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An extension of some variant of Meijer type integrals in the class of Boehmians

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Abstract

In this paper we establish certain spaces of Boehmians for some Meijer type integral transform. The extended Meijer type integral transform is defined and some desired properties are obtained in the class of Boehmians. Further theorems are also discussed with some details.

Keywords: $\mathbf{l}_{\alpha_1,\alpha_2}$ transform; generalized function; Meijer-type integral; Krätzel transform; Boehmian space

1 Introduction

During the past decades, a number of Meijer-type integral transforms have been studied by many authors such as Meijer [1], Pandey [2], Srivastava [3], Srivastava and Vyas [4] and many others. Some variants of Meijer-type integrals were given by Krätzel [5–8] and, recently, by Waphare [9]. In 1990, a new version of Meijer-type integral, referred to as an $M_{\alpha,\beta}$-integral, has been investigated and its inversion formula was duly achieved by [10]. This variant of this transform was formed to generalize those obtained by Krätzel [6], Conlan and Koh [11], and Rodriguez [12] as well.

In this article, we consider some variant defined, under conditions of regularity and convergence, by Betancor [13] as

$$\Phi(\gamma) = \mathbf{l}_{\alpha_1,\alpha_2}\big(\phi(\zeta)\big)(\gamma) = \gamma^{-\alpha_2} \int_0^\infty \zeta^{\alpha_1,\alpha_2-1} \mathbf{l}_{\alpha_1-1}(\gamma\zeta)\phi(\zeta)\,d\zeta \tag{1}$$

with the inversion formula given as

$$\mathbf{l}_{\alpha_1,\alpha_2}^{-1}\big(\phi(\gamma)\big)(\zeta) = \frac{\zeta^{-\alpha_2}}{i\pi} \int_{\Lambda_c} \gamma^{\alpha_1,\alpha_2-1} \boldsymbol{\epsilon}_{\alpha_1-1}(\gamma\zeta)\Phi(\gamma)\,d\gamma, \tag{2}$$

where $\Lambda_c = \{w \in \mathbb{C} : \operatorname{Re}\sqrt{2w} = c > 0\}$, $\alpha_1, \alpha_2 \in \mathbb{C}$, $\mathbf{l}_\nu$ and $\boldsymbol{\epsilon}_\nu$ are the modified Bessel-Clifford functions of the third and first kind of order ν, respectively.

The modified Bessel functions $\boldsymbol{\epsilon}_\nu$ and $\mathbf{l}_\nu$ are indeed related to the homologous Bessel functions $\mathbf{j}_\nu$ and $\mathbf{k}_\nu$ by the formulas

$$\boldsymbol{\epsilon}_\nu(x) = x^{\frac{-\nu}{2}} \mathbf{j}_\nu(2\sqrt{x}) \tag{3}$$

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and

$$\mathbf{l}_\nu(x) = x^{\frac{-\nu}{2}} \mathbf{k}_\nu(2\sqrt{x}), \tag{4}$$

which are solutions of the differential equation $xy'' + (\nu + 1)y' - y = 0$. They also satisfy the relations $\mathbf{l}_\nu^{(n)} = (-1)^n \mathbf{l}_{\nu+n}$ and $\boldsymbol{\epsilon}_\nu^{(n)} = \boldsymbol{\epsilon}_{\nu+n}$, for every choice of $n \in \mathbb{N}$.

In addition, the integral representation of the modified Bessel function $\mathbf{l}_{\alpha-1}$ was introduced by Rodriguez [10], Meijer [1] and Waphare [9] as

$$\mathbf{l}_{\alpha-1}(\gamma\zeta) = \frac{1}{2} \int_0^\infty \tau^{-\alpha} e^{-\tau - \frac{\gamma\zeta}{\tau}} \, d\tau. \tag{5}$$

For a complete account of $\mathbf{l}_{\alpha-1}$ and $\boldsymbol{\epsilon}_{\alpha-1}$ functions we refer to Rodriguez in [10] and Betancor in [13].

The class of Boehmians was introduced to generalize regular operators [14]. The minimal structure necessary for the abstract construction of Boehmian spaces consists of the following elements:

  (i) A nonempty set $\mathbf{a}$.

  (ii) A commutative semigroup $(\mathbf{b}, *)$.

  (iii) An operation $\odot : \mathbf{a} \times \mathbf{b} \to \mathbf{a}$ such that for each $x \in \mathbf{a}$, the following formula is satisfied:

$$x \odot (\delta_1 * \delta_2) = (x \odot \delta_1) \odot \delta_2$$

     for every choice of $\delta_1, \delta_2 \in \mathbf{b}$.

  (iv) A collection $\Delta \subset \mathbf{b}$ such that:

    (a) if $x, y \in \mathbf{a}, (\delta_n) \in \Delta, x \odot \delta_n = y \odot \delta_n$ for all n, then $x = y$;

    (b) if $(\delta_n), (\psi_n) \in \Delta$, then $(\delta_n * \psi_n) \in \Delta$.

