q G\big(\vartheta_n, \eta\big(g(\hat{u}), g(u_n)\big)\big)\\&\quad + \big\langle g(y_{q-1,n}) - g(u_n), g(\hat{u}) - g(y_{q-1,n})\big\rangle \geq 0.\end{aligned} \tag{2.23}$$

By assuming $x = g(\hat{u}) - g(y_{q-1,n})$ and $y = g(y_{q-1,n}) - g(u_n)$, and by utilizing the well-known property of the inner product, we obtain

$$\begin{aligned}&2\big\langle g(y_{q-1,n}) - g(u_n), g(\hat{u}) - g(y_{q-1,n})\big\rangle\\&\quad = \big\|g(\hat{u}) - g(u_n)\big\|^2 - \big\|g(\hat{u}) - g(y_{q-1,n})\big\|^2 - \big\|g(y_{q-1,n}) - g(u_n)\big\|^2.\end{aligned} \tag{2.24}$$

By a similar way to that of proof of (2.17), by using (2.22)-(2.24) and in light of the facts that the bifunction F is partially α-relaxed g-η-monotone with respect to T, and the bifunction G is partially β-relaxed g-η-monotone with respect to S, we can show that

$$\big\|g(\hat{u}) - g(y_{q-1,n})\big\| \leq \big\|g(\hat{u}) - g(u_n)\big\|^2 - \big(1 - 2(\alpha+\beta)\rho_q\big)\big\|g(y_{q-1,n}) - g(u_n)\big\|^2,$$

which is the required result (2.13). This completes the proof. □

In the next theorem, the strong convergence of the iterative sequences generated by Algorithm 2.2 to a solution of GMELP (2.1) is established.

Theorem 2.5 *Let $\mathcal{H}$ be a finite dimensional real Hilbert space and let $g : K \to K$ be a continuous and invertible operator. Suppose that the bifunction $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is continuous in the first argument, the bifunction $G : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is continuous in both arguments and the operator $\eta : K \times K \to \mathcal{H}$ is continuous in the second argument. Assume that the multivalued operators $S, T : K \to CB(\mathcal{H})$ are M-Lipschitz continuous with constants σ and δ, respectively. Moreover, let all the conditions of Proposition* 2.4 *hold and* GMELP$(F, G, S, T, \eta, g, K) \neq \emptyset$. *If $\rho_i \in (0, \frac{1}{2(\alpha+\beta)})$, for each $i = 1, 2, \ldots, q$, then the iterative sequences $\{u_n\}$, $\{\nu_n\}$, and $\{\vartheta_n\}$ generated by Algorithm* 2.2 *converge strongly to $\hat{u} \in K$, $\hat{\nu} \in T(\hat{u})$, and $\hat{\vartheta} \in S(\hat{u})$, respectively, and $(\hat{u}, \hat{\nu}, \hat{\vartheta})$ is a solution of* GMELP (2.1).

Proof Let $u \in K$, $\nu \in T(u)$, and $\vartheta \in S(u)$ be the solution of GMELP (2.1). In view of the fact that all the conditions of Proposition 2.4 hold, then Proposition 2.4 implies that for all $n \geq 0$

$$\big\|g(u) - g(u_{n+1})\big\|^2 \leq \big\|g(u) - g(y_{1,n})\big\|^2 - \big(1 - 2(\alpha+\beta)\rho_1\big)\big\|g(u_{n+1}) - g(y_{1,n})\big\|^2, \tag{2.25}$$

$$\big\|g(u) - g(y_{i,n})\big\|^2 \leq \big\|g(u) - g(y_{i+1,n})\big\|^2 - \big(1 - 2(\alpha+\beta)\rho_{i+1}\big)\big\|g(y_{i,n}) - g(y_{i+1,n})\big\|^2, \tag{2.26}$$

$$\big\|g(u) - g(y_{q-1,n})\big\| \leq \big\|g(u) - g(u_n)\big\|^2 - \big(1 - 2(\alpha+\beta)\rho_q\big)\big\|g(y_{q-1,n}) - g(u_n)\big\|^2, \tag{2.27}$$

where $i = 1, 2, \ldots, q-2$. From the inequalities (2.25)-(2.27), it follows that the sequence $\{\|g(u_n) - g(u)\|\}$ is nonincreasing and hence the sequence $\{g(u_n)\}$ is bounded. Since the operator g is invertible, we deduce that the sequence $\{u_n\}$ is also bounded. Furthermore, by (2.25)-(2.27), we have

