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Auxiliary principle and iterative algorithm for a system of generalized set-valued strongly nonlinear mixed implicit quasi-variational-like inequalities

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Abstract
In this paper, the auxiliary principle technique is extended to study a system of generalized set-valued strongly nonlinear mixed implicit quasi-variational-like inequalities problem in Hilbert spaces. First, we establish the existence of solutions of the corresponding system of auxiliary variational inequalities problem. Then, using the existence result, we construct a new iterative algorithm. Finally, both the existence of solutions of the original problem and the convergence of iterative sequences generated by the algorithm are proved. We give an affirmative answer to the open problem raised by Noor et al. (Korean J. Comput. Appl. Math. 1:73-89, 1998; J. Comput. Appl. Math. 47:285-312, 1993). Our results improve and extend some known results.

MSC: 47H10; 49J30

Keywords: system of generalized mixed implicit quasi-variational-like inequalities; auxiliary principle; iterative algorithm; convergence

1 Introduction

Throughout the paper, let $I = \{1, 2\}$ be an index set, let, for each $i \in I$, H_i be a Hilbert space with inner product $\langle \cdot, \cdot \rangle_i$ and norm $\| \cdot \|_i$, and $CB(H_i)$ be the family of all nonempty bounded closed subsets of H_i. Denoting by 2^{H_i} the family of all nonempty subsets of H_i, let $K_1 : H_1 \to 2^{H_1}$ and $K_2 : H_2 \to 2^{H_2}$ be two set-valued mappings such that for each $x \in H_1$, $K_1(x)$ is a closed convex subset of H_1 and for each $y \in H_2$, $K_2(y)$ is a closed convex subset of H_2. Let $N_i, \eta_i : H_i \times H_i \to H_i$ be nonlinear single-valued mappings, and let $A_i, B_i : H_i \to CB(H_i)$ and $F_i : H_1 \times H_2 \to CB(H_i)$ be set-valued mappings. We consider the following generalized set-valued strongly nonlinear mixed implicit quasi-variational-like inequalities problem: Find $(x, y) \in K_1(x) \times K_2(y)$, $u_1 \in A_1x$, $v_1 \in B_1x$, $u_2 \in A_2y$, $v_2 \in B_2y$, and $w_i \in F_i(x, y)$ such that

$$\begin{cases} \langle N_1(u_1, v_1) + w_1, \eta_1(h_1, x)\rangle_1 + b_1(x, h_1) - b_1(x, x) \geq 0, & \forall h_1 \in K_1(x), \\ \langle N_2(u_2, v_2) + w_2, \eta_2(h_2, y)\rangle_2 + b_2(y, h_2) - b_2(y, y) \geq 0, & \forall h_2 \in K_2(y), \end{cases} \tag{1.1}$$

where $b_i : H_i \times H_i \to R$, which is not necessarily differentiable, possesses the following properties:

(i) $b_i(\cdot,\cdot)$ is linear in the first argument;

(ii) $b_i(\cdot,\cdot)$ is bounded, that is, there exists a constant $\gamma_i > 0$ such that

$$b_i(u_i, v_i) \leq \gamma_i \|u_i\|_i \|v_i\|_i, \quad \forall u_i, v_i \in H_i;$$

(iii) $b_i(u_i, v_i) - b_i(u_i, w_i) \leq b_i(u_i, v_i - w_i)$, $\forall u_i, v_i, w_i \in H_i$;

(iv) $b_i(\cdot,\cdot)$ is convex in the second argument.

In many important applications, $K_1(x)$ and $K_2(y)$ have the following forms [1, 3]:

$$\begin{cases} K_1(x) = m_1(x) + K_1, & \forall x \in H_1, \\ K_2(y) = m_2(y) + K_2, & \forall y \in H_2, \end{cases} \tag{1.2}$$

where $m_i : H_i \to H_i$ are single-valued mappings, and K_i are closed convex subsets of H_i.

Choosing different mappings in problem (1.1), we have the following special cases.

(I) If $F_1(\cdot,\cdot) = F_2(\cdot,\cdot) = 0$, then problem (1.1) reduces to the problem of finding $(x, y) \in H_1 \times H_2$, $u_1 \in A_1x$, $v_1 \in B_1x$, $u_2 \in A_2y$, and $v_2 \in B_2y$ such that

$$\begin{cases} \langle N_1(u_1, v_1), \eta_1(h_1, x)\rangle_1 + b_1(x, h_1) - b_1(x, x) \geq 0, & \forall h_1 \in H_1, \\ \langle N_2(u_2, v_2), \eta_2(h_2, y)\rangle_2 + b_2(y, h_2) - b_2(y, y) \geq 0, & \forall h_2 \in H_2. \end{cases} \tag{1.3}$$

Problem (1.3) has been introduced and studied by Kazmi and Khan [4].

