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Semicontinuity and closedness of parametric generalized lexicographic quasiequilibrium problems

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Abstract
This paper is mainly concerned with the upper semicontinuity, closedness, and the lower semicontinuity of the set-valued solution mapping for a parametric lexicographic equilibrium problem where both two constraint maps and the objective bifunction depend on both the decision variable and the parameters. The sufficient conditions for the upper semicontinuity, closedness, and the lower semicontinuity of the solution map are established. Many examples are provided to ensure the essentialness of the imposed assumptions.

Keywords: parametric generalized lexicographic quasiequilibrium problem; upper semicontinuity; closedness and lower semicontinuity

1 Introduction

Equilibrium problems first considered by Blum and Oettli [1] have been playing an important role in optimization theory with many striking applications particularly in transportation, mechanics, economics, *etc.* Equilibrium models incorporate many other important problems such as: optimization problems, variational inequalities, complementarity problems, saddlepoint/minimax problems, and fixed points. Equilibrium problems with scalar and vector objective functions have been widely studied. The crucial issue of solvability (the existence of solutions) has attracted most considerable attention of researchers; see, *e.g.*, [2–5].

With regard to vector equilibrium problems, most of the existing results correspond to the case when the order is induced by a closed convex cone in a vector space. Thus, they cannot be applied to lexicographic cones, which are neither closed nor open. These cones have been extensively investigated in the framework of vector optimization; see, *e.g.*, [6–13]. For instance, Konnov and Ali [12] studied sequential problems, especially exploiting its relation with regularization methods. Bianchi *et al.* in [7] analyzed lexicographic equilibrium problems on a topological Hausdorff vector space, and their relationship with some other vector equilibrium problems. They obtained the existence results for the tangled lexicographic problem via the study of a related sequential problem.

As a unified model of vector optimization problems, vector variational inequality problems, variational inclusion problems and vector complementarity problems, vector equilibrium problems have been intensively studied. The stability analysis of the solution mapping for these problems is an important topic in vector optimization theory. Recently, a great deal of research has been devoted to the semicontinuity of the solution mapping for a parametric vector equilibrium problem. Based on the assumption of the (strong) C-inclusion property of a function, Anh and Khanh [14] obtained the upper and lower semicontinuity of the solution set map of parametric multivalued (strong) vector quasiequilibrium problems. Anh and Khanh [15] obtained the semicontinuity of a class of parametric quasiequilibrium problems by a generalized concavity assumption and a closedness of the level set of functions. Wangkeeree *et al.* [16] established the continuity of the efficient solution mappings to a parametric generalized strong vector equilibrium problem involving a set-valued mapping under the Holder relation assumption. Recently, Wangkeeree *et al.* [17] obtained the sufficient conditions for the lower semicontinuity of an approximate solution mapping for a parametric generalized vector equilibrium problem involving set-valued mappings. By using a scalarization method, they obtained the lower semicontinuity of an approximate solution mapping for such a problem without the assumptions of monotonicity and compactness. For other qualitative stability results on parametric generalized vector equilibrium problems, see [14–20] and the references therein.

It is well known that partial order plays an important role in vector optimization theory. The vector optimization problems in the previous references are studied in the partial order induced by a closed or open cone. But in some situations, the cone is neither open nor closed, such as the lexicographic cone. On the other hand, since the lexicographic order induced by the lexicographic cone is a total order, it can refine the optimal solution points to make it smaller in the theory of vector optimization. Thus, it is valuable to investigate the vector optimization problems in the lexicographic order. To the best of our knowledge, the first lower stability results of the solution set map based on the density of the solution set mapping for a parametric lexicographic vector equilibrium problem have been established by Shi-miao *et al.* [21]. Recently, Anh *et al.* [22] established the sufficient conditions for the upper semicontinuity, closedness, and continuity of the solution maps for a parametric lexicographic equilibrium problem. However, to the best of our knowledge, there is no work to study the stability analysis for a parametric lexicographic equilibrium problem where both two constraint maps and the objective bifunction depend on both the decision variable and the parameters. We observe that quasiequilibrium models are the important general models including as special cases quasivariational inequalities, complementarity problems, vector minimization problems, Nash equilibria, fixed-point and coincidence-point problems, traffic networks, *etc.* A quasioptimization problem is more general than an optimization one as constraint sets depend on the decision variable as well.

Motivated by the mentioned works, this paper is devoted to the study of closedness upper and lower of the solution map for a parametric lexicographic equilibrium problem where both two constraint maps and the objective bifunction depend on both the decision variable and the parameters. The sufficient conditions for the upper semicontinuity, closedness, and the lower semicontinuity of the solution map are established. Many examples are provided to ensure the essentialness of the imposed assumptions.

The paper is organized as follows. In Section 2, we first introduce the parametric lexicographic equilibrium problem where both two constraint maps and the objective bifunction depend on both the decision variable and the parameters, and we recall some basic definitions on semicontinuity of set-valued maps. Section 3 establishes the sufficient conditions for the upper semicontinuity and closedness of the solution map. Many examples are provided to ensure the essentialness of the imposed assumptions. Section 4 establishes the sufficient conditions for the lower semicontinuity of the solution map. Furthermore, we give also many examples ensuring the essentialness of the imposed assumptions.