The elements of Δ are called delta sequences. Denote by g the set

$$g = \left\{ (x_n, \delta_n) : x_n \in \mathbf{a}, (\delta_n) \in \Delta, x_n \odot \delta_m = x_m \odot \delta_n, \forall m, n \in \mathbb{N} \right\}.$$

Then, if $(x_n, \delta_n), (y_n, \psi_n) \in g, x_n \odot \psi_m = y_m \odot \delta_n, \forall m, n \in \mathbb{N}$, then we say $(x_n, \delta_n) \sim (y_n, \psi_n)$. The relation $\sim$ is an equivalence relation in g. The space of equivalence classes in g is denoted by $\boldsymbol{\alpha}$. Elements of $\boldsymbol{\alpha}$ are called Boehmians. Between $\mathbf{a}$ and $\boldsymbol{\alpha}$ there is a canonical embedding expressed as

$$x \to \frac{x \odot \delta_n}{\delta_n}.$$

The operation $\odot$ can be extended to $\boldsymbol{\alpha} \times \mathbf{a}$ by

$$\frac{x_n}{\delta_n} \odot t = \frac{x_n \odot t}{\delta_n}.$$

In $\boldsymbol{\alpha}$, we have two types of convergence.

δ-convergence: A sequence (β_n) in $\boldsymbol{\alpha}$ is said to be δ-convergent to β in $\boldsymbol{\alpha}$, denoted by $\beta_n \xrightarrow{\delta} \beta$, if there exists a delta sequence (δ_n) such that $(\beta_n \odot \delta_n), (\beta \odot \delta_n) \in \mathbf{a}, \forall k, n \in \mathbb{N}$, and $(\beta_n \odot \delta_k) \to (\beta \odot \delta_k)$ as $n \to \infty$, in $\mathbf{a}$, for every $k \in \mathbb{N}$.

Δ-convergence: A sequence (β_n) in $\boldsymbol{\alpha}$ is said to be Δ-convergent to β in $\boldsymbol{\alpha}$, denoted by $\beta_n \xrightarrow{\Delta} \beta$, if there exists a $(\delta_n) \in \Delta$ such that $(\beta_n - \beta) \odot \delta_n \in \mathbf{a}$, $\forall n \in \mathbb{N}$, and $(\beta_n - \beta) \odot \delta_n \to 0$ as $n \to \infty$ in $\mathbf{a}$.

For the abstract construction of Boehmian spaces, we refer to Karunakaran and Roopkumar [15], Karunakaran and Vembu [16], Mikusinski [17], Al-Omari [18–20], Al-Omari and Kilicman [21], Nemzer [22], and some others.

2 Some notations and necessary theorems

Denote by $\Omega(0, \infty)$ the Schwartz' space of test functions of compact supports defined on $(0, \infty)$ and by $F_{\text{loc}}((0, \infty))$, the set of all continuous functions of bounded supports over $(0, \infty)$ such that

$$\int_k \bigl|f(x)\bigr| \, dx \tag{6}$$

is finite for every compact subset k of $(0, \infty)$.

The following are two definitions needed for our next investigation.

Definition 1 The Mellin-type convolution product of first kind is defined by Karunakaran and Roopkumar as [15]

$$(\phi * \psi)(\zeta) = \int_0^\infty x^{-1} \phi\bigl(\zeta x^{-1}\bigr) \psi(x) \, dx. \tag{7}$$

The general properties of $*$ are given by
(i) $(\phi * \psi)(t) = (\psi * \phi)(t)$;
(ii) $((\phi + \psi) * \varphi)(t) = (\phi * \varphi)(t) + (\psi * \varphi)(t)$;
(iii) $(\alpha\phi * \psi)(t) = \alpha(\psi * \phi)(t)$, α is complex number;
(iv) $((\phi * \psi) * \varphi)(t) = (\phi * (\psi * \varphi))(t)$.

Definition 2 Let $\mathbf{l}^1$ denote the space of those functions which are complex-valued Lebesgue and integrable on $(0, \infty)$. Between the two functions ϕ and ψ in $\mathbf{l}^1$ we define an operation $\circledast$ in terms of the integral equation

$$(\phi \circledast \psi)(\gamma) = \int_0^\infty x^{\alpha_1 + \alpha_2 - 1} \phi(\gamma x) \psi(x) \, dx \tag{8}$$

for every choice of α_1 and α_2.

The integrals (7) and (8) are clearly related by the following theorem.

Theorem 3 *Let ϕ and ψ be complex-valued integrable functions in $\mathbf{l}^1$ and $\gamma \in (0, \infty)$. Then we have*

$$\mathbf{l}_{\alpha_1, \alpha_2}(\phi * \psi)(\gamma) = (\mathbf{l}_{\alpha_1, \alpha_2}\phi \circledast \psi)(\gamma),$$

where $\alpha_1, \alpha_2 \in \mathbb{C}$.