$$\left(1-2(\alpha+\beta)\sum_{i=1}^{q}\rho_i\right)\left(\left\|g(u_{n+1})-g(y_{1,n})\right\|^2+\sum_{i=1}^{q-2}\left\|g(y_{i,n})-g(y_{i+1,n})\right\|^2 \right.$$
$$\left. +\left\|g(y_{q-1,n})-g(u_n)\right\|^2\right)\le\left\|g(u)-g(u_n)\right\|^2-\left\|g(u)-g(u_{n+1})\right\|^2,$$

which implies that

$$\sum_{n=0}^{\infty}\left(1-2(\alpha+\beta)\sum_{i=1}^{q}\rho_i\right)\left(\left\|g(u_{n+1})-g(y_{1,n})\right\|^2+\sum_{i=1}^{q-2}\left\|g(y_{i,n})-g(y_{i+1,n})\right\|^2 \right.$$
$$\left. +\left\|g(y_{q-1,n})-g(u_n)\right\|^2\right)\le\left\|g(u)-g(u_0)\right\|^2. \tag{2.28}$$

The inequality (2.28) guarantees that

$$\left\|g(u_{n+1})-g(y_{1,n})\right\|\to 0,\qquad \left\|g(y_{i,n})-g(y_{i+1,n})\right\|\to 0,\qquad \left\|g(y_{q-1,n})-g(u_n)\right\|\to 0,$$

for each $i=1,2,\ldots,q-2$, as $n\to\infty$. Let $\hat{u}$ be a cluster point of the sequence $\{u_n\}$. Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j}\to\hat{u}$, as $j\to\infty$. Taking into consideration the fact that the multivalued operators S and T are M-Lipschitz continuous with constants σ and δ, respectively, in virtue of the inequalities (2.9) and (2.10), we have

$$\begin{aligned}\|\nu_{n_j+1}-\nu_{n_j}\| &\le \left(1+(1+n_j)^{-1}\right)M\left(T(u_{n_j+1}),T(u_{n_j})\right)\\ &\le \left(1+(1+n_j)^{-1}\right)\delta\|u_{n_j+1}-u_{n_j}\|\end{aligned}\tag{2.29}$$

and

$$\begin{aligned}\|\vartheta_{n_j+1}-\vartheta_{n_j}\| &\le \left(1+(1+n_j)^{-1}\right)M\left(S(u_{n_j+1}),S(u_{n_j})\right)\\ &\le \left(1+(1+n_j)^{-1}\right)\sigma\|u_{n_j+1}-u_{n_j}\|.\end{aligned}\tag{2.30}$$

The inequalities (2.29) and (2.30) imply that $\|\nu_{n_j+1}-\nu_{n_j}\|\to 0$ and $\|\vartheta_{n_j+1}-\vartheta_{n_j}\|\to 0$, as $j\to\infty$, that is, $\{\nu_{n_j}\}$ and $\{\vartheta_{n_j}\}$ are Cauchy sequences in $\mathcal{H}$. Thus, $\nu_{n_j}\to\hat{\nu}$ and $\vartheta_{n_j}\to\hat{\vartheta}$ for some $\hat{\nu},\hat{\vartheta}\in\mathcal{H}$, as $j\to\infty$. By using (2.6), we have

$$\begin{aligned}&\rho_q F\left(\nu_{n_j},g(\nu)\right)+\rho_q G\left(\vartheta_{n_j},\eta\left(g(\nu),g(u_{n_j})\right)\right)\\ &\quad+\left\langle g(y_{q-1,n_j})-g(u_{n_j}),g(\nu)-g(y_{q-1,n_j})\right\rangle\ge 0.\end{aligned}\tag{2.31}$$

In view of the fact that F is continuous in the first argument, G is continuous in both arguments, η is continuous in the second argument, g is continuous and $\|g(y_{q-1,n})-g(u_n)\|\to 0$, as $n\to\infty$, letting $i\to\infty$ and by using (2.31), we deduce that

$$F\left(\hat{\nu},g(\nu)\right)+G\left(\hat{\vartheta},\eta\left(g(\nu),g(\hat{u})\right)\right)\ge 0,\quad \forall\nu\in K.$$

In the meantime, from the M-Lipschitz continuity of T with constant δ, it follows that

$$\begin{aligned} d\bigl(\hat{\nu}, T(\hat{u})\bigr) &= \inf\bigl\{\|\hat{\nu} - z\| : z \in T(\hat{u})\bigr\} \\ &\leq \|\hat{\nu} - \nu_{n_j}\| + d\bigl(\nu_{n_j}, T(\hat{u})\bigr) \\ &\leq \|\hat{\nu} - \nu_{n_j}\| + M\bigl(T(u_{n_j}), T(\hat{u})\bigr) \\ &\leq \|\hat{\nu} - \nu_{n_j}\| + \delta\|u_{n_j} - \hat{u}\|. \end{aligned} \tag{2.32}$$