2 Preliminaries

Throughout this paper, if not otherwise specified, let X and Λ be Hausdorff topological vector spaces. Let $A \subseteq X$ be nonempty. Let $K_1, K_2 : A \times \Lambda \to 2^X$ be two multivalued constraint maps and $f := (f_1, f_2, \dots, f_n) : A \times A \times \Lambda \to \mathbb{R}^n$ a vector-valued function where, for each $i \in I_n := \{1, 2, \dots, n\}$, $f_i : A \times A \times \Lambda \to \mathbb{R}$ is a real valued function. We assume that, for every $x \in X$ and $i \in I_n$, $f_i(x, x, \lambda) = 0$, *i.e.*, f_i is an equilibrium function. Set $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_- = -\mathbb{R}_+$ and $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. For a subset A of X, $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{bd} A$ stand for the interior, closure, and boundary of A, respectively. For any given $\alpha \in \mathbb{R}$, *the upper α-level set* and *the lower α-level set* of the function $f : X \to \bar{\mathbb{R}}$ are denoted, respectively, by

$$\operatorname{lev}_{\geq\alpha} f := \{x \in X | f(x) \geq \alpha\}$$

and

$$\operatorname{lev}_{\leq\alpha} f := \{x \in X | f(x) \leq \alpha\}.$$

Recall that the lexicographic cone of $\mathbb{R}^n$, denoted by C_L, is defined as

$$C_L := \{0\} \cup \{x \in \mathbb{R}^n | \exists i \in I_n : x_i > 0, \forall j < i, x_j = 0\}.$$

We observe that it is neither closed nor open. Indeed, when comparing with the cone $C_1 := \{x \in \mathbb{R}^n | x_1 \geq 0\}$, we have

$$\operatorname{int} C_1 \subsetneq C_L \subsetneq C_1, \qquad \operatorname{int} C_L = \operatorname{int} C_1 \quad \text{and} \quad \operatorname{cl} C_L = C_1.$$

However, it is worth noticing that the lexicographic cone is convex, pointed, and total ('total' means that $C_L \cup (-C_L) = \mathbb{R}^n$). The lexicographic order, $\geq_L$, in C_L is defined by

$$x \geq_L y \quad \Longleftrightarrow \quad x - y \in C_L.$$

This is a total (called also linear) order, *i.e.*, any pair of elements is comparable. In [23], it was shown that, for a fixed orthogonal base, the lexicographic order is the unique total order. We will see later that this causes difficulties in studies of many topics related to ordering cones.

Next, we shall introduce and study a problem where both the two constraint maps and the bifunction depend on parameters. For a given $\lambda \in \Lambda$, *the parametric generalized lexicographic quasiequilibrium problem*, denoted by GLQEP_λ, is

$$(\mathrm{GLQEP}_\lambda) \begin{cases} \text{finding } \bar{x} \in K_1(\bar{x}, \lambda) \text{ such that, for all } y \in K_2(\bar{x}, \lambda), \\ f(\bar{x}, y, \lambda) \geq_L 0. \end{cases}$$

Remark 2.1 When $K_1 = K_2 := K : \Lambda \to 2^X$, the (GLQEP$_\lambda$) collapses to the lexicographic vector quasiequilibrium problem (LEP$_\lambda$): for each $\lambda \in \Lambda$,

$$(\mathrm{LEP}_\lambda)\begin{cases}\text{finding } \bar{x} \in K(\lambda) \text{ such that}\\ f(\bar{x}, y, \lambda) \geq_L 0, \forall y \in K(\lambda).\end{cases}$$

The stability analysis of the set-valued solution mapping for (LEP$_\lambda$) are studied in Anh *et al.* [6] and Shi-miao *et al.* [21].

Let the set-valued mappings $E : \Lambda \to 2^X$ and $S_{f_1} : \Lambda \to 2^X$ be defined by

$$E(\lambda) = \{x \in A : x \in K_1(x, \lambda)\}$$

and

$$S_{f_1}(\lambda) = \{x \in E(\lambda) : f_1(x, y, \lambda) \geq 0, \forall y \in K_2(x, \lambda)\}.$$

Furthermore, let a mapping $Z : S_{f_1}(\lambda) \times \Lambda \to 2^X$ be given by

$$Z(x, \lambda) := \{y \in K_2(x, \lambda) \mid f_1(x, y, \lambda) = 0\}.$$

For the sake of simplicity, we consider the case $n = 2$, since the general case is similar. Then GLQEP$_\lambda$ collapses to: find $\bar{x} \in K_1(\bar{x}, \lambda)$ such that

$$\begin{cases} f_1(\bar{x}, y, \lambda) \geq 0, & \forall y \in K_2(\bar{x}, \lambda),\\ f_2(\bar{x}, z, \lambda) \geq 0, & \forall z \in Z(\bar{x}, \lambda).\end{cases}$$

Thus, GLQEP$_\lambda$ can be rewritten as

$$\text{find } \bar{x} \in S_{f_1}(\lambda) \text{ such that } f_2(\bar{x}, y, \lambda) \geq 0, \text{ for all } y \in Z(\bar{x}, \lambda). \tag{2.1}$$

The solution mapping for GLQEP$_\lambda$ is denoted by S_f. We denote the whole family of problems, say of GLQEP$_\lambda$, for $\lambda \in \Lambda$, by $(\mathrm{GLQEP}_\lambda)_{\lambda \in \Lambda}$. We first observe some basic facts about lexicographic equilibrium problems. The lexicographic cone C_L contains clearly all pointed closed and convex cones C included in the closed half space $\{x \in \mathbb{R}^n : x_1 \geq 0\}$. Then, for an ordering cone C, we consider some kinds of parametric equilibrium problems: *the parametric generalized quasiequilibrium problem* [23], denoted by GQEP$_\lambda$, is

$$(\mathrm{GQEP}_\lambda)\begin{cases}\text{finding } \bar{x} \in K_1(\bar{x}, \lambda) \text{ such that, for all } y \in K_2(\bar{x}, \lambda),\\ f(\bar{x}, y, \lambda) \in C.\end{cases}$$

The solution mapping for GQEP$_\lambda$ is denoted by S_{GQEP}. Therefore, for any pointed closed and convex cones C included in the closed half space $\{x \in \mathbb{R}^n : x_1 \geq 0\}$, we can get the following fact: $S_{GQEP} \subseteq C_L$. Hence, the existence results of solutions for *GLQEP* can be obtained by the nonemptiness of S_{GQEP}. Next, we need to recall some well-known definitions.