Proof Let the hypothesis of the theorem be satisfied by some functions ϕ and ψ in $\mathbf{l}^1$. Then by invoking (1) in (7) we get

$$
\begin{aligned}
\mathbf{l}_{\alpha_1,\alpha_2}(\phi * \psi)(\gamma) &= \gamma^{-\alpha_2} \int_0^\infty \zeta^{\alpha_1+\alpha_2-1} \mathbf{l}_{\alpha_1-1}(\gamma\zeta)(\phi * \psi)(\zeta)\, d\zeta \\
&= \gamma^{-\alpha_2} \int_0^\infty \zeta^{\alpha_1+\alpha_2-1} \mathbf{l}_{\alpha_1-1}(\gamma\zeta) \int_0^\infty x^{-1}\phi(\zeta x^{-1})\psi(x)\, dx\, d\zeta \\
&= \gamma^{-\alpha_2} \int_0^\infty x^{-1}\psi(x) \int_0^\infty \zeta^{\alpha_1+\alpha_2-1} \mathbf{l}_{\alpha_1-1}(\gamma\zeta)\phi(\zeta x^{-1})\, dx\, d\zeta.
\end{aligned}
$$

By Fubini's theorem and setting variables, we obtain the integral equation

$$
\mathbf{l}_{\alpha_1,\alpha_2}(\phi * \psi)(\gamma) = \gamma^{-\alpha_2} \int_0^\infty \psi(x) x^{\alpha_1+\alpha_2-1} y^{\alpha_1+\alpha_2-1} \mathbf{l}_{\alpha_1-1}\big(\gamma(xy)\big)\phi(y)\, dy\, dx.
$$

Hence, simplifications yield

$$
\begin{aligned}
\mathbf{l}_{\alpha_1,\alpha_2}(\phi * \psi)(\gamma) &= \int_0^\infty \psi(x) x^{\alpha_1+\alpha_2-1} \\
&\quad \times \left((\gamma x)^{-\alpha_2} \int_0^\infty y^{\alpha_1+\alpha_2-1} \mathbf{l}_{\alpha_1-1}\big((\gamma x)y\big)\phi(y)\, dy \right) dx \\
&= \int_0^\infty x^{\alpha_1+\alpha_2-1} \mathbf{l}_{\alpha_1,\alpha_2}\phi(\gamma x)\psi(x)\, dx.
\end{aligned}
$$

Therefore, by the aid of (8), we get

$$
\mathbf{l}_{\alpha_1,\alpha_2}(\phi * \psi)(\gamma) = (\mathbf{l}_{\alpha_1,\alpha_2}\phi \circledast \psi)(\gamma).
$$

This completes the proof of the theorem. $\qquad\square$

Theorem 4 *Let ϕ, φ, and ψ be integrable functions in $\mathbf{l}^1$ and $y \in (0,\infty)$. Then we have*

$$
\big(\phi \circledast (\varphi * \psi)\big)(y) = \big((\phi \circledast \varphi) \circledast \psi\big)(y).
$$

Proof By taking into account (7) and (8) we by Fubini's theorem obtain

$$
\begin{aligned}
\big(\phi \circledast (\varphi * \psi)\big)(y) &= \int_0^\infty \zeta^{\alpha_1+2\alpha_2-1}\phi(y\zeta)(\varphi * \psi)(\zeta)\, d\zeta \\
&= \int_0^\infty \zeta^{\alpha_1+2\alpha_2-1}\phi(y\zeta) \int_0^\infty x^{-1}\varphi(\zeta x^{-1})\psi(x)\, dx\, d\zeta \\
&= \int_0^\infty x^{\alpha_1+2\alpha_2-1}\psi(x) \int_0^\infty \phi(yx\tau)\varphi(z)\tau^{\alpha_1+2\alpha_2-1}\, d\tau\, dx.
\end{aligned}
$$

Setting variables gives

$$
\big(\phi \circledast (\varphi * \psi)\big)(y) = \int_0^\infty x^{\alpha_1+2\alpha_2-1}(\phi \circledast \varphi)(yx)\psi(x)\, dx.
$$

Hence, equation (8) reveals

$$\big(\phi \circledast (\varphi * \psi)\big)(y) = \big((\phi \circledast \varphi) \circledast \psi\big)(y).$$

This completes the proof of the theorem. □

Theorem 5 *Let $\phi \in F_{\text{loc}}(0,\infty)$. Then $\mathbf{l}_{\alpha_1,\alpha_2}(\phi) \in F_{\text{loc}}(0,\infty)$, for $\alpha_1 < 2$, $\alpha_2 \neq 2 - \alpha_1$.*