(II) If $H_1 = H_2 = H$, $b_1(\cdot,\cdot) = b_2(\cdot,\cdot) = 0$, $\eta_1(h_1, x) = h_1 - x$, $\eta_2(h_2, y) = h_2 - y$, and $A_i = B_i = I_i$, then problem (1.1) reduces to the problem of finding $(x, y) \in K_1(x) \times K_2(y)$ and $w_i \in F_i(x, y)$ such that

$$\begin{cases} \langle N_1(x, y) + w_1, h_1 - x\rangle \geq 0, & \forall h_1 \in K_1(x), \\ \langle N_2(x, y) + w_2, h_2 - y\rangle \geq 0, & \forall h_2 \in K_2(y). \end{cases} \tag{1.4}$$

Problem (1.4) has been introduced and studied by Qiu et al. [5].

(III) If $N_1 = N_2 = N$, $A_1 = A_2 = A$, $B_1 = B_2 = B$, $K_1 = K_2 = K$, $F_1 = F_2 = 0$, $\eta_1(h_1, x) = \eta_2(h_2, y) = h - x$, and $b_1 = b_2 = b$, then problem (1.1) reduces to the problem of finding $x \in K(x)$, $u \in Ax$, and $v \in Bx$ such that

$$\bigl\langle N(u, v), h - x\bigr\rangle + b(x, h) - b(x, x) \geq 0, \quad \forall h \in K(x). \tag{1.5}$$

The problem has been introduced by Ding [3].

(IV) If $N_1(\cdot,\cdot) = N_2(\cdot,\cdot) = (V \circ A + B)(\cdot)$, $F_1(\cdot,\cdot) = F_2(\cdot,\cdot) = F(\cdot)$, $K_1 = K_2 = K$, and $A_i = B_i = I_i$, then problem (1.1) is equivalent to finding $x \in K(x)$ such that $w \in F(x)$, $u \in A(x)$, and

$$\langle \omega + Vu + Bx, h - x\rangle \geq 0, \quad \forall h \in K(x), \tag{1.6}$$

which is due to Noor [1].

(V) If $N_1 = N_2 = N$, $A_1 = A_2 = A$, $B_1 = B_2 = B$, $\eta_1 = \eta_2 = \eta$, $F_1 = F_2 = 0$, and $b_1 = b_2 = b$, then problem (1.1) reduces to the problem of finding $x \in H$, $u \in Ax$, and $v \in Bx$ such that

$$\bigl\langle N(u, v), \eta(h, x)\bigr\rangle + b(x, h) - b(x, x) \geq 0, \quad \forall h \in H. \tag{1.7}$$

The problem was introduced by Zeng et al. [6] and Huang et al. [7].

There are many special cases of problems (1.1) and (1.3)-(1.7), which can be also found in Cohen [8], Noor [2, 9, 10], and the references therein. In the system of the generalized set-valued strongly nonlinear mixed implicit quasi-variational-like inequalities problem (1.1), the two convex sets depend on the solution implicitly or explicitly, and b_i is a nonlinear mapping, so the projection method cannot be applied to it. This fact motivated many authors to develop the auxiliary principle technique to study the existence of solutions of generalized mixed type quasi-variational inequalities and also to develop a large number of numerical methods for solving various variational inequalities, complementarity problems, and optimization problems. The auxiliary principle technique was first introduced by Glowinski et al. [11]. Then Noor [12] extended it to study the existence and uniqueness of solutions when A and B are compact set-valued mappings. Regretfully, A is actually a single-valued mapping in his Theorem 4.1, as it was pointed out in Liu and Li [13]. Noor [1, 2] put forward that extending the projection methods and its variant forms for generalized set-valued mixed nonlinear variational inequalities involving the nonlinear form $b(\cdot,\cdot)$ satisfying properties (i), (ii), and (iii) is still an open problem. The theory of quasi-variational inequalities is not developed to an extent providing a complete framework for studying these problems, and this is another direction for future research in this fascinating and elegant area. Also, extending the auxiliary principle technique for quasi-variational inequalities is still an open problem, and this needs further research efforts.

