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Boundedness of homogeneous fractional integral operator on Morrey space

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Abstract

For $0<\alpha<n$, the homogeneous fractional integral operator $T_{\Omega,\alpha}$ is defined by

$$T_{\Omega,\alpha}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)\,dy.$$

In this paper we prove that if Ω satisfies some smoothness conditions on S^{n-1}, then $T_{\Omega,\alpha}$ is bounded from $L^{\frac{\lambda}{\alpha},\lambda}(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$, and from $L^{p,\lambda}(\mathbb{R}^n)$ ($\frac{\lambda}{\alpha}<p<\infty$) to a class of the Campanato spaces $\mathcal{L}_{l,\lambda}(\mathbb{R}^n)$, respectively.

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1 Introduction

Before going into the next sections addressing details, let us agree to some conventions. The n-dimensional Euclidean space $\mathbb{R}^n$, $Q=Q(x_0,d)$ is a cube with its sides parallel to the coordinate axes and center at x_0, diameter $d>0$.

For $1\le l\le\infty$, $-\frac{n}{l}\le\lambda\le 1$, we denote

$$\|f\|_{\mathcal{L}_{l,\lambda}}=\sup_Q\frac{1}{|Q|^{\lambda/n}}\left(\frac{1}{|Q|}\int_Q\left|f(x)-f_Q\right|^l\,dx\right)^{1/l},$$

where $f_Q=\frac{1}{|Q|}\int_Q f(y)\,dy$. Then the Campanato space $\mathcal{L}_{l,\lambda}(\mathbb{R}^n)$ is defined by

$$\mathcal{L}_{l,\lambda}(\mathbb{R}^n)=\left\{f\in L^l_{\mathrm{loc}}(\mathbb{R}^n):\|f\|_{\mathcal{L}_{l,\lambda}}<\infty\right\}.$$

If we identify functions that differ by a constant, then $\mathcal{L}_{l,\lambda}$ becomes a Banach space with the norm $\|\cdot\|_{\mathcal{L}_{l,\lambda}}$. It is well known that

$$\mathcal{L}_{l,\lambda}(\mathbb{R}^n)\quad\sim\quad\begin{cases}\mathrm{Lip}_\lambda(\mathbb{R}^n), & \text{for } 0<\lambda<1,\\ \mathrm{BMO}(\mathbb{R}^n), & \text{for } \lambda=0,\\ \text{Morrey space } L^{p,n+l\lambda}(\mathbb{R}^n), & \text{for } -n/l\le\lambda<0.\end{cases}$$

On the other properties of the spaces $\mathcal{L}_{l,\lambda}(\mathbb{R}^n)$, we refer the reader to [1].

The Morrey space, which was introduced by Morrey in 1938, connects with certain problems in elliptic PDE [2, 3]. Later, there were many applications of Morrey space to the Navier-Stokes equations (see [4]), the Schrödinger equations (see [5] and [6]) and the elliptic problems with discontinuous coefficients (see [7–9] and [10]).

For $1 \le p < \infty$ and $0 < \lambda \le n$, the Morrey space is defined by

$$L^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\mathrm{loc}} : \|f\|_{L^{p,\lambda}} = \left[\sup_{x\in\mathbb{R}^n, d>0} d^{\lambda-n} \int_{Q(x,d)} \left|f(y)\right|^p dy \right]^{\frac{1}{p}} < \infty \right\},$$

where $Q(x,d)$ denotes the cube centered at x and with diameter $d > 0$. The space $L^{p,\lambda}(\mathbb{R}^n)$ becomes a Banach space with norm $\|\cdot\|_{L^{p,\lambda}}$. Moreover, for $\lambda = 0$ and $\lambda = n$, the Morrey spaces $L^{p,0}(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n)$ coincide (with equality of norms) with the space $L^\infty(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$, respectively.

The boundedness of the Hardy-Littlewood maximal operator, the fractional integral operator, and the Calderón-Zygmund singular integral operator on Morrey space can be found in [11–15]. It is well known that further properties and applications of the classical Morrey space have been widely studied by many authors. (For example, see [8, 16–19].)

A function $g \in \mathrm{BMO}(\mathbb{R}^n)$ (see [20]), if there is a constant $C > 0$ such that for any cube $Q \in \mathbb{R}^n$,

$$\|g\|_{\mathrm{BMO}} = \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|Q|} \int_Q \left|g(x) - g_Q\right| dx \right) < \infty,$$

where $g_Q = \frac{1}{|Q|} \int_Q g(y)\, dy$.

The Hardy-Littlewood-Sobolev theorem showed that the Riesz potential operator I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Here

$$I_\alpha f(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \quad \text{and} \quad \gamma(\alpha) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma(\alpha/2)}{\Gamma(\frac{n-\alpha}{2})}.$$

In 1974, Muckenhoupt and Wheeden [21] gave the weighted boundedness of I_α from $L^{\frac{n}{\alpha}}(w, \mathbb{R}^n)$ to $\mathrm{BMO}_v(\mathbb{R}^n)$.