Proof Let $\phi \in F_{\text{loc}}(0,\infty)$ be arbitrary and k_1 be a compact subset in $(0,\infty)$. Then, invoking (5) in (1) suggests

$$\int_{\mathsf{k}_1} \big|\mathbf{l}_{\alpha_1,\alpha_2}\big(\phi(\zeta)\big)(\gamma)\big| \, d\gamma \leq \int_{\mathsf{k}_1} \gamma^{-\alpha_2} \int_0^\infty \zeta^{\alpha_1 + 2\alpha_2 - 1} \big|\mathbf{l}_{\alpha_1 - 1}(\gamma\zeta)\big| \big|\phi(\zeta)\big| \, d\zeta \, d\gamma$$

$$\leq \frac{1}{2} \int_{\mathsf{k}_1} \gamma^{-\alpha_2} \int_0^\infty \zeta^{\alpha_1 + 2\alpha_2 - 1} \left| \int_0^\infty \tau^{-\alpha_1} e^{-\tau - \frac{\gamma\zeta}{\tau}} \, d\tau \right|$$

$$\times \big|\phi(\zeta)\big| \, d\zeta \, d\gamma$$

$$= \frac{1}{2} \int_{\mathsf{k}_1} \zeta^{\alpha_1 + 2\alpha_2 - 1}$$

$$\times \left| \int_0^\infty (\gamma\zeta y^{-1})^{-\alpha_1} e^{-(\gamma\zeta)^{-1} y - y} (\gamma\zeta) y^{-2} \, dy \right| \big|\phi(\zeta)\big| \, d\zeta \, d\gamma.$$

Setting variables reveals

$$\int_{\mathsf{k}_1} \big|\mathbf{l}_{\alpha_1,\alpha_2}\big(\phi(\zeta)\big)(\gamma)\big| \, d\gamma \leq \frac{1}{2} \int_{\mathsf{k}_1} \gamma^{-\alpha_2 - \alpha_1 + 1} \int_0^\infty \zeta^{\alpha_2} \left| \int_0^\infty y^{\alpha_1 - 2} e^{-(\gamma\zeta y^{-1}) - y} \, dy \right|$$

$$\times \big|\phi(\zeta)\big| \, d\zeta \, d\gamma.$$

By the fact that $e^{-(\gamma\zeta y^{-1}) - y} < e^{-y}$ we have

$$\int_{\mathsf{k}_1} \big|\mathbf{l}_{\alpha_1,\alpha_2}\big(\phi(\zeta)\big)(\gamma)\big| \, d\gamma \leq \frac{1}{2} \int_{\mathsf{k}_1} \gamma^{-\alpha_2 - \alpha_1 + 1} \int_0^\infty \zeta^{\alpha_2} \left| \int_0^\infty y^{\alpha_1 - 2} e^{-y} \, dy \right| \big|\phi(\zeta)\big| \, d\zeta \, d\gamma$$

$$< \frac{1}{2} \int_{\mathsf{k}_1} \gamma^{-\alpha_2 - \alpha_1 + 1} \int_0^\infty \zeta^{\alpha_2} \left(\int_0^\infty e^{-y} \, dy \right) \big|\phi(\zeta)\big| \, d\zeta \, d\gamma.$$

By the fact that $|y^{\alpha_1 - 2}| < 1$, for $\alpha_1 < 1$, we get

$$\int_{\mathsf{k}_1} \big|\mathbf{l}_{\alpha_1,\alpha_2}\big(\phi(\zeta)\big)(\gamma)\big| \, d\gamma \leq \frac{1}{2} \int_{\mathsf{k}_1} \gamma^{-\alpha_2 - \alpha_1 + 1} \int_0^\infty \zeta^{\alpha_2} \big|\phi(\zeta)\big| \, d\zeta \, d\gamma.$$

The last equation follows from the fact that $\int_0^\infty e^{-y} \, dy = 1$.

Now, let $\mathsf{k}_2 = [c,d]$, $0 < c < d$, be a compact set containing the support of ϕ, *i.e.*, supp $\phi \subseteq [c,d]$, and $\mathsf{k}_1 = [a,b]$, $0 < a < b$. Then we get

$$\int_{\mathsf{k}_1} \big|\mathbf{l}_{\alpha_1,\alpha_2}\big(\phi(\zeta)\big)(\gamma)\big| \, d\gamma < \frac{1}{2} \int_a^b \gamma^{-\alpha_2 - \alpha_1 + 1} \left(\int_c^d \zeta^{\alpha_2} \big|\phi(\zeta)\big| \, d\zeta \right) d\eta$$

$$< \frac{M}{2} \int_a^b \gamma^{-\alpha_2 - \alpha_1 + 1} \, d\gamma,$$

where $M = \int_c^d \zeta^{\alpha_2} |\phi(\zeta)| \, d\zeta$. Hence,

$$\int_{k_1} \left| \mathbf{l}_{\alpha_1,\alpha_2}(\phi(\zeta))(\gamma) \right| d\gamma < \frac{M}{2} \frac{(b^{-\alpha_2-\alpha_1+2} - b^{-\alpha_1-\alpha_1+2})}{-\alpha_2 - \alpha_1 + 2}$$
$$< \infty.$$