In this paper, we extend the auxiliary principle technique to study the generalized set-valued strongly nonlinear mixed implicit quasi-variational-like inequalities problem (1.1) in Hilbert spaces. First, we establish the existence of solutions of the corresponding system of auxiliary variational inequalities (2.1). Then, using the existence result, we construct a new iterative algorithm. Finally, both the existence of solutions of the original problem and the convergence of iterative sequences generated by the algorithm are proved. Our results improve and extend some known results.

We first recall some concepts and definitions.

Definition 1.1 The mapping $\eta : H \times H \to H$ is said to be ε-Lipschitz continuous, if there exists a constant $\varepsilon > 0$ such that

$$\bigl\|\eta(x,y)\bigr\| \le \varepsilon\|x-y\|, \quad \forall x,y \in H.$$

Definition 1.2 Let $N : H \times H \to H$ be a single-valued mapping, and $A, B : H \to CB(H)$ be set-valued mappings.

(1) N is said to be (β,ξ)-Lipschitz continuous if there exists a pair of constants $\beta, \xi > 0$ such that

$$\bigl\|N(u_1,v_1) - N(u_2,v_2)\bigr\| \le \beta\|u_1 - u_2\| + \xi\|v_1 - v_2\|, \quad \forall u_1, v_1, u_2, v_2 \in H;$$

(2) N is said to be α-strongly mixed monotone with set-valued mappings A and B if there exists a constant $\alpha > 0$ such that

$$\bigl\langle N(u_1,v_1) - N(u_2,v_2), x-y\bigr\rangle \ge \alpha\|x-y\|^2,$$
$$\forall x,y \in H, u_1 \in A(x), v_1 \in B(x), u_2 \in A(y), v_2 \in B(y).$$

(3) A is said to be λ-H-Lipschitz continuous if there exists a constant $\lambda > 0$ such that

$$H(Au, Av) \leq \lambda \|u - v\|, \quad \forall u, v \in H,$$

where $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(H)$;

Definition 1.3 The mapping $m : H \to H$ is said to be δ-Lipschitz continuous if there exists a constant $\delta > 0$ such that

$$\|mx - my\| \leq \delta \|x - y\|, \quad \forall x, y \in H.$$

Definition 1.4 The set-valued mapping $F(\cdot,\cdot) : H \times H \to CB(H)$ is said to be (l,k)-H-Lipschitz continuous if there exist constants $l, k > 0$ such that

$$H\big(F(x_1, y_1), F(x_2, y_2)\big) \leq l\|x_1 - x_2\| + k\|y_1 - y_2\|, \quad \forall x_1, x_2, y_1, y_2 \in H,$$

where $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(H)$.

To obtain our results, we need the following assumption.

Assumption 1.1 The mapping $\eta_i : H_i \times H_i \to H_i$ satisfies the following conditions:

(1) $\eta_i(u, v) = \eta_i(u, z) + \eta_i(z, v)$, $\forall u, v \in H_i$;

(2) $\eta_i(u + v, w) = -\eta_i(w - u, v)$, $\forall u, v, w \in H_i$;

(3) the functionals $h_1 \mapsto \langle N_1(u_1, v_1) + w_1, \eta_1(h_1, x)\rangle_1$ and $h_2 \mapsto \langle N_2(u_2, v_2) + w_2, \eta_2(h_2, y)\rangle_2$ are both continuous and linear for all $h_1 \in H_1$ and $h_2 \in H_2$.

Remark 1.1 If $\eta_i(u, v) = u - v$, obviously, Assumption 1.1(2) holds. In a word, Assumption 1.1 is justified.

We also need the following lemma.

Lemma 1.1 [15] *Let E be a normed vector space, $CB(E)$ be the family of all closed bounded subsets of E, and $T : E \to CB(E)$ be a set-valued mapping. Then for any given $\varepsilon > 0$, $x, y \in E$, and $u \in Tx$, there exists $v \in Ty$ such that*

$$d(u, v) \leq (1 + \varepsilon)H(Tx, Ty),$$

where $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$.

2 Auxiliary problem and iterative algorithm

In this section, we extend the auxiliary principle technique to study problem (1.1) and prove the existence of solutions of the auxiliary problem for (1.1). Then, by using the existence theorem we construct an iterative algorithm for problem (1.1).