In 1975, Adams proved the following theorem in [11].

Theorem A (Adams) ([11]) *Let $\alpha \in (0,n)$ and $\lambda \in (0,n]$, there is a constant $C > 0$, such that, if $1 < p = \frac{\lambda}{\alpha}$, then*

$$\|I_\alpha f\|_{\mathrm{BMO}} \le C\|f\|_{L^{p,\lambda}}.$$

On the other hand, many scholars have investigated the various map properties of the homogeneous fractional integral operator $T_{\Omega,\alpha}$, which is defined by

$$T_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)\, dy,$$

where $0 < \alpha < n$, Ω is homogeneous of degree zero on $\mathbb{R}^n$ with $\Omega \in L^s(S^{n-1})$ $(s \ge 1)$ and S^{n-1} denotes the unit sphere of $\mathbb{R}^n$. For instance, the weighted (L^p, L^q)-boundedness of $T_{\Omega,\alpha}$ for

$1 < p < \frac{n}{\alpha}$ had been studied in [22] (for power weights) and in [23] (for $A(p,q)$ weights). The weak boundedness of $T_{\Omega,\alpha}$ when $p = 1$ can be found in [24] (unweighed) and in [25] (with power weights). In 2002, Ding [26] proved that $T_{\Omega,\alpha}$ is bounded from $L^{\frac{n}{\alpha}}(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$ when Ω satisfies some smoothness conditions on S^{n-1}.

Inspired by the $(L^{p,\lambda}(\mathbb{R}^n), \mathrm{BMO}(\mathbb{R}^n))$-boundedness of Riesz potential integral operator I_α for $p = \frac{\lambda}{\alpha}$. We will prove the $(L^{p,\lambda}(\mathbb{R}^n), \mathrm{BMO}(\mathbb{R}^n))$-boundedness of homogeneous fractional integral operator $T_{\Omega,\alpha}$ for $p = \frac{\lambda}{\alpha}$. Then we find that $T_{\Omega,\alpha}$ is also bounded from $L^{p,\lambda}(\mathbb{R}^n)$ ($\frac{\lambda}{\alpha} < p < \infty$) to a class of the Campanato spaces $\mathcal{L}_{l,\lambda}(\mathbb{R}^n)$.

We say that Ω satisfies the L^s-Dini condition if Ω is homogeneous of degree zero on $\mathbb{R}^n$ with $\Omega \in L^s(S^{n-1})$ ($s \geq 1$), and

$$\int_0^1 \omega_s(\delta) \frac{d\delta}{\delta} < \infty,$$

where $\omega_s(\delta)$ denotes the integral modulus of continuity of order s of Ω defined by

$$\omega_s(\delta) = \sup_{|\rho|<\delta} \left(\int_{S^{n-1}} \left|\Omega(\rho x') - \Omega(x')\right|^s dx' \right)^{\frac{1}{s}},$$

and ρ is a rotation in $\mathbb{R}^n$ and $|\rho| = \|\rho - I\|$.

Now, let us formulate our result as follows.

Theorem 1.1 *Let* $0 < \alpha$, $\lambda < n$, *if* Ω *satisfies the* L^s*-Dini condition* ($s > 1$), *then there is a constant* $C > 0$ *such that*

$$\|T_{\Omega,\alpha} f\|_{\mathrm{BMO}} \leq C \|f\|_{L^{\frac{\lambda}{\alpha},\lambda}}. \tag{1.1}$$

Remark 1.2 If $\Omega \equiv 1$, $s = \infty$, and $\lambda = 0$, then $T_{\Omega,\alpha}$ is a Riesz potential I_α, and Theorem 1.1 becomes Theorem A (Adams) [3].

The following theorem shows that $T_{\Omega,\alpha}$ is a bounded map from $L^{p,\lambda}(\mathbb{R}^n)$ ($\frac{\lambda}{\alpha} < p < \infty$) to the Campanato spaces $\mathcal{L}_{l,\lambda}(\mathbb{R}^n)$ for appropriate indices $\lambda > 0$ and $l \geq 1$.

Theorem 1.3 *Let* $0 < \alpha < 1$, $0 < \lambda < n$, $\lambda/\alpha < p < \infty$, *and* $s > \lambda/(\lambda - \alpha)$. *If for some* $\beta > \alpha - \lambda/p$, *the integral modulus of continuity* $\omega_s(\delta)$ *of order* s *of* Ω *satisfies*

$$\int_0^1 \omega_s(\delta) \frac{d\delta}{\delta^{1+\beta}} < \infty,$$

then there is a $C > 0$ *such that for* $1 \leq l \leq \lambda/(\lambda - \alpha)$,

$$\|T_{\Omega,\alpha} f\|_{\mathcal{L}_{l,n(\frac{\alpha}{n} - \frac{1}{p}\frac{\lambda}{n})}} \leq C \|f\|_{L^{p,\lambda}}. \tag{1.2}$$

Remark 1.4 If we take $\Omega \equiv 1$, then $T_{\Omega,\alpha}$ is the Riesz potential I_α, and Theorem 1.3 is even new for the Riesz potential I_α.