Notice that the right side of the above inequality tends to zero as $j \to \infty$. Since $T(\hat{u}) \in CB(\mathcal{H})$ it follows that $\hat{\nu} \in T(\hat{u})$. Taking into account of the fact that the multivalued operator S is M-Lipschitz continuous with constant σ, in a similar way to that of proof of (2.32), one can deduce that

$$d\bigl(\hat{\vartheta}, S(\hat{u})\bigr) \leq \|\hat{\vartheta} - \vartheta_{n_j}\| + \sigma\|u_{n_j} - \hat{u}\|,$$

which relying on the fact that $S(\hat{u}) \in CB(\mathcal{H})$ implies that $\hat{\vartheta} \in S(\hat{u})$. Hence, $\hat{u} \in K$, $\hat{\nu} \in T(\hat{u})$, and $\hat{\vartheta} \in S(\hat{u})$ are the solution of GMELP (2.1). Now, the inequalities (2.11)-(2.13) imply that

$$\bigl\|g(u_{n+1} - g(\hat{u})\bigr\| \leq \bigl\|g(u_n) - g(\hat{u})\bigr\|, \quad \forall n \geq 0. \tag{2.33}$$

The inequality (2.33) guarantees that $g(u_n) \to g(\hat{u})$, as $n \to \infty$ and hence $u_n \to \hat{u}$, as $n \to \infty$, since g is continuous and invertible. Consequently, the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$. Considering the facts that S and T are M-Lipschitz continuous with constants σ and δ, respectively, by using (2.9) and (2.10), we have

$$\begin{aligned} \|\nu_{n+1} - \nu_n\| &\leq \bigl(1 + (1+n)^{-1}\bigr)M\bigl(T(u_{n+1}), T(u_n)\bigr) \\ &\leq \bigl(1 + (1+n)^{-1}\bigr)\delta\|u_{n+1} - u_n\| \end{aligned} \tag{2.34}$$

and

$$\begin{aligned} \|\vartheta_{n+1} - \vartheta_n\| &\leq \bigl(1 + (1+n)^{-1}\bigr)M\bigl(S(u_{n+1}), S(u_n)\bigr) \\ &\leq \bigl(1 + (1+n)^{-1}\bigr)\sigma\|u_{n+1} - u_n\|. \end{aligned} \tag{2.35}$$

The inequalities (2.34) and (2.35) imply that $\{\nu_n\}$ and $\{\vartheta_n\}$ are Cauchy sequences in $\mathcal{H}$. Since $\hat{\nu}$ and $\hat{\vartheta}$ are cluster points of the sequences $\{\nu_n\}$ and $\{\vartheta_n\}$, respectively, it follows that $\nu_n \to \hat{\nu}$ and $\vartheta_n \to \hat{\vartheta}$, as $n \to \infty$, that is, the sequences $\{\nu_n\}$ and $\{\vartheta_n\}$ have exactly one cluster point $\hat{\nu}$ and $\hat{\vartheta}$, respectively. The proof is completed. □

The next proposition is a main tool for studying the convergence analysis of the iterative sequences generated by Algorithm 2.3.

Proposition 2.6 *Let F, G, S, T, and g be the same as in* GMHEP (2.2) *and let $\hat{u} \in K$, $\hat{\nu} \in T(u)$, and $\hat{\vartheta} \in G(\hat{u})$ be the solution of* GMHEP (2.2). *Furthermore, let $\{u_n\}$ and $\{y_{i,n}\}$ $(i = 1, 2, \ldots, q-1)$ be the sequences generated by Algorithm* 2.3. *If F is partially α-relaxed g-monotone with respect to T, and G is partially β-relaxed g-monotone with respect to S, then the inequalities* (2.11)-(2.13) *hold for all $n \geq 0$.*

Proof It follows from Proposition 2.4 by defining the operator $\eta : K \times K \to \mathcal{H}$ as $\eta(x, y) = x - y$ for all $x, y \in K$. □

The next assertion provides us the required conditions under which the iterative sequences generated by Algorithm 2.3 converge strongly to a solution of GMHEP (2.2).