The theorem is therefore completely proved. □

3 Constructed spaces of Boehmians

Here we begin by the following spaces of Boehmians, named $\boldsymbol{\alpha}(F_{\text{loc}}, \Omega, *, *)$, and $\boldsymbol{\alpha}(F_{\text{loc}}, \Omega, *, \circledast)$, where Δ is a collection of delta sequence (δ_n) from Ω satisfying

$$\Delta_1 \colon \int_0^\infty \delta_n(x) \, dx = 1;$$

$$\Delta_2 \colon \int_0^\infty |\delta_n(x)| \, dx < M, \quad M \in \mathbb{R}, M > 0;$$

$$\Delta_3 \colon \operatorname{supp} \delta_n(x) \to 0 \quad \text{as } n \to \infty.$$

Let us now establish that $\boldsymbol{\alpha}(F_{\text{loc}}, \Omega, *, \circledast)$ is a Boehmian space. We prefer to omit the details of establishing $\boldsymbol{\alpha}(F_{\text{loc}}, \Omega, *, *)$ since the construction of the latter is straightforward from the properties of $*$.

We prove certain theorems that are needed for this construction.

Theorem 6 *Let $\phi \in F_{\text{loc}}(0, \infty)$ and $\varphi, \psi \in \Omega(0, \infty)$. Then $\phi \circledast (\varphi * \psi) = (\phi \circledast \varphi) \circledast \psi$.*

The proof of this theorem follows from Theorem 4. Details are thus omitted.

Theorem 7 *Let $\phi \in F_{\text{loc}}(0, \infty)$ and $\varphi \in \Omega(0, \infty)$. Then $\phi \circledast \varphi \in F_{\text{loc}}(0, \infty)$.*

Proof Let k_1 be a compact subset in $(0, \infty)$. Then we have

$$\int_{k_1} |(\phi \circledast \varphi)(\gamma)| \, d\gamma = \int_{k_1} \left| \int_0^\infty x^{\alpha_1 + 2\alpha_2 - 1} \phi(\gamma x) \varphi(x) \, dx \right| d\gamma$$

$$\leq \int_{k_1} \int_k \left| \int_0^\infty x^{\alpha_1 + 2\alpha_2 - 1} \phi(\gamma x) \right| |\varphi(x)| \, dx \, d\gamma$$

$$\text{(by Fubini's theorem)}$$

$$\leq \int_k |\varphi(x)| \int_{k_1} |x^{\alpha_1 + 2\alpha_2 - 1} \phi(\gamma x)| \, d\gamma \, dx,$$

where k is a compact set containing the $\operatorname{supp} \varphi$.

By Δ_1, we obtain

$$\int_{k_1} |(\phi \circledast \varphi)(\gamma)| \, d\gamma \leq M M_1,$$

where $M_1 = \int_{k_1} |x^{\alpha_1 + 2\alpha_2 - 1} \phi(\gamma x)| \, d\gamma$ and $M = \int_k |\varphi(x)| \, dx$.

Hence, the theorem is completely proved. □

Theorem 8 *Let* $\phi_1, \phi_2 \in F_{\text{loc}}(0, \infty)$ *and* $\psi_1, \psi_2 \in \Omega(0, \infty)$. *Then the following hold*:

(i) $(\phi_1 + \phi_2) \circledast \psi_1 = \phi_1 \circledast \psi_1 + \phi_2 \circledast \psi_1$.

(ii) *If* $\phi_n \to \phi$ *in* $F_{\text{loc}}(0, \infty)$ *and* $\psi \in \Omega(0, \infty)$, *then*

$$\phi_n \circledast \psi \to \phi \circledast \psi \quad \text{as } n \to \infty.$$

(iii) *If* $(\delta_n), (\psi_n) \in \Delta$ *then* $(\delta_n * \psi_n) \in \Delta$.

Proof The proof of parts (i) and (ii) follows from a simple integration. The proof of part (iii) follows from Roopkumar [23].

Hence, the theorem is proved. □

Finally in this construction, we prove the following theorem.

Theorem 9 *Let* $\phi \in F_{\text{loc}}^{\text{K}}(0, \infty)$ *and* $(\delta_n) \in \Delta$. *Then* $\phi \circledast \delta_n \to \phi$ *as* $n \to \infty$.