Below the letter 'C' will denote a constant not necessarily the same at each occurrence.

2 Proof of Theorem 1.1

In this section we will give the proof of Theorem 1.1. Let us recall the following conclusion.

Lemma 2.1 ([26]) Suppose that $0 < \alpha < n$, $s > 1$, Ω satisfies the L^s-Dini condition. There is a constant $0 < a_0 < \frac{1}{2}$ such that if $|x| < a_0 R$, then

$$\begin{aligned}&\left(\int_{R<|y|<2R}\left|\frac{\Omega(y-x)}{|y-x|^{n-\alpha}}-\frac{\Omega(y)}{|y|^{n-\alpha}}\right|^s dy\right)^{\frac{1}{s}}\\&\quad\le CR^{n/s-(n-\alpha)}\left\{\frac{|x|}{R}+\int_{|x|/2R<\delta<|x|/R}\omega_s(\delta)\frac{d\delta}{\delta}\right\}.\end{aligned}$$

Proof of Theorem 1.1 Fix a cube $Q\subset\mathbb{R}^n$, we denote the center and the diameter of Q by x_0 and d, respectively. We write

$$\begin{aligned}T_{\Omega,\alpha}f&=\int_B\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)\,dy+\int_{R^n\setminus B}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)\,dy\\&:=T_1f(x)+T_2f(x),\end{aligned}$$

where $B=\{y\in\mathbb{R}^n;|y-x_0|<d\}$. It is sufficient to prove (1.1) for $T_1f(x)$ and $T_2f(x)$, respectively.

First let us consider $T_1f(x)$. We have

$$\begin{aligned}\frac{1}{|Q|}\int_Q\left|T_1f(x)-(T_1f)_Q\right|dx&\le\frac{1}{|Q|}\int_Q\int_B\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}\left|f(y)\right|dy\,dx\\&\quad+\frac{1}{|Q|}\int_Q\left(\frac{1}{|Q|}\int_Q\int_B\frac{|\Omega(z-y)|}{|z-y|^{n-\alpha}}\left|f(y)\right|dy\,dz\right)dx\\&\le\frac{2}{|Q|}\int_B\left|f(y)\right|\int_Q\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}\,dx\,dy\\&\le\frac{2}{|Q|}\int_B\left|f(y)\right|\int_{|x-y|<2d}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}\,dx\,dy.\end{aligned}\tag{2.1}$$

Note that $\Omega(x')\in L^s(S^{n-1})$, $\|\Omega\|_{L^s(S^{n-1})}=(\int_{S^{n-1}}|\Omega(y')|^s\,d\sigma(y'))^{\frac{1}{s}}$, we get

$$\begin{aligned}\int_{|x-y|<2d}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}\,dx&\le Cd^\alpha\|\Omega\|_{L^s(S^{n-1})}\\&\le C|Q|^{\frac{\alpha}{n}}\|\Omega\|_{L^s(S^{n-1})}.\end{aligned}\tag{2.2}$$

On the other hand, since $p'<\frac{1}{\frac{1}{p}(\frac{\lambda}{n}-1)-\frac{\alpha}{n}}$, by using the Hölder inequality, we get

$$\begin{aligned}\int_B\left|f(y)\right|dy&\le\left(\int_B\left|f(y)\right|^p dy\right)^{\frac{1}{p}}\left(\int_B 1^{p'}\,dy\right)^{\frac{1}{p'}}\\&\le\left(\int_B\left|f(y)\right|^p dy\right)^{\frac{1}{p}}\left(\int_B 1^{\frac{1}{\frac{1}{p}(\frac{\lambda}{n}-1)-\frac{\alpha}{n}}}\,dy\right)^{\frac{1}{p}(\frac{\lambda}{n}-1)-\frac{\alpha}{n}}\\&=|B|^{\frac{1}{p}(\frac{\lambda}{n}-1)-\frac{\alpha}{n}}\left(\int_B\left|f(y)\right|^p dy\right)^{\frac{1}{p}}\end{aligned}$$

$$
\begin{aligned}
&= |B|^{\frac{1}{p}(\frac{\lambda}{n}-1)-\frac{\alpha}{n}}|B|^{\frac{1}{p}(1-\frac{\lambda}{n})}\left(d^{\lambda-n}\int_B |f(y)|^p\,dy\right)^{\frac{1}{p}} \\
&= |B|^{-\frac{\alpha}{n}}\|f\|_{L^{p,\lambda}}. \qquad (2.3)
\end{aligned}
$$

Here and below we denote $p = \frac{\lambda}{\alpha}$ in the proof of Theorem 1.1. Plugging (2.2) and (2.3) into (2.1), we obtain

$$
\begin{aligned}
\frac{1}{|Q|}\int_Q \left|T_1f(x)-(T_1f)_Q\right|dx &\le C|Q|^{\frac{\alpha}{n}}|B|^{-\frac{\alpha}{n}}\|f\|_{L^{p,\lambda}} \\
&\le C\|f\|_{L^{p,\lambda}}. \qquad (2.4)
\end{aligned}
$$