Definition 2.2 [24] Let $\{A_n\}$ be a sequence of subsets of X. Then

(i) the *upper limit* or *outer limit* of the sequence $\{A_n\}$ is a subset of X given by

$$\limsup_{n\to\infty} A_n = \Bigl\{x \in X \Bigm| \liminf_{n\to\infty} \operatorname{dist}(x, A_n) = 0\Bigr\};$$

(ii) the *lower limit* or *inner limit* of the sequence $\{A_n\}$ is a subset of X given by

$$\liminf_{n\to\infty} A_n = \Bigl\{x \in X \Bigm| \lim_{n\to\infty} \operatorname{dist}(x, A_n) = 0\Bigr\};$$

(iii) if $\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$, then we say that the *limit of* $\{A_n\}$ exist and

$$\lim_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n.$$

Consequently, we have the following result.

Proposition 2.3 *Let $\{A_n\}$ be a sequence of subsets of X. Then*

(i) $\limsup_{n\to\infty} A_n = \{x \in A | x_{n_k} \in A_{n_k} : x_{n_k} \to x\}$;

(ii) $\liminf_{n\to\infty} A_n = \{x \in A | x_n \in A_n : x_n \to x\}$.

Definition 2.4 [25] Let X and Y be Hausdorff topological vector spaces and $S : X \to 2^Y$ a given set-valued map.

(i) S is said to be *upper semicontinuous* (usc, for short) at $x_0 \in X$ iff for any open set $V \subset Y$, where $S(x_0) \subset V$, there exists a neighborhood $U \subset X$ of x_0 such that

$$S(x) \subset V, \quad \forall x \in U.$$

The map $S(\cdot)$ is said to be u.s.c. on X if it is u.s.c. at every $x \in X$.

(ii) S is said to be *lower semicontinuous* (lsc, for short) at $x_0 \in X$ iff for any open set $V \subset Y$ such that $S(x_0) \cap V \neq \emptyset$, there exists a neighborhood $U \subset X$ of x_0 such that

$$S(x) \cap V \neq \emptyset, \quad \forall x \in U.$$

The map $S(\cdot)$ is said to be l.s.c. on X if $S(\cdot)$ is l.s.c. at every $x \in X$.

(iii) S is said to be *closed* at x_0 if from (x_n, y_n) in the graph $\operatorname{gr} S := \{(x, y) \in X \times Y \mid y \in S(x)\}$ of S and tends to (x_0, y_0) it follows that $(x_0, y_0) \in \operatorname{gr} S$.

We will often use the well-known fact: if $S(x)$ is compact, then S is usc at x if and only if for any sequence $\{x_n\}$ in X converging to x and $y_n \in Q(x_n)$, there is a subsequence of $\{y_n\}$ converging to a point $y \in Q(x)$. Next we give equivalent forms of the lower semicontinuity of S.

For a set-valued map $Q : X \to 2^Y$ between two linear spaces, Q is called *concave* [15] on a convex subset $A \subseteq X$ if, for each $x_1, x_2 \in A$ and $t \in [0, 1]$,

$$Q\bigl((1-t)x_1 + tx_2\bigr) \subseteq tQ(x_1) + (1-t)Q(x_2).$$

Lemma 2.5 *Let $S : X \to 2^Y$ be a given set-valued map. The following are equivalent*:

(i) *S is lsc at x_0*;

(ii) *if $\{x_n\}$ is any sequence such that $x_n \to x_0$ and $V \subset Y$ an open subset such that $S(x_0) \cap V \neq \emptyset$, then*

$$\exists N \geq 1 : S(x_n) \cap V \neq \emptyset, \quad \forall n \geq N;$$

(iii) *if $\{x_n\}$ is a sequence such that $x_n \to x_0$ and $y_0 \in S(x_0)$ arbitrary, then there is a sequence $\{y_n\}$ with $y_n \in S(x_n)$ such that $y_n \to y_0$ as $n \to \infty$.*

From Proposition 2.3 and Lemma 2.5 we can obtain the following lemma immediately.

Lemma 2.6 *Let $S : X \to 2^Y$ be a given set-valued map. Then S is lsc at x_0 iff for any sequence $\{x_n\} \subseteq X$ converging to x_0,*

$$S(x_0) \subset \liminf_{n\to\infty} S(x_n).$$

The following relaxed continuity properties are also needed and can be found in [26].

Definition 2.7 ([26]) Let X be a topological space and $g : X \to \bar{\mathbb{R}}$ be a function on X.

(i) g is said to be (*sequentially*) *upper pseudocontinuous* at $x_0 \in X$ if for any sequence $\{x_n\}$ in X converging to x_0 and for each $x \in X$ such that $g(x) > g(x_0)$,

$$g(x) > \limsup_{n\to\infty} g(x_n).$$

(ii) g is called (*sequentially*) *lower pseudocontinuous* at $x_0 \in X$ if for any sequence $\{x_n\}$ in X converging to x_0 and for each $x \in X$ such that $g(x) < g(x_0)$,

$$g(x) < \liminf_{n\to\infty} g(x_n).$$

(iii) g is *pseudocontinuous* at $x_0 \in X$ if it is both lower and upper pseudocontinuous at this point.

The class of the pseudocontinuous functions strictly contains that of the semicontinuous functions as shown by the following.

Example 2.8 The function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -2, & \text{if } x < 0, \end{cases}$$

is pseudocontinuous, but neither upper nor lower semicontinuous at 0.