Proof Let k be a compact subset of $(0, \infty)$. Then, by Δ_1, we write

$$\int_k \left| (\phi \circledast \delta_n)(\gamma) - \phi(\gamma) \right| d\gamma = \int_k \left| \int_0^\infty x^{\alpha_1 + 2\alpha_2 - 1} \phi(\gamma x) \delta_n(x)\, dx - \phi(\gamma) \right| d\gamma$$

$$= \int_k \left| \int_0^\infty x^{\alpha_1 + 2\alpha_2 - 1} \phi(\gamma x) \delta_n(x)\, dx - \int_0^\infty \phi(\gamma) \delta_n(x)\, dx \right| d\gamma$$

$$\leq \int_k \int_0^\infty \left| x^{\alpha_1 + 2\alpha_2 - 1} \phi(\gamma x) - \phi(\gamma) \right| \left| \delta_n(x) \right| dx\, d\gamma.$$

Let $h(\gamma) = x^{\alpha_1 + 2\alpha_2 - 1} \phi(\gamma x) - \phi(\gamma)$. Then $h(\gamma)$ is uniformly continuous for each $\gamma \in (0, \infty)$. Hence

$$\int_k \left| (\phi \circledast \delta_n)(\gamma) - \phi(\gamma) \right| d\gamma \to 0 \quad \text{as } n \to \infty.$$

Therefore $\phi \circledast \delta_n \to \phi$ as $n \to \infty$.

This completes the proof of the theorem. □

The space $\boldsymbol{\alpha}(F_{\text{loc}}, \Omega, *, \circledast)$ is therefore constructed. Construction of the space $\boldsymbol{\alpha}(F_{\text{loc}}, \Omega, *, *)$ is almost similar.

We define the sum and multiplication by a scalar of two Boehmians in $\boldsymbol{\alpha}(F_{\text{loc}}, \Omega, *, \circledast)$ in the natural way by

$$\left[\frac{(\phi_n)}{(\delta_n)} \right] + \left[\frac{(g_n)}{(\varphi_n)} \right] = \left[\frac{(\phi_n) \circledast (\varphi_n) + (g_n) \circledast (\delta_n)}{(\delta_n) * (\varphi_n)} \right]$$

and $\rho[\frac{(\phi_n)}{(\delta_n)}] = [\frac{(\rho \phi_n)}{(\delta_n)}]$, ρ is a complex number.

Between F_{loc} and $\boldsymbol{\alpha}(F_{\text{loc}}, \Omega, *, \circledast)$ there is a canonical embedding expressed as

$$x \to \frac{x \circledast \delta_n}{\delta_n}.$$

The operation $\circledast$ can be extended to $\boldsymbol{\alpha}(F_{\text{loc}}, \Omega, *, \circledast) \times F_{\text{loc}}$ by

$$\frac{x_n}{\delta_n} \circledast t = \frac{x_n \circledast t}{\delta_n}.$$

In $\boldsymbol{\alpha}(F_{\text{loc}}, \Omega, *, \circledast)$, we have two types of convergence.

δ-convergence: A sequence (β_n) in $\boldsymbol{\alpha}(F_{\text{loc}}, \Omega, *, \circledast)$ is said to be δ-convergent to β in $\boldsymbol{\alpha}(F_{\text{loc}}, \Omega, *, \circledast)$ if there can be a delta sequence (δ_n) such that $(\beta_n \circledast \delta_n), (\beta \circledast \delta_n) \in F_{\text{loc}}$, $\forall k, n \in \mathbb{N}$, and $(\beta_n \circledast \delta_k) \to (\beta \circledast \delta_k)$ as $n \to \infty$, in F_{loc}, for every $k \in \mathbb{N}$.

Δ-convergence: A sequence (β_n) in $\boldsymbol{\alpha}(F_{\text{loc}}, \Omega, *, \circledast)$ is said to be Δ-convergent to β in $\boldsymbol{\alpha}(F_{\text{loc}}, \Omega, *, \circledast)$ if there can be a $(\delta_n) \in \Delta$ such that $(\beta_n - \beta) \circledast \delta_n \in F_{\text{loc}}, \forall n \in \mathbb{N}$, and $(\beta_n - \beta) \circledast \delta_n \to 0$ as $n \to \infty$ in F_{loc}.

4 Introduction of the generalized Meijer-type integral

Here we introduce a generalized Meijer-type integral. Then we show its properties in the space of Boehmians.

Definition 10 Let $[\frac{(\phi_n)}{(\delta_n)}] \in \boldsymbol{\alpha}(F_{\text{loc}}, \Omega, *, *)$. Then we define the generalized Meijer-type integral transform of $[\frac{(\phi_n)}{(\delta_n)}]$ as $\mathbf{l}_{\alpha_1,\alpha_2}^{ge}[\frac{(\phi_n)}{(\delta_n)}] = [\frac{\mathbf{l}_{\alpha_1,\alpha_2}(\phi_n)}{(\delta_n)}]$ in $\boldsymbol{\alpha}(F_{\text{loc}}^{\text{K}}, \Omega, *, \circledast)$.