Now, let us turn to the estimate for $T_2f(x)$. In this case we have

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q \left|T_2f(x)-(T_2f)_Q\right|dx \\
&\quad = \frac{1}{|Q|}\int_Q \left|\frac{1}{|Q|}\int_Q\left\{\int_{|y-x_0|\ge d} f(y)\left[\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right]dy\right\}dz\right|dx \\
&\quad \le \frac{1}{|Q|}\int_Q \frac{1}{|Q|}\int_Q\left\{\sum_{j=0}^{\infty}\int_{2^jd\le|y-x_0|<2^{j+1}d}|f(y)|\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right|dy\right\}dz\,dx. \quad (2.5)
\end{aligned}
$$

By Hölder's inequality, we get

$$
\begin{aligned}
&\int_{2^jd\le|y-x_0|<2^{j+1}d}|f(y)|\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right|dy \\
&\quad \le \left(\int_{2^jd\le|y-x_0|<2^{j+1}d}|f(y)|^{s'}dy\right)^{\frac{1}{s'}} \\
&\qquad \times \left(\int_{2^jd\le|y-x_0|<2^{j+1}d}\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right|^s dy\right)^{\frac{1}{s}}. \qquad (2.6)
\end{aligned}
$$

Since

$$
\begin{aligned}
\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right| &= \left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(y-x_0)}{|y-x_o|^{n-\alpha}}+\frac{\Omega(y-x_0)}{|y-x_o|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right| \\
&\le \left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(y-x_0)}{|y-x_o|^{n-\alpha}}\right|+\left|\frac{\Omega(y-x_0)}{|y-x_o|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right|,
\end{aligned}
$$

we have

$$
\begin{aligned}
&\left(\int_{2^jd\le|y-x_0|<2^{j+1}d}\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right|^s dy\right)^{\frac{1}{s}} \\
&\quad \le \left(\int_{2^jd\le|y-x_0|<2^{j+1}d}\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(y-x_0)}{|y-x_0|^{n-\alpha}}\right|^s dy\right)^{\frac{1}{s}} \\
&\qquad + \left(\int_{2^jd\le|y-x_0|<2^{j+1}d}\left|\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}-\frac{\Omega(y-x_0)}{|y-x_0|^{n-\alpha}}\right|^s dy\right)^{\frac{1}{s}} \\
&\quad := J_1 + J_2. \qquad (2.7)
\end{aligned}
$$

Let us give the estimates of J_1 and J_2, respectively. We write J_1 as

$$J_1 = \left(\int_{2^j d\le|y-x_0|<2^{j+1}d}\left|\frac{\Omega((x-x_0)-y)}{|x-x_0-y|^{n-\alpha}}-\frac{\Omega(y)}{|y|^{n-\alpha}}\right|^s dy\right)^{\frac{1}{s}}.$$

Note that $x\in Q$, if taking $R=2^j d$, then $|x-x_0|<\frac{1}{2^{j+1}}R$. Applying Lemma 2.1 to J_1, we get

$$\begin{aligned} J_1 &\le C\bigl(2^j d\bigr)^{n/s-(n-\alpha)}\left\{\frac{|x-x_0|}{2^j d}+\int_{|x-x_0|/2^{j+1}d<\delta<|x-x_0|/2^j d}\omega_s(\delta)\frac{d\delta}{\delta}\right\} \\ &\le C\bigl(2^j d\bigr)^{n/s-(n-\alpha)}\left\{\frac{1}{2^{j+1}}+\int_{|x-x_0|/2^{j+1}d<\delta<|x-x_0|/2^j d}\omega_s(\delta)\frac{d\delta}{\delta}\right\}. \end{aligned} \tag{2.8}$$

By $z\in Q$ and using a similar method, we have

$$J_2 \le C\bigl(2^j d\bigr)^{n/s-(n-\alpha)}\left\{\frac{1}{2^{j+1}}+\int_{|z-x_0|/2^{j+1}d<\delta<|z-x_0|/2^j d}\omega_s(\delta)\frac{d\delta}{\delta}\right\}. \tag{2.9}$$

Since $p=\frac{\lambda}{\alpha}$ and $\frac{n}{s}-(n-\alpha)<-\frac{n}{s'(p/s')'}$, we get

$$\bigl(2^j d\bigr)^{n/s-(n-\alpha)}\le C\bigl|2^{j+1}\sqrt{n}Q\bigr|^{-\frac{1}{s'(p/s')'}},$$

where $2^{j+1}\sqrt{n}Q$ denote the cube with the center at x_0 and the diameter $2^{j+1}\sqrt{n}d$.