Corollary 2.7 *Suppose that $\mathcal{H}$ is a finite dimensional real Hilbert space and let $g : K \to K$ be a continuous and invertible operator. Assume that the bifunction $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is continuous in the first argument and the bifunction $G : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is continuous in both arguments. Let the multivalued operators $S, T : K \to CB(\mathcal{H})$ be M-Lipschitz continuous with constants σ and δ, respectively. Furthermore, let all the conditions of Proposition* 2.6 *hold and* GMHEP$(F, G, S, T, g, K) \neq \emptyset$. *If $\rho_i \in (0, \frac{1}{2(\alpha+\beta)})$, for each $i = 1, 2, \ldots, q$, then the iterative sequences $\{u_n\}$, $\{v_n\}$, and $\{\vartheta_n\}$ generated by Algorithm* 2.3 *converge strongly to $\hat{u} \in K$, $\hat{v} \in T(\hat{u})$ and $\hat{\vartheta} \in S(\hat{u})$, respectively, and $(\hat{u}, \hat{v}, \hat{\vartheta})$ is a solution of* GMHEP (2.2).

Proof By defining the operator $\eta : K \times K \to \mathcal{H}$ as $\eta(x, y) = x - y$ for all $x, y \in K$, the desired result follows from Theorem 2.5. □

It is well known that to implement the proximal point methods, one has to calculate the approximate solution implicitly, which is itself a difficult problem. In order to overcome this drawback, we consider another auxiliary problem and with the help of it, we construct an iterative algorithm for solving the problem (2.1).

Let S, T, F, G, η, and g be the same as in GMELP (2.1). For given $u \in K$, $v \in T(u)$, and $\vartheta \in S(u)$, we consider the auxiliary generalized multivalued equilibrium-like problem of finding $w \in K$, $\xi \in T(w)$, and $\gamma \in S(w)$ such that

$$\rho F(\xi, g(v)) + \rho G(\gamma, \eta(g(v), g(u))) + \langle g(w) - g(u), g(v) - g(w) \rangle \geq 0, \quad \forall v \in K, \tag{2.36}$$

where $\rho > 0$ is a constant. It should be pointed out that the two problems (2.3) and (2.36) are quite different. If $w = u$, then clearly (w, ξ, γ) is a solution of GMELP (2.1). By using this observation and Nadler's technique [34], we are able to suggest the following predictor-corrector method for solving GMELP (2.1).

Algorithm 2.8 Let F, G, S, T, η, and g be the same as in GMELP (2.1). For given $u_0 \in K$, $\xi_0 \in T(u_0)$, and $\gamma_0 \in S(u_0)$, compute the iterative sequences $\{u_n\}$, $\{\xi_n\}$, and $\{\gamma_n\}$ by the iterative schemes

$$\begin{aligned}
&\rho_1 F(\xi_{1,n}, g(v)) + \rho_1 G(\gamma_{1,n}, \eta(g(v), g(y_{1,n}))) \\
&\quad + \langle g(u_{n+1}) - g(y_{1,n}), g(v) - g(u_{n+1}) \rangle \geq 0, \quad \forall v \in K, &(2.37)\\
&\rho_{i+1} F(\xi_{i+1,n}, g(v)) + \rho_{i+1} G(\gamma_{i+1,n}, \eta(g(v), g(y_{i+1,n}))) \\
&\quad + \langle g(y_{i,n}) - g(y_{i+1,n}), g(v) - g(y_{i,n}) \rangle \geq 0, \quad i = 1, 2, \ldots, q-2, \forall v \in K, &(2.38)\\
&\rho_q F(\xi_n, g(v)) + \rho_q G(\gamma_n, \eta(g(v), g(u_n))) \\
&\quad + \langle g(y_{q-1,n}) - g(u_n), g(v) - g(y_{q-1,n}) \rangle \geq 0, \quad \forall v \in K, &(2.39)\\
&\xi_{i,n} \in T(y_{i,n}) : \|\xi_{i,n+1} - \xi_{i,n}\| \leq (1 + (1+n)^{-1}) M(T(y_{i,n+1}), T(y_{i,n})), \\
&\quad i = 1, 2, \ldots, q-1, &(2.40)
\end{aligned}$$

$$\begin{aligned} &\gamma_{i,n} \in S(y_{i,n}) : \|\gamma_{i,n+1} - \gamma_{i,n}\| \le \bigl(1 + (1+n)^{-1}\bigr) M\bigl(S(y_{i,n+1}), S(y_{i,n})\bigr), \\ &\quad i = 1, 2, \ldots, q-1, \end{aligned} \tag{2.41}$$

$$\xi_n \in T(u_n) : \|\xi_{n+1} - \xi_n\| \le \bigl(1 + (1+n)^{-1}\bigr) M\bigl(T(u_{n+1}), T(u_n)\bigr), \tag{2.42}$$

$$\gamma_n \in S(u_n) : \|\gamma_{n+1} - \gamma_n\| \le \bigl(1 + (1+n)^{-1}\bigr) M\bigl(S(u_{n+1}), S(u_n)\bigr), \tag{2.43}$$

where $\rho_i > 0$ $(i = 1, 2, \ldots, q)$ are constants and $n = 0, 1, 2, \ldots$.