Theorem 5 shows that Definition 10 is well defined.

We discuss now some properties of $\mathbf{l}_{\alpha_1,\alpha_2}^{ge}$.

Theorem 11 *Let $\beta_1, \beta_2 \in \boldsymbol{\alpha}(F_{\text{loc}}, \Omega, *, *)$. Then $\mathbf{l}_{\alpha_1,\alpha_2}^{ge}(\beta_1 * \beta_2) = \mathbf{l}_{\alpha_1,\alpha_2}^{ge} \beta_1 \circledast \beta_2$ in $\boldsymbol{\alpha}(F_{\text{loc}}, \Omega, *, \circledast)$.*

Proof Assume the hypothesis of the theorem is satisfied for some $\beta_1, \beta_2 \in \boldsymbol{\alpha}(F_{\text{loc}}, \Omega, *, *)$. Therefore there are $(\phi_n), (\kappa_n) \in F_{\text{loc}}(0, \infty)$ and $(\varphi_n), (\delta_n) \in \Delta$ such that $\beta_1 = [\frac{(\phi_n)}{(\varphi_n)}]$ and $\beta_2 = [\frac{(\kappa_n)}{(\delta_n)}]$. Therefore,

$$\mathbf{l}_{\alpha_1,\alpha_2}^{ge}(\beta_1 * \beta_2) = \mathbf{l}_{\alpha_1,\alpha_2}^{ge}\left(\left[\frac{(\phi_n) * (\kappa_n)}{(\varphi_n) * (\delta_n)}\right]\right).$$

Definition 10 gives

$$\mathbf{l}_{\alpha_1,\alpha_2}^{ge}(\beta_1 * \beta_2) = \left[\frac{\mathbf{l}_{\alpha_1,\alpha_2}((\phi_n) * (\kappa_n))}{(\varphi_n) * (\delta_n)}\right].$$

By the aid of Theorem 3 we obtain

$$\mathbf{l}_{\alpha_1,\alpha_2}^{ge}(\beta_1 * \beta_2) = \left[\frac{(\mathbf{l}_{\alpha_1,\alpha_2}\phi_n) \circledast (\kappa_n)}{(\varphi_n) \circledast (\delta_n)}\right].$$

On separating quotients this yields

$$\mathbf{l}_{\alpha_1,\alpha_2}^{ge}(\beta_1 * \beta_2) = \left[\frac{(\mathbf{l}_{\alpha_1,\alpha_2}\phi_n)}{(\varphi_n)}\right] \circledast \left[\frac{(\kappa_n)}{(\delta_n)}\right].$$

Hence, $\mathbf{l}_{\alpha_1,\alpha_2}^{ge}(\beta_1 * \beta_2) = \mathbf{l}_{\alpha_1,\alpha_2}^{ge}(\beta_1) \circledast \beta_2$.

This completes the proof of the theorem. □

Theorem 12 $\mathbf{l}_{\alpha_1,\alpha_2}^{ge}$ *defines a linear mapping from* $\boldsymbol{\alpha}(F_{\mathrm{loc}}, \Omega, *, *)$ *into* $\boldsymbol{\alpha}(F_{\mathrm{loc}}, \Omega, *, \circledast)$.

The proof is straightforward. Details are therefore omitted.

Theorem 13 *Let* $[\frac{(\phi_n)}{(\delta_n)}] \in \boldsymbol{\alpha}(F_{\mathrm{loc}}, \Omega, *, *)$ *and* $\delta \in \Omega(0, \infty)$. *Then we have*

$$\mathbf{l}_{\alpha_1,\alpha_2}^{ge}\left(\left[\frac{(\phi_n)}{(\delta_n)}\right] * \delta\right) = \left[\frac{(\mathbf{l}_{\alpha_1,\alpha_2}\phi_n)}{(\delta_n)}\right] \circledast \delta.$$

Proof Let $[\frac{(\phi_n)}{(\delta_n)}] \in \boldsymbol{\alpha}(F_{\mathrm{loc}}, \Omega, *, *)$ and $\delta \in \Omega(0, \infty)$. By virtue of Definition 10 we write

$$\mathbf{l}_{\alpha_1,\alpha_2}^{ge}\left(\left[\frac{(\phi_n)}{(\delta_n)}\right] * \delta\right) = \left[\frac{\mathbf{l}_{\alpha_1,\alpha_2}((\phi_n) * \delta)}{(\delta_n)}\right].$$

Once again, Definition 10 and Theorem 3 give

$$\mathbf{l}_{\alpha_1,\alpha_2}^{ge}\left(\left[\frac{(\phi_n)}{(\delta_n)}\right] * \delta\right) = \left[\frac{(\mathbf{l}_{\alpha_1,\alpha_2}\phi_n) \circledast \delta}{(\delta_n)}\right] = \left[\frac{(\mathbf{l}_{\alpha_1,\alpha_2}\phi_n)}{(\delta_n)}\right] \circledast \delta.$$

This completes the proof of the theorem. □

Theorem 14 *The transform* $\mathbf{l}_{\alpha_1,\alpha_2}^{ge}$ *is consistent with* $\mathbf{l}_{\alpha_1,\alpha_2}^{ge} : F_{\mathrm{loc}}(0, \infty) \to F_{\mathrm{loc}}(0, \infty)$.