Thus, plugging (2.8) and (2.9) into (2.7), we have

$$\begin{aligned} &\left(\int_{2^j d\le|y-x_0|<2^{j+1}d}\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right|^s dy\right)^{\frac{1}{s}} \\ &\quad\le C\bigl|2^{j+1}\sqrt{n}Q\bigr|^{n/s-(n-\alpha)}\left\{\frac{1}{2^j}+\int_{|x-x_0|/2^{j+1}d<\delta<|x-x_0|/2^j d}\omega_s(\delta)\frac{d\delta}{\delta}\right. \\ &\qquad\left.+\int_{|z-x_0|/2^{j+1}d<\delta<|z-x_0|/2^j d}\omega_s(\delta)\frac{d\delta}{\delta}\right\} \\ &\quad\le C\bigl|2^{j+1}\sqrt{n}Q\bigr|^{-\frac{1}{s'(p/s')'}}\left\{\frac{1}{2^j}+\int_{|x-x_0|/2^{j+1}d<\delta<|x-x_0|/2^j d}\omega_s(\delta)\frac{d\delta}{\delta}\right. \\ &\qquad\left.+\int_{|z-x_0|/2^{j+1}d<\delta<|z-x_0|/2^j d}\omega_s(\delta)\frac{d\delta}{\delta}\right\}. \end{aligned} \tag{2.10}$$

On the other hand, we estimate $(\int_{2^j d\le|y-x_0|<2^{j+1}d}|f(y)|^{s'}dy)^{\frac{1}{s'}}$, since $p'<s'(\frac{p}{s'})'$ and $s'(\frac{p}{s'})'<\frac{1}{\frac{1}{p}(\frac{\lambda}{n}-1)+\frac{1}{(p/s')'s'}}$, by using Hölder's inequality again, we get

$$\begin{aligned} &\left(\int_{2^j d\le|y-x_0|<2^{j+1}d}\bigl|f(y)\bigr|^{s'}dy\right)^{\frac{1}{s'}} \\ &\quad\le\left(\int_{2^j d\le|y-x_0|<2^{j+1}d}\bigl|f(y)\bigr|^{p}dy\right)^{\frac{1}{p}}\left(\int_{|y-x_0|<2^{j+1}d}1^{s'(\frac{p}{s'})'}dy\right)^{\frac{1}{(p/s')'s'}} \\ &\quad\le C\left(\int_{|y-x_0|<2^{j+1}d}\bigl|f(y)\bigr|^{p}dy\right)^{\frac{1}{p}}\left(\int_{|y-x_0|<2^{j+1}d}1^{\frac{1}{\frac{1}{p}(\frac{\lambda}{n}-1)+\frac{1}{(p/s')'s'}}}dy\right)^{\frac{1}{p}(\frac{\lambda}{n}-1)+\frac{1}{(p/s')'s'}} \end{aligned}$$

$$\begin{aligned}
&\leq C|B_1|^{\frac{1}{p}(\frac{\lambda}{n}-1)+\frac{1}{(p/s')'s'}}|B_1|^{\frac{1}{p}(1-\frac{\lambda}{n})}\left(\left(2^{j+1}d\right)^{\lambda-n}\int_{|y-x_0|<2^{j+1}d}\left|f(y)\right|^p dy\right)^{\frac{1}{p}}\\
&\leq C|B_1|^{\frac{1}{(p/s')'s'}}\|f\|_{L^{p,\lambda}}, \qquad (2.11)
\end{aligned}$$

where $B_1=\{y\in\mathbb{R}^n; |y-x_0|<2^{j+1}d\}$.

Plugging (2.10) and (2.11) into (2.6) we obtain

$$\begin{aligned}
&\sum_{j=0}^{\infty}\int_{2^jd\leq|y-x_0|<2^{j+1}d}\left|f(y)\right|\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right|dy\\
&\quad\leq C\sum_{j=0}^{\infty}\|f\|_{L^{p,\lambda}}|B_1|^{\frac{1}{(p/s')'s'}}\left|2^{j+1}\sqrt{n}Q\right|^{n/s-(n-\alpha)}\left\{\frac{1}{2^j}+\int_{|x-x_0|/2^{j+1}d<\delta<|x-x_0|/2^jd}\omega_s(\delta)\frac{d\delta}{\delta}\right.\\
&\quad\quad\left.+\int_{|z-x_0|/2^{j+1}d<\delta<|z-x_0|/2^jd}\omega_s(\delta)\frac{d\delta}{\delta}\right\}\\
&\quad\leq C\|f\|_{L^{p,\lambda}}\sum_{j=0}^{\infty}|B_1|^{\frac{1}{(p/s')'s'}}|B_1|^{-\frac{1}{(p/s')'s'}}\left\{\frac{1}{2^j}+\int_{|x-x_0|/2^{j+1}d<\delta<|x-x_0|/2^jd}\omega_s(\delta)\frac{d\delta}{\delta}\right.\\
&\quad\quad\left.+\int_{|z-x_0|/2^{j+1}d<\delta<|z-x_0|/2^jd}\omega_s(\delta)\frac{d\delta}{\delta}\right\}\\
&\quad\leq C\|f\|_{L^{p,\lambda}}\left\{2+2\int_0^1\omega_s(\delta)\frac{d\delta}{\delta}\right\}\\
&\quad\leq C\|f\|_{L^{p,\lambda}}. \qquad (2.12)
\end{aligned}$$