If $\eta(x,y) = x - y$, for all $x, y \in K$, then Algorithm 2.8 reduces to the following predictor-corrector method.

Algorithm 2.9 For given $u_0 \in K$, $\xi_0 \in T(u_0)$, and $\gamma_0 \in S(u_0)$, compute the iterative sequences $\{u_n\}$, $\{\xi_n\}$, and $\{\gamma_n\}$ by the iterative schemes

$$\begin{aligned} &\rho_1 F\bigl(\xi_{1,n}, g(v)\bigr) + \rho_1 G\bigl(\gamma_{1,n}, g(v) - g(y_{1,n})\bigr) \\ &\quad + \bigl\langle g(u_{n+1}) - g(y_{1,n}), g(v) - g(u_{n+1})\bigr\rangle \ge 0, \quad \forall v \in K, \\ &\rho_{i+1} F\bigl(\xi_{i+1,n}, g(v)\bigr) + \rho_{i+1} G\bigl(\gamma_{i+1,n}, g(v) - g(y_{i+1,n})\bigr) \\ &\quad + \bigl\langle g(y_{i,n}) - g(y_{i+1,n}), g(v) - g(y_{i,n})\bigr\rangle \ge 0, \quad i = 1, 2, \ldots, q-2, \forall v \in K, \\ &\rho_q F\bigl(\xi_n, g(v)\bigr) + \rho_q G\bigl(\gamma_n, g(v) - g(u_n)\bigr) \\ &\quad + \bigl\langle g(y_{q-1,n}) - g(u_n), g(v) - g(y_{q-1,n})\bigr\rangle \ge 0, \quad \forall v \in K, \\ &\xi_{i,n} \in T(y_{i,n}) : \|\xi_{i,n+1} - \xi_{i,n}\| \le \bigl(1 + (1+n)^{-1}\bigr) M\bigl(T(y_{i,n+1}), T(y_{i,n})\bigr), \quad i = 1, 2, \ldots, q-1, \\ &\gamma_{i,n} \in S(y_{i,n}) : \|\gamma_{i,n+1} - \gamma_{i,n}\| \le \bigl(1 + (1+n)^{-1}\bigr) M\bigl(S(y_{i,n+1}), S(y_{i,n})\bigr), \quad i = 1, 2, \ldots, q-1, \\ &\xi_n \in T(u_n) : \|\xi_{n+1} - \xi_n\| \le \bigl(1 + (1+n)^{-1}\bigr) M\bigl(T(u_{n+1}), T(u_n)\bigr), \\ &\gamma_n \in S(u_n) : \|\gamma_{n+1} - \gamma_n\| \le \bigl(1 + (1+n)^{-1}\bigr) M\bigl(S(u_{n+1}), S(u_n)\bigr), \end{aligned}$$

where $\rho_i > 0$ $(i = 1, 2, \ldots, q)$ are constants and $n = 0, 1, 2, \ldots$.

To prove the strong convergence of the sequences generated by Algorithm 2.8 to a solution of GMELP (2.1), we need the following definition.

Definition 2.4 Let $S, T : K \to CB(\mathcal{H})$, $\eta : K \times K \to \mathcal{H}$, and $g : K \to K$ be operators. The bifunctions $F, G : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ are said to be *jointly g-η-pseudomonotone* with respect to S and T, if

$$\begin{aligned} &F\bigl(w_1, g(u_2)\bigr) + G\bigl(\vartheta_1, \eta\bigl(g(u_2), g(u_1)\bigr)\bigr) \ge 0, \quad \text{implies that} \\ &F\bigl(w_2, g(u_1)\bigr) + G\bigl(\vartheta_2, \eta\bigl(g(u_1), g(u_2)\bigr)\bigr) \le 0, \\ &\quad \forall u_1, u_2 \in K, w_1 \in T(u_1), w_2 \in T(u_2), \vartheta_1 \in S(u_1), \vartheta_2 \in S(u_2). \end{aligned}$$

It should be remarked that if the operator $\eta : K \times K \to \mathcal{H}$ is defined as $\eta(x,y) = x - y$, for all $x, y \in K$, then Definition 2.4 reduces to the definition of jointly g-pseudomonotonicity of the bifunctions F and G with respect to the multivalued operators S and T.