Lemma 2.9 ([16]) *Let X be a topological space. Then $g : X \to \overline{\mathbb{R}}$ is pseudocontinuous in X if and only if, for all sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$*

and $g(y) < g(x)$,

$$\limsup_{n\to\infty} g(y_n) < \liminf_{n\to\infty} g(x_n).$$

The following important definition can be found in [15].

Definition 2.10 Let $g : X \times Z \to \mathbb{R}$ and $\Delta \subset \mathbb{R}$, where $\operatorname{int}\Delta \neq \emptyset$. g is called generalized Δ-concave in a convex set $A \subset Z$, if for each $x \in X$ and $z_1, z_2 \in A$ satisfying $g(x, z_1) \in \Delta$ and $g(x, z_2) \in \operatorname{int}\Delta$, it follows that, for all $t \in (0, 1)$,

$$g\big(x, (1-t)z_1 + tz_2\big) \in \operatorname{int}\Delta.$$

3 The upper semicontinuouity and closedness of S_f

In this section, we discuss the upper semicontinuity and closedness of the solution mapping S_f. Since there have been a number of contributions to existence issues, focusing on stability we always assume that $S_{f_1}(\lambda)$ and $S_f(\lambda)$ are nonempty for all λ in a neighborhood of the considered point $\bar{\lambda}$. First of all, we shall establish the upper semicontinuity and closedness of the solution mapping S_{f_1}.

Lemma 3.1 *For* $(\mathrm{GLQEP}_\lambda)_{\lambda\in\Lambda}$ *assume that*

(i) *E is usc at $\bar{\lambda}$ and $E(\bar{\lambda})$ is compact*;
(ii) *K_2 is lsc in $K_1(A, \Lambda) \times \{\bar{\lambda}\}$*;
(iii) *$\mathrm{lev}_{\geq 0} f_i(\cdot, \cdot, \bar{\lambda})$ is closed in $K_1(A, \Lambda) \times K_2(A, \Lambda) \times \{\bar{\lambda}\}$ for $i = 1, 2$.*

Then the solution map S_{f_1} is both usc and closed at $\bar{\lambda}$.

Proof We first prove that the solution map S_{f_1} is usc at $\bar{\lambda}$. Suppose on the contrary that there exists an open set $U \supseteq S_{f_1}(\bar{\lambda})$ such that for any neighborhood $N(\bar{\lambda})$ of $\bar{\lambda}$, there exists $\lambda \in N(\bar{\lambda})$ such that $S_{f_1}(\lambda) \nsubseteq U$. In particular, for each $n \in \mathbb{N}$, there exist sequences $\{\lambda_n\} \subseteq \Lambda$ converging to $\bar{\lambda}$ and $\{x_n\} \subseteq S_{f_1}(\lambda_n) \subseteq E(\lambda_n)$ with $x_n \notin U$. By the upper semicontinuity of E and the compactness of $E(\bar{\lambda})$, one can assume that $x_n \to x_0$, for some $x_0 \in E(\bar{\lambda})$. Next, we claim that $x_0 \in S_{f_1}(\bar{\lambda})$. Again suppose on the contrary that there exists $y_0 \in K_2(x_0, \bar{\lambda})$ such that $f_1(x_0, y_0, \bar{\lambda}) < 0$. The lower semicontinuity of K_2 at $(x_0, \bar{\lambda})$, by Lemma 2.5, implies that there exists a sequence $\{y_n\}$ in $K_2(x_n, \lambda_n)$ such that $y_n \to y_0$ as $n \to \infty$. For each $n \in \mathbb{N}$, since $x_n \in S_{f_1}(\lambda_n)$, we have

$$f_1(x_n, y_n, \lambda_n) \geq 0.$$

It follows from the closedness of $\mathrm{lev}_{\geq 0} f_i(\cdot, \cdot, \bar{\lambda})$ that $f_1(x_0, y_0, \bar{\lambda}) \geq 0$, which leads to a contradiction. Therefore, $x_0 \in S_{f_1}(\bar{\lambda}) \subseteq U$, again a contradiction, since $x_n \notin U$ for all n. Thus, S_{f_1} is usc at $\bar{\lambda}$.

Next, we prove that S_{f_1} is closed at $\bar{\lambda}$. We suppose on the contrary that S_{f_1} is not closed at $\bar{\lambda}$, *i.e.*, there a sequence $\{\lambda_n\}$ converging to $\bar{\lambda}$ and $\{x_n\} \subseteq S_{f_1}(\lambda_n)$ with $x_n \to x_0$ but $x_0 \notin S_{f_1}(\bar{\lambda})$. The same argument as above ensures that $x_0 \in S_{f_1}(\bar{\lambda})$, which gives a contradiction. Therefore, we can conclude that S_{f_1} is closed at $\bar{\lambda}$. □

Now, we are in the position to discuss the upper semicontinuity and closedness of the solution mapping S_f.

Theorem 3.2 *For* $(\mathrm{GLQEP}_\lambda)_{\lambda\in\Lambda}$ *assume that*

(i) *E is usc at $\bar{\lambda}$ and $E(\bar{\lambda})$ is compact*;

(ii) *K_2 is lsc in $K_1(A,\Lambda)\times\{\bar{\lambda}\}$*;

(iii) *$\mathrm{lev}_{\geq 0}f_i(\cdot,\cdot,\bar{\lambda})$ is closed in $K_1(A,\Lambda)\times K_2(A,\Lambda)\times\{\bar{\lambda}\}$ for $i=1,2$*;

(iv) *Z is lsc in $S_{f_1}(\bar{\lambda})\times\{\bar{\lambda}\}$*.

Then the solution map S_f is both usc and closed at $\bar{\lambda}$.