Therefore, applying (2.12) into (2.5) we obtain

$$\frac{1}{|Q|}\int_Q\left|T_2f(x)-(T_2f)_Q\right|dx\leq C\|f\|_{L^{p,\lambda}}. \qquad (2.13)$$

Combining (2.4) and (2.13), we get

$$\begin{aligned}
\|T_{\Omega,\alpha}f\|_{\mathrm{BMO}}&=\frac{1}{|Q|}\int_Q\left|T_{\Omega,\alpha}f(y)-(T_{\Omega,\alpha}f)_Q\right|dy\\
&\leq\frac{1}{|Q|}\int_Q\left|T_1f(y)-(T_1f)_Q\right|dy+\frac{1}{|Q|}\int_Q\left|T_2f(y)-(T_2f)_Q\right|dy\\
&\leq C\|f\|_{L^{p,\lambda}}.
\end{aligned}$$

Thus, we complete the proof of Theorem 1.1. □

3 Proof of Theorem 1.3

Similarly to the proof of Theorem 1.1. We need only to prove (1.2) for T_1 and T_2, respectively. First let us consider $T_1f(x)$. We have

$$\begin{aligned}
&\frac{1}{|Q|^{\frac{\alpha}{n}-\frac{1}{p}\frac{\lambda}{n}}}\left(\frac{1}{|Q|}\int_Q\left|T_1f(x)-(T_1f)_Q\right|^l dx\right)^{\frac{1}{l}}\\
&\quad\leq\frac{2}{|Q|^{\frac{\alpha}{n}-\frac{1}{p}\frac{\lambda}{n}}}\left(\frac{1}{|Q|}\int_Q\left|T_1f(x)\right|^l dx\right)^{\frac{1}{l}}
\end{aligned}$$

$$= \frac{2}{|Q|^{\frac{\alpha}{n}-\frac{1}{p}\frac{\lambda}{n}}}\left(\frac{1}{|Q|}\int_Q\left|\int_B \frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)\,dy\right|^l dx\right)^{\frac{1}{l}}$$
$$\leq \frac{2}{|Q|^{\frac{\alpha}{n}-\frac{1}{p}\frac{\lambda}{n}}}\frac{1}{|Q|^{\frac{1}{l}}}\int_B\left|f(y)\right|\left(\int_{|y-x|<2d}\left(\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}\right)^l dx\right)^{\frac{1}{l}} dy. \tag{3.1}$$

Note that $\Omega(x') \in L^s(S^{n-1})$, $\|\Omega\|_{L^s(S^{n-1})} = (\int_{S^{n-1}} |\Omega(y')|^s\,d\sigma(y'))^{\frac{1}{s}}$, and $s > \frac{\lambda}{\lambda-\alpha} \geq l$, hence

$$\left(\int_{|y-x|<2d}\left(\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}\right)^l dx\right)^{\frac{1}{l}} \leq Cd^{\frac{n}{l}-(n-\alpha)}\|\Omega\|_{L^s(S^{n-1})}$$
$$\leq C|Q|^{\frac{1}{l}-(1-\frac{\alpha}{n})}\|\Omega\|_{L^s(S^{n-1})}. \tag{3.2}$$

On the other hand, by Hölder's inequality,

$$\int_B\left|f(y)\right| dy \leq \left(\int_B\left|f(y)\right|^p dy\right)^{\frac{1}{p}}\left(\int_B 1^{p'}\,dy\right)^{\frac{1}{p'}}$$
$$= |B|^{\frac{1}{p}(1-\frac{\lambda}{n})}\left(|B|^{\frac{\lambda}{n}-1}\int_B\left|f(y)\right|^p dy\right)^{\frac{1}{p}}$$
$$\times\left(\int_B 1^{\frac{1}{1-\frac{\lambda}{n}(1-\frac{1}{p'})+\frac{1}{p}(\frac{\lambda}{n}-1)}}\,dy\right)^{1-\frac{\lambda}{n}(1-\frac{1}{p'})+\frac{1}{p}(\frac{\lambda}{n}-1)}$$
$$\leq |B|^{1-\frac{\lambda}{n}(1-\frac{1}{p'})+\frac{1}{p}(\frac{\lambda}{n}-1)}|B|^{\frac{1}{p}(1-\frac{\lambda}{n})}\|f\|_{L^{p,\lambda}}$$
$$= |B|^{1-\frac{\lambda}{n}(1-\frac{1}{p'})}\|f\|_{L^{p,\lambda}}. \tag{3.3}$$