Proof For every $\phi \in F_{\mathrm{loc}}(0, \infty)$, let $\beta \in \boldsymbol{\alpha}(F_{\mathrm{loc}}, \Omega, *, *)$ be the representative in the space $\boldsymbol{\alpha}(F_{\mathrm{loc}}, \Omega, *, *)$, then $\forall n \in \mathbb{N}$, $(\delta_n) \in \Delta$, $\beta = [\frac{\phi*(\delta_n)}{(\delta_n)}]$. For all $n \in \mathbb{N}$ it is clear that (δ_n) is independent from the choice of the representative.

We have

$$\mathbf{l}_{\alpha_1,\alpha_2}^{ge}(\beta) = \mathbf{l}_{\alpha_1,\alpha_2}^{ge}\left(\left[\frac{\phi * (\delta_n)}{(\delta_n)}\right]\right) = \left[\frac{\mathbf{l}_{\alpha_1,\alpha_2}(\phi * (\delta_n))}{(\delta_n)}\right] = \left[\frac{\mathbf{l}_{\alpha_1,\alpha_2}\phi * (\delta_n)}{(\delta_n)}\right],$$

which is the representative of $\mathbf{l}_{\alpha_1,\alpha_2}^{ge}\phi$ in the space $F_{\mathrm{loc}}(0, \infty)$.

Hence the proof is completed. □

Theorem 15 *The necessary and sufficient condition for* $[\frac{(g_n)}{(\psi_n)}] \in \boldsymbol{\alpha}(F_{\mathrm{loc}}, \Omega, *, \circledast)$ *to be in the range of* $\mathbf{l}_{\alpha_1,\alpha_2}^{ge}$ *is that* g_n *belongs to the range of* $\mathbf{l}_{\alpha_1,\alpha_2}$ *for every* $n \in \mathbb{N}$.

Proof Let $[\frac{(g_n)}{(\psi_n)}]$ be in the range of $\mathbf{l}_{\alpha_1,\alpha_2}^{ge}$. Then of course g_n belongs to the range of $\mathbf{l}_{\alpha_1,\alpha_2}$, $\forall n \in \mathbb{N}$. To establish the converse, let g_n be in the range of $\mathbf{l}_{\alpha_1,\alpha_2}$, $\forall n \in \mathbb{N}$. Then there is $\phi_n \in F_{\mathrm{loc}}(0, \infty)$ such that $\mathbf{l}_{\alpha_1,\alpha_2}\phi_n = g_n$, $n \in \mathbb{N}$.

Since $[\frac{(g_n)}{(\psi_n)}] \in \boldsymbol{\alpha}(F_{\mathrm{loc}}, \Omega, *, \circledast)$ we get $g_n \circledast \psi_m = g_m \circledast \psi_n$, $\forall m, n \in \mathbb{N}$. Therefore, Theorem 3 yields

$$\mathbf{l}_{\alpha_1,\alpha_2}(\phi_n * \delta_m) = \mathbf{l}_{\alpha_1,\alpha_2}(\phi_m * \delta_n), \quad \forall m, n \in \mathbb{N},$$

where $\phi_n \in F_{\mathrm{loc}}(0, \infty)$ and $(\delta_n) \in \Delta$, $\forall n \in \mathbb{N}$. Thus $\phi_n * \delta_m = \phi_m * \delta_n$, $m, \forall n \in \mathbb{N}$. Hence,

$$\left[\frac{(\phi_n)}{(\delta_n)}\right] \in \boldsymbol{\alpha}(F_{\mathrm{loc}}, \Omega, *, *) \quad \text{and} \quad \mathbf{l}_{\alpha_1,\alpha_2}^{ge}\left(\left[\frac{(\phi_n)}{(\delta_n)}\right]\right) = \left[\frac{(g_n)}{(\psi_n)}\right].$$

The theorem is therefore completely proved. □

Theorem 16 *The mappings $\mathbf{l}^{ge}_{\alpha_1,\alpha_2}$ are continuous with respect to δ and Δ-convergence.*

The proof of this theorem is given by various papers of many authors. A detailed proof is therefore omitted.

5 Concluding remark

The present study defined the space of Boehmians for some Meijer type integral transform and desired some properties in the class of Boehmians. The results given in this paper are seen to generalize those given by Roopkumar in [15, 23].

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