Before turning to the study of convergence analysis of the iterative sequences generated by Algorithm 2.8, we would like to present the following proposition which plays an important and key role in it.

Proposition 2.10 *Let F, G, S, T, η, and g be the same as in* GMELP (2.1) *and let $\hat{u} \in K$, $\hat{\nu} \in T(\hat{u})$, and $\hat{\vartheta} \in S(\hat{u})$ be the solution of* GMELP (2.1). *Assume further that $\{u_n\}$ and $\{y_{i,n}\}$ $(i = 1, 2, \dots, q-1)$ are the sequences generated by Algorithm* 2.8. *If F and G are jointly g-η-pseudomonotone with respect to S and T, then for all $n \geq 0$*

$$\|g(\hat{u}) - g(u_{n+1})\|^2 \leq \|g(\hat{u}) - g(y_{1,n})\|^2 - \|g(u_{n+1}) - g(y_{1,n})\|^2, \tag{2.44}$$

$$\begin{aligned}&\|g(\hat{u}) - g(y_{i,n})\|^2 \leq \|g(\hat{u}) - g(y_{i+1,n})\|^2 - \|g(y_{i,n}) - g(y_{i+1,n})\|^2,\\ &\quad i = 1, 2, \dots, q-2,\end{aligned} \tag{2.45}$$

$$\|g(\hat{u}) - g(y_{q-1,n})\|^2 \leq \|g(\hat{u}) - g(u_n)\|^2 - \|g(y_{q-1,n}) - g(u_n)\|^2. \tag{2.46}$$

Proof Since $\hat{u} \in K$, $\hat{\nu} \in T(\hat{u})$, and $\hat{\vartheta} \in S(\hat{u})$ are the solution of GMELP (2.1), it follows that $(\hat{u}, \hat{\nu}, \hat{\vartheta})$ satisfies (2.14). Taking $v = y_{1,n}$ in (2.14), we have

$$F(\hat{\nu}, g(y_{1,n})) + G(\hat{\vartheta}, \eta(g(y_{1,n}), g(\hat{u}))) \geq 0. \tag{2.47}$$

Taking into consideration the fact that the bifunctions F and G are jointly g-η-pseudomonotone with respect to S and T, from (2.47), we conclude that

$$F(\xi_{1,n}, g(\hat{u})) + G(\gamma_{1,n}, \eta(g(\hat{u}), g(y_{1,n}))) \leq 0. \tag{2.48}$$

Taking $v = \hat{u}$ in (2.37), we obtain

$$\begin{aligned}&\rho_1 F(\xi_{1,n}, g(\hat{u})) + \rho_1 G(\gamma_{1,n}, \eta(g(\hat{u}), g(y_{1,n})))\\ &\quad + \langle g(u_{n+1}) - g(y_{1,n}), g(\hat{u}) - g(u_{n+1})\rangle \geq 0.\end{aligned} \tag{2.49}$$

By combining (2.48) and (2.49), we get

$$\begin{aligned}\langle g(u_{n+1}) - g(y_{1,n}), g(\hat{u}) - g(u_{n+1})\rangle &\geq -\rho_1 F(\xi_{1,n}, g(\hat{u})) - \rho_1 G(\gamma_{1,n}, \eta(g(\hat{u}), g(y_{1,n})))\\ &\geq 0.\end{aligned} \tag{2.50}$$

Relying on (2.18) and (2.50), we get

$$\|g(\hat{u}) - g(u_{n+1})\|^2 \leq \|g(\hat{u}) - g(y_{1,n})\|^2 - \|g(u_{n+1}) - g(y_{1,n})\|^2,$$

which is the required result (2.44).

Taking $v = y_{i+1,n}$ $(i = 1, 2, \dots, q-2)$ in (2.14), we have

$$F(\hat{\nu}, g(y_{i+1,n})) + G(\hat{\vartheta}, \eta(g(y_{i+1,n}), g(\hat{u}))) \geq 0. \tag{2.51}$$

Considering the fact that F and G are jointly g-η-pseudomonotone with respect to S and T, the inequality (2.51) implies that, for each $i = 1, 2, \dots, q-2$,

$$F\bigl(\xi_{i+1,n}, g(\hat{u})\bigr) + G\bigl(\gamma_{i+1,n}, \eta\bigl(g(\hat{u}), g(y_{i+1,n})\bigr)\bigr) \leq 0. \tag{2.52}$$