Proof We first claim that the solution map S_f is usc at $\bar{\lambda}$. Suppose there exist an open set $U\supseteq S_f(\bar{\lambda})$, $\{\lambda_n\}\to\bar{\lambda}$, and $\{x_n\}\subseteq S_f(\lambda_n)$ such that $x_n\notin U$ for all n. By the upper semicontinuity of S_{f_1} at $\bar{\lambda}$ and the compactness of $S_{f_1}(\bar{\lambda})$, without loss of generality we can assume that $x_n\to x_0$ as $n\to\infty$ for some $x_0\in S_{f_1}(\bar{\lambda})$. If $x_0\notin S_f(\bar{\lambda})$, there exists $y_0\in Z(x_0,\bar{\lambda})$ such that $f_2(x_0,y_0,\bar{\lambda})<0$. The lower semicontinuity of Z in turn yields $y_n\in Z(x_n,\lambda_n)$ tending to y_0. Notice that for each $n\in\mathbb{N}$, $f_2(x_n,y_n,\lambda_n)\geq 0$. This together with the closedness of $\mathrm{lev}_{\geq 0}f_2(\cdot,\cdot,\bar{\lambda})$ in $K_1(A,\Lambda)\times K_2(A,\Lambda)\times\{\bar{\lambda}\}$ implies that $f_2(x_0,y_0,\bar{\lambda})\geq 0$, which gives a contradiction. If $x_0\in S_f(\bar{\lambda})\subseteq U$, one has another contradiction, since $x_n\notin U$ for all n. Thus, S_f is usc at $\bar{\lambda}$.

Now we prove that S_f is closed at $\bar{\lambda}$. Suppose on the contrary that there exists a sequence $\{(\lambda_n,x_n)\}$ converging to $(\bar{\lambda},x_0)$ with $x_n\in S_f(\lambda_n)$ but $x_0\notin S_f(\bar{\lambda})$. Then $f_2(x_0,y_0,\bar{\lambda})<0$ for some $y_0\in Z(x_0,\bar{\lambda})$. Due to the lower semicontinuity of Z, there is $y_n\in Z(x_n,\lambda_n)$ such that $y_n\to y_0$. Since $x_n\in S_f(\lambda_n)$, $f_2(x_n,y_n,\lambda_n)\geq 0$. By the closedness of the set $\mathrm{lev}_{\geq 0}f_2$, $f_2(x_0,y_0,\bar{\lambda})\geq 0$, which is impossible since $f_2(x_0,y_0,\bar{\lambda})<0$. Therefore, S_f is closed at $\bar{\lambda}$. □

Corollary 3.3 *For GLQEP, suppose that the conditions* (i), (ii), *and* (iv) *given in Theorem* 3.2 *are satisfied. Further, for each $i=1,2$, assume that f_i is upper pseudocontinuous in $K_1(A,\Lambda)\times K_2(A,\Lambda)\times\{\bar{\lambda}\}$. Then the solution map S_f is both usc and closed at $\bar{\lambda}$.*

Proof It is suffice to derive the condition (iii) that given in Theorem 3.2. For $i=1,2$, suppose $\{(x_n,y_n,\lambda_n)\}$ is any sequence in $\mathrm{lev}_{\geq 0}f_i(\cdot,\cdot,\bar{\lambda})$ such that $(x_n,y_n,\lambda_n)\to(\bar{x},\bar{y},\bar{\lambda})$ as $n\to\infty$. Assume that on the contrary that $(\bar{x},\bar{y},\bar{\lambda})\notin\mathrm{lev}_{\geq 0}f_i(\cdot,\cdot,\bar{\lambda})$, which implies that $f_i(\bar{x},\bar{y},\bar{\lambda})<0=f_i(\bar{x},\bar{x},\bar{\lambda})$. The upper pseudocontinuity of f_i at $(\bar{x},\bar{y},\bar{\lambda})$ implies that

$$0=f_i(\bar{x},\bar{x},\bar{\lambda})>\limsup_{n\to\infty}f_i(x_n,y_n,\lambda_n)\geq 0,$$

which gives a contradiction. Hence, we can conclude that $(\bar{x},\bar{y},\bar{\lambda})\in\mathrm{lev}_{\geq 0}f_i(\cdot,\cdot,\bar{\lambda})$. Now, the closedness of $\mathrm{lev}_{\geq 0}f_i(\cdot,\cdot,\bar{\lambda})$ is proved. Applying Theorem 3.2, we get the desired result. □

The following examples show that all assumptions imposed in Theorem 3.2 are very essential and cannot be relaxed.

Example 3.4 (The upper semicontinuity and compactness in (i) are crucial) Let $A=X=\mathbb{R}$, $\Lambda=[0,1]$, $\bar{\lambda}=0$. Define the mappings K_1, K_2, and f by

$$K_1(x,\lambda)=(\lambda,1+\lambda]\quad\text{and}\quad K_2(x,\lambda)=(0,1]$$

and

$$f(x,y,\lambda)=\bigl(x(x-y)^2(1+\lambda),2^{\lambda+xy}x(x-y)\bigr).$$

Then we have

$$E(0) = (0,1] \quad \text{and} \quad E(\lambda) = (\lambda, 1+\lambda], \quad \forall \lambda \in (0,1].$$

Hence, $E(0)$ is not compact and E is not usc. Indeed, we choose an open set $U := (0,3/2) \supseteq E(0) = (0,1]$. We observe that, for any $\varepsilon > 0$, we can choose $\lambda' = -\varepsilon/2 \in N(0,\varepsilon)$ such that

$$E(\lambda') = (-\varepsilon/2, \varepsilon/2] \nsubseteq U.$$

Clearly, the conditions (ii) and (iii) are all satisfied. Easy calculations yield

$$S_{f_1}(\lambda) = \begin{cases} (0,1], & \text{if } \lambda = 0, \\ (\lambda, 1+\lambda], & \text{if } \lambda \neq 0, \end{cases}$$

and $Z(x,\lambda) = \{x\}$. Hence, assumption (iv) is satisfied. Direct computations give

$$S_f(\lambda) = \begin{cases} (0,1], & \text{if } \lambda = 0, \\ (\lambda, 1+\lambda], & \text{if } \lambda \neq 0. \end{cases}$$

It is evident that S_f is neither usc nor closed at $\bar{\lambda} = 0$. This is caused by the fact that E is neither upper semicontinuous nor compact-valued at $\bar{\lambda} = 0$.