Plugging (3.2) and (3.3) into (3.1) we get

$$\frac{1}{|Q|^{\frac{\alpha}{n}-\frac{1}{p}\frac{\lambda}{n}}}\left(\frac{1}{|Q|}\int_Q\left|T_1f(x)-(T_1f)_Q\right|^l dx\right)^{\frac{1}{l}}$$
$$\leq C|Q|^{\frac{\lambda}{n}\frac{1}{p}-\frac{\alpha}{n}-\frac{1}{l}+1-\frac{\lambda}{n}(1-\frac{1}{p'})+\frac{1}{l}-(1-\frac{\alpha}{n})}\|\Omega\|_{L^s(S^{n-1})}\|f\|_{L^{p,\lambda}}$$
$$\leq C\|f\|_{L^{p,\lambda}}. \tag{3.4}$$

Now, let us turn to the estimate for $T_2f(x)$. In this case we have

$$\frac{1}{|Q|^{\frac{\alpha}{n}-\frac{1}{p}\frac{\lambda}{n}}}\left(\frac{1}{|Q|}\int_Q\left|T_2f(x)-(T_2f)_Q\right|^l dx\right)^{\frac{1}{l}}$$
$$= \frac{1}{|Q|^{\frac{\alpha}{n}-\frac{1}{p}\frac{\lambda}{n}}}\left(\frac{1}{|Q|}\int_Q\left|\frac{1}{|Q|}\int_Q\left\{\sum_{j=0}^{\infty}\int_{2^jd\leq|y-x_0|<2^{j+1}d} f(y)\right.\right.\right.$$
$$\left.\left.\left.\times\left[\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right]dy\right\}dz\right|^l dx\right)^{\frac{1}{l}}. \tag{3.5}$$

By Hölder's inequality and the proof of Theorem 1.1, $s' < \frac{\lambda}{\alpha} < p$,

$$\begin{aligned}
&\int_{2^j d \le |y-x_0| < 2^{j+1} d} |f(y)| \left| \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(z-y)}{|z-y|^{n-\alpha}} \right| dy \\
&\quad \le \left(\int_{2^j d \le |y-x_0| < 2^{j+1} d} |f(y)|^{s'} dy \right)^{\frac{1}{s'}} (J_1 + J_2) \\
&\quad \le \left(\int_{2^j d \le |y-x_0| < 2^{j+1} d} |f(y)|^{p} dy \right)^{\frac{1}{p}} \left(\int_{|y-x_0| < 2^{j+1} d} 1^{s'(p/s')'} dy \right)^{\frac{1}{s'(p/s')'}} (J_1 + J_2) \\
&\quad = \left(2^{j+1} d\right)^{-\frac{\lambda-n}{p}} \left[\left(2^{j+1} d\right)^{\lambda-n} \int_{|y-x_0| < 2^{j+1} d} |f(y)|^p dy \right]^{\frac{1}{p}} \\
&\qquad \times \left(\int_{|y-x_0| < 2^{j+1} d} 1^{\frac{1}{\frac{1}{s'(p/s')'} + \frac{\lambda-n}{np} + \frac{1}{p}(1-\frac{\lambda}{n})}} dy \right)^{\frac{1}{s'(p/s')'} + \frac{\lambda-n}{np} + \frac{1}{p}(1-\frac{\lambda}{n})} (J_1 + J_2) \\
&\quad \le C \|f\|_{L^{p,\lambda}} |B_1|^{-\frac{\lambda-n}{np}} |B_1|^{\frac{1}{s'(p/s')'} + \frac{\lambda-n}{np} + \frac{1}{p}(1-\frac{\lambda}{n})} (J_1 + J_2) \\
&\quad = C \|f\|_{L^{p,\lambda}} \left(2^j d\right)^{\frac{n}{s'(p/s')'} + \frac{n}{p}(1-\frac{\lambda}{n})} (J_1 + J_2) \\
&\quad = C \|f\|_{L^{p,\lambda}} \left(2^j d\right)^{\frac{n}{s'(p/s')'}} 2^{j(\frac{n}{p} - \frac{\lambda}{p})} |Q|^{\frac{1}{p} - \frac{1}{p}\frac{\lambda}{n}} (J_1 + J_2),
\end{aligned} \tag{3.6}$$

where $B_1 = \{y \in \mathbb{R}^n; |y - x_0| < 2^{j+1} d\}$.

Since the integral modulus of continuity $\omega_s(\delta)$ of order s of Ω satisfies (1.2) and

$$\int_o^1 \omega_s(\delta) \frac{d\delta}{\delta} < \int_o^1 \omega_s(\delta) \frac{d\delta}{\delta^{1+\beta}} < \infty,$$

we know that Ω satisfies also the L^s-Dini condition. From Lemma 2.1 and the proof of Theorem 1.1, we get

$$\begin{aligned}
J_1 + J_2 \le{}& C\left(2^j d\right)^{n/s-(n-\alpha)} \times \left\{ \frac{1}{2^j} + \int_{|x-x_0|/2^{j+1} d < \delta < |x-x_0|/2^j d} \omega_s(\delta) \frac{d\delta}{\delta} \right. \\
&\left. + \int_{|z-x_0|/2^{j+1} d < \delta < |z-x_0|/2^j d} \omega_s(\delta) \frac{d\delta}{\delta} \right\}.
\end{aligned} \tag{3.7}$$