Letting $v = \hat{u}$ in (2.38), for each $i = 1, 2, \dots, q-2$, we obtain

$$\begin{aligned} &\rho_{i+1} F\bigl(\xi_{i+1,n}, g(\hat{u})\bigr) + \rho_{i+1} G\bigl(\gamma_{i+1,n}, \eta\bigl(g(\hat{u}), g(y_{i+1,n})\bigr)\bigr) \\ &\quad + \bigl\langle g(y_{i,n}) - g(y_{i+1,n}), g(\hat{u}) - g(y_{i,n})\bigr\rangle \geq 0. \end{aligned} \tag{2.53}$$

It follows from (2.18), (2.52), and (2.53) that, for each $i = 1, 2, \dots, q-2$,

$$\bigl\|g(\hat{u}) - g(y_{i,n})\bigr\|^2 \leq \bigl\|g(\hat{u}) - g(y_{i+1,n})\bigr\|^2 - \bigl\|g(y_{i,n}) - g(y_{i+1,n})\bigr\|^2,$$

which is the required result (2.45).

Taking $v = y_{q,n}$ in (2.14), we have

$$F\bigl(\hat{\nu}, g(y_{q,n})\bigr) + G\bigl(\hat{\vartheta}, \eta\bigl(g(y_{q,n}), g(\hat{u})\bigr)\bigr) \geq 0. \tag{2.54}$$

In light of the fact that F and G are jointly g-η-pseudomotone with respect to S and T, we deduce that

$$F\bigl(\xi_n, g(\hat{u})\bigr) + G\bigl(\gamma_n, \eta\bigl(g(\hat{u}), g(u_n)\bigr)\bigr) \leq 0. \tag{2.55}$$

Letting $v = \hat{u}$ in (2.39), we get

$$\rho_q F\bigl(\xi_n, g(\hat{u})\bigr) + \rho_q G\bigl(\gamma_n, \eta\bigl(g(\hat{u}), g(u_n)\bigr)\bigr) + \bigl\langle g(y_{q-1,n}) - g(u_n), g(\hat{u}) - g(y_{q-1,n})\bigr\rangle \geq 0. \tag{2.56}$$

Applying (2.18), (2.55) and (2.56), one can deduce that

$$\bigl\|g(\hat{u}) - g(y_{q-1,n})\bigr\| \leq \bigl\|g(\hat{u}) - g(u_n)\bigr\|^2 - \bigl\|g(y_{q-1,n}) - g(u_n)\bigr\|^2,$$

which is the required result (2.46). This completes the proof. □

Now we establish the strong convergence of the iterative sequences generated by Algorithm 2.8 to a solution of GMELP (2.1).

Theorem 2.11 *Let $\mathcal{H}$ be a finite dimensional real Hilbert space and let $g : K \to K$ be a continuous and invertible operator. Suppose that the bifunction $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is continuous in the first argument, the bifunction $G : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is continuous in both arguments and the operator $\eta : K \times K \to \mathcal{H}$ is continuous in the second argument. Assume that the multivalued operators $S, T : K \to CB(\mathcal{H})$ are M-Lipschitz continuous with constants σ and δ, respectively. Further, let all the conditions of Proposition* 2.10 *hold and* GMELP$(F, G, S, T, \eta, g, K) \neq \emptyset$. *Then the iterative sequences $\{u_n\}$, $\{\xi_n\}$, and $\{\gamma_n\}$ generated by Algorithm* 2.8 *converge strongly to $\hat{u} \in K$, $\hat{\nu} \in T(\hat{u})$, and $\hat{\vartheta} \in S(\hat{u})$, respectively, and $(\hat{u}, \hat{\nu}, \hat{\vartheta})$ is a solution of* GMELP (2.1).

Proof Let $u \in K$, $\nu \in T(u)$, and $\vartheta \in S(u)$ be the solution of GMELP (2.1). Since all the conditions of Proposition 2.10 hold, according to Proposition 2.10, for all $n \geq 0$ we have

$$\|g(u) - g(u_{n+1})\|^2 \leq \|g(u) - g(y_{1,n})\|^2 - \|g(u_{n+1}) - g(y_{1,n})\|^2, \tag{2.57}$$

$$\|g(u) - g(y_{i,n})\|^2 \leq \|g(u) - g(y_{i+1,n})\|^2 - \|g(y_{i,n}) - g(y_{i+1,n})\|^2, \quad i = 1, 2, \dots, q-2, \tag{2.58}$$

$$\|g(u) - g(y_{q-1,n})\|^2 \leq \|g(u) - g(u_n)\|^2 - \|g(y_{q-1,n}) - g(u_n)\|^2. \tag{2.59}$$