Example 3.5 (The lower semicontinuity of K_2 in $K_1(A,\Lambda) \times \{\bar{\lambda}\}$ is essential) Let $A = X = \mathbb{R}$, $\Lambda = [0,1]$, $\bar{\lambda} = 0$. Define the mappings K_1, K_2, and f by

$$K_1(x,\lambda) = K_2(x,\lambda) = \begin{cases} [-1,0], & \text{if } \lambda = 0, \\ [-\frac{1}{2}, 0], & \text{if } \lambda \neq 0, \end{cases}$$

and

$$f(x,y,\lambda) = \bigl((1+\lambda)(y-x), 2^{\lambda y}(y-x)\bigr).$$

Then we have

$$E(\lambda) = \begin{cases} [-1,0], & \text{if } \lambda = 0, \\ [-\frac{1}{2}, 0], & \text{if } \lambda \neq 0, \end{cases}$$

which shows that E is usc at 0 and $E(0)$ is compact, that is, (i) is satisfied. Clearly, the condition (iii) in Theorem 3.2 is satisfied. Furthermore, easy calculations yield

$$S_{f_1}(\lambda) = \begin{cases} \{-1\}, & \text{if } \lambda = 0, \\ \{-\frac{1}{2}\}, & \text{if } \lambda \neq 0, \end{cases}$$

and $Z(x,\lambda) = \{x\}$, which is lsc in $S_{f_1}(\bar{\lambda}) \times \{\bar{\lambda}\}$. Direct calculation gives

$$S_f(\lambda) = \begin{cases} \{-1\}, & \text{if } \lambda = 0, \\ \{-\frac{1}{2}\}, & \text{if } \lambda \neq 0. \end{cases}$$

It is evident that S_f is neither usc nor closed at $\bar{\lambda} = 0$. This is caused by the fact that K_2 is not lsc at $\bar{\lambda} = 0$. Indeed, we observe that $(0,0) \in K_1(A,\Lambda) \times \{0\}$ and $\{(\frac{1}{n}, \frac{1}{n})\} \to (0,0)$. We choose $y := -1 \in K_2(0,0) = [-1,0]$ such that there is not any sequence $\{y_n\}$ in $K_2(\frac{1}{n}, \frac{1}{n}) = [-\frac{1}{2}, 0]$ converging to y.

Example 3.6 (The lower semicontinuity of Z in $S_{f_1}(\bar{\lambda}) \times \{\bar{\lambda}\}$ are crucial) Let $A = X = \mathbb{R}$, $\Lambda = [0,1]$, $\bar{\lambda} = 0$. Define the mappings K_1, K_2, and f by

$$K_1(x,\lambda) = K_2(x,\lambda) = [0,1]$$

and

$$f(x,y,\lambda) = \big(\lambda(x-y), 2^{\lambda y}(y-x)\big).$$

Hence Conditions (i), (ii), and (iii) clearly hold. By direct calculations, we can get

$$S_{f_1}(\lambda) = \begin{cases} [0,1], & \text{if } \lambda = 0, \\ \{1\}, & \text{if } \lambda \neq 0, \end{cases}$$

$$Z(x,\lambda) = \begin{cases} [0,1], & \text{if } \lambda = 0, \\ \{x\}, & \text{if } \lambda \neq 0. \end{cases}$$

Hence Z is not lsc in $[0,1] \times \{0\}$. Further

$$S_f(\lambda) = \begin{cases} \{0\}, & \text{if } \lambda = 0, \\ \{1\}, & \text{if } \lambda \neq 0. \end{cases}$$

It is evident that S_f is neither usc nor closed at $\bar{\lambda} = 0$.

4 The lower semicontinuouity of S_f

For investigation the lower semicontinuity of the solution mapping S_f, as an auxiliary problem we consider, for a given $\lambda \in \Lambda$, *an auxiliary parametric generalized lexicographic quasiequilibrium problem*, denoted by AGLQEP_λ:

$$(\text{AGLQEP}_\lambda) \begin{cases} \text{finding } \bar{x} \in K_1(\bar{x},\lambda) \text{ such that} \\ f_1(\bar{x},y,\lambda) > 0, \ \text{for all } y \in K_2(\bar{x},\lambda). \end{cases}$$

Let the set-valued mappings $E : \Lambda \to 2^X$ and $S_{AGQEP} : \Lambda \to 2^X$ be defined by

$$E(\lambda) = \big\{x \in A : x \in K_1(x,\lambda)\big\},$$

and the solution mapping for AGLQEP_λ is denoted by $S_{AGLQEP}(\lambda)$, *i.e.*

$$S_{AGQEP}(\lambda) = \big\{x \in E(\lambda) : f_1(x,y,\lambda) > 0, \forall y \in K_2(x,\lambda)\big\}.$$

First, we establish the lower semicontinuity of the solution mapping S_{AGQEP}.