Note that

$$\begin{aligned}
\left(2^j d\right)^{n/[s'(p/s')'] + n/s - (n-\alpha)} 2^{j(\frac{n}{p} - \frac{\lambda}{p})} &= \left(2^j d\right)^{n(\frac{\alpha}{n} - \frac{1}{p})} 2^{j(\frac{n}{p} - \frac{\lambda}{p})} \\
&\le C|Q|^{\frac{\alpha}{n} - \frac{1}{p}} 2^{jn(\frac{\alpha}{n} - \frac{1}{p}) + j(\frac{n}{p} - \frac{\lambda}{p})} \\
&= C|Q|^{\frac{\alpha}{n} - \frac{1}{p}} 2^{jn(\frac{\alpha}{n} - \frac{1}{p}\frac{\lambda}{n})}.
\end{aligned} \tag{3.8}$$

Moreover,

$$\begin{aligned}
2^{jn(\frac{\alpha}{n} - \frac{1}{p}\frac{\lambda}{n})} \int_{|x-x_0|/2^{j+1} d}^{|x-x_0|/2^j d} \omega_s(\delta) \frac{d\delta}{\delta} &\le 2^{jn(\frac{\alpha}{n} - \frac{1}{p}\frac{\lambda}{n})} \left(|x - x_0|/2^j d\right)^{\beta} \int_{|x-x_0|/2^{j+1} d}^{|x-x_0|/2^j d} \omega_s(\delta) \frac{d\delta}{\delta^{1+\beta}} \\
&\le C 2^{j[n(\frac{\alpha}{n} - \frac{1}{p}\frac{\lambda}{n}) - \beta]} \int_0^1 \omega_s(\delta) \frac{d\delta}{\delta^{1+\beta}}.
\end{aligned} \tag{3.9}$$

By $0 < \alpha < 1$ and $\beta > \alpha - \frac{\lambda}{p}$, we have $n(\frac{\alpha}{n} - \frac{1}{p}\frac{\lambda}{n}) - 1 < 0$ and $n(\frac{\alpha}{n} - \frac{1}{p}\frac{\lambda}{n}) - \beta < 0$, respectively. Applying (3.7), (3.8), and (3.9) to (3.6) we obtain

$$\begin{aligned}
&\sum_{j=0}^{\infty} \int_{2^j d \le |y-x_0| < 2^{j+1} d} f(y) \left[\frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(z-y)}{|z-y|^{n-\alpha}} \right] dy \\
&\quad \le C\|f\|_{L^{p,\lambda}} |Q|^{\frac{1}{p} - \frac{1}{p}\frac{\lambda}{n}} |Q|^{\frac{\alpha}{n} - \frac{1}{p}} \sum_{j=0}^{\infty} \left\{ 2^{j[n(\frac{\alpha}{n} - \frac{1}{p}\frac{\lambda}{n}) - 1]} + C 2^{j[n(\frac{\alpha}{n} - \frac{1}{p}\frac{\lambda}{n}) - \beta]} \int_0^1 \omega_s(\delta) \frac{d\delta}{\delta^{1+\beta}} \right\} \\
&\quad \le C\|f\|_{L^{p,\lambda}} |Q|^{\frac{\alpha}{n} - \frac{1}{p}\frac{\lambda}{n}}.
\end{aligned} \tag{3.10}$$

Plugging (3.10) into (3.5), we obtain

$$\frac{1}{|Q|^{\frac{\alpha}{n} - \frac{1}{p}\frac{\lambda}{n}}} \left(\frac{1}{|Q|} \int_Q \left| T_2 f(x) - (T_2 f)_Q \right|^l dx \right)^{\frac{1}{l}} \le C\|f\|_{L^{p,\lambda}}. \tag{3.11}$$

Then by (3.4) and (3.11) we get

$$\begin{aligned}
\|T_{\Omega,\alpha} f\|_{\mathcal{L}_{l,n(\frac{\alpha}{n} - \frac{1}{p}\frac{\lambda}{n})}} &\le \frac{1}{|Q|^{\frac{\alpha}{n} - \frac{1}{p}\frac{\lambda}{n}}} \left(\frac{1}{|Q|} \int_Q \left| T_1 f(x) - (T_1 f)_Q \right|^l dx \right)^{\frac{1}{l}} \\
&\quad + \frac{1}{|Q|^{\frac{\alpha}{n} - \frac{1}{p}\frac{\lambda}{n}}} \left(\frac{1}{|Q|} \int_Q \left| T_2 f(x) - (T_2 f)_Q \right|^l dx \right)^{\frac{1}{l}} \\
&\le C\|f\|_{L^{p,\lambda}}.
\end{aligned}$$

Thus, we complete the proof of Theorem 1.3. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MS carried out the boundedness of the homogeneous fractional integral operator on Morrey space studies and drafted the manuscript. YC participated in the study of fractional integral operator and helped to check the manuscript. All authors read and approved the final manuscript.

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