It is easy to see that the inequalities (2.57)-(2.59) imply that the sequence $\{\|g(u_n) - g(u)\|\}$ is nonincreasing and hence the sequence $\{g(u_n)\}$ is bounded. Considering the fact that the operator g is invertible, it follows that the sequence $\{u_n\}$ is also bounded. Meanwhile, relying on (2.57)-(2.59), we have

$$\begin{aligned} &\|g(u_{n+1}) - g(y_{1,n})\|^2 + \sum_{i=1}^{q-2} \|g(y_{i,n}) - g(y_{i+1,n})\|^2 + \|g(y_{q-1,n}) - g(u_n)\|^2 \\ &\quad \leq \|g(u) - g(u_n)\|^2 - \|g(u) - g(u_{n+1})\|^2, \end{aligned}$$

whence we deduce that

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(\|g(u_{n+1}) - g(y_{1,n})\|^2 + \sum_{i=1}^{q-2} \|g(y_{i,n}) - g(y_{i+1,n})\|^2 + \|g(y_{q-1,n}) - g(u_n)\|^2 \right) \\ &\quad \leq \|g(u) - g(u_0)\|^2. \end{aligned} \tag{2.60}$$

From (2.60), it follows that

$$\|g(u_{n+1}) - g(y_{1,n})\| \to 0, \qquad \|g(y_{i,n}) - g(y_{i+1,n})\| \to 0, \qquad \|g(y_{q-1,n}) - g(u_n)\| \to 0,$$

for each $i = 1, 2, \dots, q-2$, as $n \to \infty$. Let $\hat{u}$ be a cluster point of the sequence $\{u_n\}$. Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \to \hat{u}$, as $j \to \infty$. In a similar way to that of proof of Theorem 2.5, one can establish that $\{\xi_{n_j}\}$ and $\{\gamma_{n_j}\}$ are Cauchy sequences in $\mathcal{H}$ and $\xi_{n_j} \to \hat{\nu}$, and $\gamma_{n_j} \to \hat{\vartheta}$ for some $\hat{\nu}, \hat{\vartheta} \in \mathcal{H}$, as $j \to \infty$. Furthermore, $\hat{u} \in K$, $\hat{\nu} \in T(\hat{u})$, and $\hat{\vartheta} \in S(\hat{u})$ are the solution of GMELP (2.1) and the sequences $\{u_n\}$, $\{\xi_n\}$, and $\{\gamma_n\}$ have exactly one cluster point $\hat{u}$, $\hat{\nu}$, and $\hat{\vartheta}$, respectively. This gives us the desired result. □

The next proposition plays a crucial role in the study of convergence analysis of the iterative sequences generated by Algorithm 2.9.

Proposition 2.12 *Let F, G, S, T, and g be the same as in* GMHEP (2.2) *and let $\hat{u} \in K$, $\hat{\nu} \in T(\hat{u})$, and $\hat{\vartheta} \in S(\hat{u})$ be the solution of* GMHEP (2.2). *Moreover, let $\{u_n\}$ and $\{y_{i,n}\}$* ($i = 1, 2, \dots, q-1$) *be the sequences generated by Algorithm* 2.9. *If the bifunctions F and G are jointly g-pseudomonotone with respect to S and T, then the inequalities* (2.44)-(2.46) *hold.*

Proof By defining the operator $\eta : K \times K \to \mathcal{H}$ as $\eta(x, y) = x - y$ for all $x, y \in K$, we get the desired result from Proposition 2.10. □

We now conclude this paper with the following result in which the strong convergence of the iterative sequence generated by Algorithm 2.9 is established.

Corollary 2.13 *Assume that $\mathcal{H}$ is a finite dimensional real Hilbert space and let $g : K \to K$ be a continuous and invertible operator. Let the bifunction $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be continuous in the first argument and the bifunction $G : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be continuous in both arguments. Suppose that the multivalued operators $S, T : K \to CB(\mathcal{H})$ are M-Lipschitz continuous with constants σ and δ, respectively. Moreover, let all the conditions of Proposition* 2.12 *hold and* GMHEP$(F, G, S, T, g, K) \neq \emptyset$. *Then the iterative sequences $\{u_n\}$, $\{\xi_n\}$, and $\{\gamma_n\}$ generated by Algorithm* 2.9 *converge strongly to $\hat{u} \in K$, $\hat{\nu} \in T(\hat{u})$, and $\hat{\vartheta} \in S(\hat{u})$, respectively, and $(\hat{u}, \hat{\nu}, \hat{\vartheta})$ is a solution of* GMHEP (2.2).

Proof We obtain the desired result from Theorem 2.11 by defining $\eta : K \times K \to \mathcal{H}$ as $\eta(x, y) = x - y$ for all $x, y \in K$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in this paper. All authors read and approved the final manuscript.

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