Lemma 4.1 *For AGLQEP, assume that the following conditions are satisfied*:
(i) *E is lsc at $\bar{\lambda}$*;
(ii) *K_2 is usc and compact-valued in $K_1(A, \Lambda) \times \{\bar{\lambda}\}$*;
(iii) *$\mathrm{lev}_{\leq 0} f_1(\cdot, \cdot, \bar{\lambda})$ is closed in $K_1(A, \Lambda) \times K_2(A, \Lambda) \times \{\bar{\lambda}\}$.*
Then the solution map S_{AGQEP} is lsc at $\bar{\lambda}$.

Proof Suppose to the contrary that S_{AGQEP} is not lsc at $\bar{\lambda}$, *i.e.*, there a sequence $\{\lambda_n\} \subseteq \Lambda$ with $\lambda_n \to \bar{\lambda}$ and there exists $\bar{x} \in S_{AGQEP}(\bar{\lambda}) \subseteq E(\bar{\lambda})$ such that,

$$\text{for all sequence } \{y_n\} \subseteq S_{AGQEP}(\lambda_n) \subseteq E(\lambda_n), \quad y_n \nrightarrow \bar{x}. \tag{4.1}$$

Since E is lsc at $\bar{\lambda}$, there exists a sequence $\{x_n\}$ in $E(\lambda_n)$ with $x_n \to \bar{x}$ as $n \to \infty$. It follows from (4.1) that there exists $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \notin S_{AGQEP_1}(\lambda_{n_k})$ for all $k \in \mathbb{N}$. This implies that there $y_{n_k} \in K_2(x_{n_k}, \lambda_{n_k})$ satisfying

$$f_1(x_{n_k}, y_{n_k}, \lambda_{n_k}) \leq 0, \quad \text{for each } k \in \mathbb{N}.$$

As K_2 is usc at $(\bar{x}, \bar{\lambda})$ and $K_2(\bar{x}, \bar{\lambda})$ is compact, there exists $\bar{y} \in K_2(\bar{x}, \bar{\lambda})$ such that

$$y_{n_k} \to \bar{y} \quad \text{as } k \to \infty \text{ (taking a subsequence if necessary).}$$

It follows from the closedness of $\mathrm{lev}_{\leq 0} f_1(\cdot, \cdot, \bar{\lambda})$ that $f_1(\bar{x}, \bar{y}, \bar{\lambda}) \leq 0$, which is impossible since $\bar{x} \in S_{AGQEP}(\bar{\lambda})$. The proof is completed. □

Now, we establish the lower semicontinuity of the solution mapping S_f.

Theorem 4.2 *For* (GLQEP) *let the following conditions be satisfied*:
(i) *E is lsc at $\bar{\lambda}$ and $E(\bar{\lambda})$ is convex*;
(ii) *$K_2(\cdot, \bar{\lambda})$ is usc and compact-valued in $K_1(A, \Lambda) \times \{\bar{\lambda}\}$; $K_2(\cdot, \bar{\lambda})$ is concave in $E(\bar{\lambda})$*;
(iii) *$\mathrm{lev}_{\leq 0} f_1(\cdot, \cdot, \bar{\lambda})$ is closed in $K_1(A, \Lambda) \times K_2(A, \Lambda) \times \{\bar{\lambda}\}$*;
(iv) *$f_1(\cdot, \cdot, \bar{\lambda})$ is generalized $\mathbb{R}_+$-concave in $E(\bar{\lambda}) \times K_2(A, \bar{\lambda})$.*
Then the solution map S_f is lsc at $\bar{\lambda}$.

Proof First, we notice that $S_f(\bar{\lambda}) \subseteq \mathrm{cl}\, S_{AGLQEP}(\bar{\lambda})$. Indeed, let $\bar{x} \in S_f(\bar{\lambda})$ be arbitrary. For any $\bar{x}_A \in S_{AGLQEP}(\bar{\lambda})$ and $t \in (0,1)$, define $x_t = (1-t)\bar{x} + t\bar{x}_A$. Clearly, $x_t \to \bar{x}$ as $t \downarrow 0$ and by the virtue of the convexity of $E(\bar{\lambda})$, $x_t \in K_1(x_t, \bar{\lambda})$. Further, for all $y \in K_2(x_t, \bar{\lambda})$, the concavity of $K_2(\cdot, \bar{\lambda})$ implies that there exist $\bar{y} \in K_2(\bar{x}, \bar{\lambda})$ and $\bar{y}_A \in K_2(\bar{x}_A, \bar{\lambda})$ such that $y = (1-t)\bar{y} + t\bar{y}_A$. It is clear that $f_1(\bar{x}, \bar{y}, \bar{\lambda}) \geq 0$ and $f_1(\bar{x}_A, \bar{y}_A, \bar{\lambda}) > 0$. It follows from the generalized $\mathbb{R}_+$-concavity of $f_1(\cdot, \cdot, \bar{\lambda})$ that $f_1(x_t, y, \bar{\lambda}) > 0$, *i.e.*, $x_t \in S_{AGQEP}(\bar{\lambda})$. Therefore, we conclude that $\bar{x} \in \mathrm{cl}\, S_{AGQEP}(\bar{\lambda})$, which shows that $S_f(\bar{\lambda}) \subseteq \mathrm{cl}\, S_{AGQEP}(\bar{\lambda})$. Next, for any sequence $\{\lambda_n\}$ satisfying $\lambda_n \to \bar{\lambda}$ as $n \to \infty$, by the lower semicontinuity of S_{AGQEP} at $\bar{\lambda}$ given in Lemma 4.1, we have

$$S_f(\bar{\lambda}) \subseteq \mathrm{cl}\, S_{AGQEP}(\bar{\lambda}) \subseteq \mathrm{cl} \liminf_{n \to \infty} S_{AGQEP}(\lambda_n) \subseteq \mathrm{cl} \liminf_{n \to \infty} S_f(\lambda_n),$$

which gives the lower semicontinuity of S_f at $\bar{\lambda}$ since Lemma 2.6. The proof is completed. □

Corollary 4.3 *For GLQEP, suppose that the conditions* (i), (ii), *and* (iv) *given in Theorem* 4.2 *are satisfied. Further, assume that* f_1 *is lower pseudocontinuous in* $K_1(A,\Lambda) \times K_2(A,\Lambda) \times \{\bar{\lambda}\}$. *Then the solution map* S_f *is lsc at* $\bar{\lambda}$.

Proof It is suffice to derive the condition (iii) that imposed in Theorem 4.2. Let $\{(x_n, y_n, \lambda_n)\}$ be any sequence in $\operatorname{lev}_{\leq 0} f_1(\cdot,\cdot,\bar{\lambda})$ such that $(x_n, y_n, \lambda_n) \to (\bar{x}, \bar{y}, \bar{\lambda})$ as $n \to \infty$. Assume that on the contrary that $(\bar{x}, \bar{y}, \bar{\lambda}) \notin \operatorname{lev}_{\leq 0} f_1$, which implies that $f_1(\bar{x}, \bar{y}, \bar{\lambda}) > 0 = f_1(\bar{x}, \bar{x}, \bar{\lambda})$. It follows from the lower pseudocontinuity of f_1 at $(\bar{x}, \bar{y}, \bar{\lambda})$ that

$$0 = f_1(\bar{x}, \bar{x}, \bar{\lambda}) < \liminf_{n\to\infty} f_1(x_n, y_n, \lambda_n) \leq 0,$$

which gives a contradiction. Hence, we can conclude that $(\bar{x}, \bar{y}, \bar{\lambda}) \in \operatorname{lev}_{\leq 0} f_1$. Now, the closedness of $\operatorname{lev}_{\leq 0} f_1(\cdot,\cdot,\bar{\lambda})$ is proved. Applying Theorem 4.2 we obtain the desired result. $\square$

The following example illustrates that the lower semicontinuity assumption for the set E cannot be relaxed in Theorem 4.2.

Example 4.4 (The lower semicontinuity of E at $\bar{\lambda}$ is crucial) Let $A = X = \mathbb{R}$, $\Lambda = [0,1]$, $\bar{\lambda} = 0$. Define the mappings K_1, K_2, and f by

$$K_1(x,\lambda) = \begin{cases} [-1,1], & \text{if } \lambda = 0, \\ [-1,0] \cup \{1\}, & \text{if } \lambda \neq 0, \end{cases} \qquad K_2(x,\lambda) = [-1,0],$$

and

$$f(x,y,\lambda) = \bigl((1+\lambda)(x-y), 2^{\lambda y}(x-2y)\bigr).$$

Hence conditions (ii)-(iv) clearly hold. However, E is not lsc at $\bar{\lambda} = 0$. Indeed, we choose a sequence $\{1/n\} \subseteq \Lambda$ such that $1/n \to 0$ and $1/2 \in E(0) = [-1,1]$. We can see that, for all sequences $\{y_n\} \subseteq E(1/n) := [-1,0] \cup \{1\}$, $y_n \nrightarrow 1/2$ as $n \to \infty$. By direct calculations, we can get

$$S_{AGQEP}(\lambda) = S_f(\lambda) = \begin{cases} (0,1], & \text{if } \lambda = 0, \\ \{1\}, & \text{if } \lambda \neq 0. \end{cases}$$

Hence S_f is not lsc at $\bar{\lambda} = 0$, indeed, we choose $\lambda_n = \frac{1}{n} \to 0$ and $\frac{1}{2} \in S_f(0)$ but we cannot find a sequence in $S_f(\frac{1}{n})$ which converges to $\frac{1}{2}$.

The next example indicates the essential role of the upper semicontinuity assumption for the set K_2 in Theorem 4.2.

Example 4.5 (The upper semicontinuity of K_2 is crucial) Let $A = X = \mathbb{R}$, $\Lambda = [0,1]$, $\bar{\lambda} = 0$. Define the mappings K_1, K_2, and f by

$$K_1(x,\lambda) = [0,1], \qquad K_2(x,\lambda) = \begin{cases} \{-\frac{1}{2}\} \cup [0,1], & \text{if } \lambda = 0, \\ [-\frac{2}{3}, \frac{2}{3}], & \text{if } \lambda \neq 0, \end{cases}$$

and

$$f(x,y,\lambda) = (x+y, x-y).$$

It is clear that the upper semicontinuity of K_2 is not satisfied. Indeed, for each $n \in \mathbb{N}$, we choose

$$(x_n, \lambda_n) = \left(\frac{1}{n}, \frac{1}{n}\right) \quad \text{and} \quad y_n = -\frac{2}{3} \in K_2(x_n, \lambda_n).$$

It is obvious that there is not any subsequence of $\{y_n\}$ converging to an element in $\{-\frac{1}{2}\} \cup [0,1] := K(0,0)$. However, all conditions (i), (iii)-(v) of Theorem 4.2 are satisfied. By direct calculations, we have

$$S_{AGQEP}(\lambda) = S_f(\lambda) = \begin{cases} (\frac{1}{2}, 1], & \text{if } \lambda = 0, \\ (\frac{2}{3}, 1], & \text{if } \lambda \neq 0. \end{cases}$$

Hence S_f is not lsc at $\bar{\lambda} = 0$. The cause is that (ii) is not fulfilled.

5 Conclusion

We presented the upper semicontinuity, closedness, and the lower semicontinuity of the set-valued solution mapping for a parametric lexicographic equilibrium problem where both two constraint maps and the objective bifunction depend on both the decision variable and the parameters. The sufficient conditions for the upper semicontinuity, closedness, and the lower semicontinuity of the solution map are established. Many examples are provided to ensure the essentialness of the imposed assumptions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

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