For each $i \in I$, given $(x,y) \in K_1(x) \times K_2(y)$, $u_1 \in A_1x$, $v_1 \in B_1x$, $u_2 \in A_2y$, $v_2 \in A_2y$, and $w_i \in F_i(x,y)$, we consider the following problem: Find $(p_1,p_2) \in K_1(x) \times K_2(y)$ such that

$$\begin{cases} \langle p_1, h_1 - p_1\rangle_1 \geq \langle x, h_1 - p_1\rangle_1 - \rho_1\langle N_1(u_1,v_1) + w_1, \eta_1(h_1,p_1)\rangle_1 \\ \qquad + \rho_1[b_1(x,p_1) - b_1(x,h_1)], \quad \forall h_1 \in K_1(x), \\ \langle p_2, h_2 - p_2\rangle_2 \geq \langle y, h_2 - p_2\rangle_2 - \rho_2\langle N_2(u_2,v_2) + w_2, \eta_2(h_2,p_2)\rangle_2 \\ \qquad + \rho_2[b_2(y,p_2) - b_2(y,h_2)], \quad \forall h_2 \in K_2(y), \end{cases} \tag{2.1}$$

where $\rho_1, \rho_2 > 0$ are constants. Problem (2.1) is called the system of auxiliary variational inequalities for problem (1.1).

Theorem 2.1 *For each $i \in I$, let $K_1 : H_1 \to 2^{H_1}$ and $K_2 : H_2 \to 2^{H_2}$ be two set-valued mappings such that for each $x \in H_1$, $K_1(x)$ is a nonempty closed convex subset of H_1 and for each $y \in H_2$, $K_2(y)$ is also a nonempty closed convex subset of H_2. Let $N_i, \eta_i : H_i \times H_i \to H_i$ be nonlinear single-valued mappings, $A_i, B_i : H_i \to CB(H_i)$, $F_i : H_1 \times H_2 \to CB(H_i)$ be set-valued mappings, and $b_i : H_i \times H_i \to R$ be a mapping such that for any given $(x,y) \in H_1 \times H_2$, the functionals $h_1 \mapsto b_1(x,h_1)$ and $h_2 \mapsto b_2(y,h_2)$ are proper convex and lower semicontinuous. If Assumption* 1.1 *holds, then for any given $(x,y) \in H_1 \times H_2$, $u_1 \in A_1x$, $v_1 \in B_1x$, $u_2 \in A_2y$, $v_2 \in B_2y$, and $w_i \in F_i(x,y)$, define the functionals $J_1 : K_1(x) \to R$ and $J_2 : K_2(y) \to R$ as follows*:

$$J_1(h_1) = \frac{1}{2}\langle h_1, h_1\rangle_1 + j_1(h_1),$$

where

$$j_1(h_1) = \rho_1\bigl\langle N_1(u_1,v_1) + w_1, \eta_1(h_1,x)\bigr\rangle_1 + \rho_1 b_1(x,h_1) - \langle x, h_1\rangle_1,$$
$$J_2(h_2) = \frac{1}{2}\langle h_2, h_2\rangle_2 + j_2(h_2),$$

where

$$j_2(h_2) = \rho_2\bigl\langle N_2(u_2,v_2) + w_2, \eta_2(h_2,y)\bigr\rangle_2 + \rho_2 b_2(y,h_2) - \langle y, h_2\rangle_2.$$

Then we have:

(i) *J_1 has a unique minimum point p_1 in $K_1(x)$, and J_2 has a unique minimum point p_2 in $K_2(y)$.*

(ii) *J_1 and J_2 have unique minimum points p_1 in $K_1(x)$ and p_2 in $K_2(y)$, respectively, if and only if (p_1,p_2) is a unique solution of the system of auxiliary variational inequalities* (2.1).

Proof By Assumption 1.1(3) the functional $h_1 \mapsto \langle N_1(u_1,v_1) + w_1, \eta_1(h_1,x)\rangle_1$ is continuous and linear. Since $h_1 \mapsto b_1(x,h_1)$ is proper convex lower semicontinuous on $K_1(x)$, it is easy to see that $j_1(h_1)$ is proper convex lower semicontinuous on $K_1(x)$, and so $J_1(h_1)$ is a strictly convex and lower semicontinuous functional on $K_1(x)$. Thus, $j_1(h_1)$ is bounded from below by a hyperplane $\langle q_1, h_1\rangle_1 + a_1$ (see [14]), where $q_1 \in H_1$ and $a_1 \in R$. Hence, we have

$$J_1(h_1) = \frac{1}{2}\langle h_1, h_1\rangle_1 + j_1(h_1) \geq \frac{1}{2}\|h_1\|_1^2 + \langle q_1, h_1\rangle_1 + a_1 = \frac{1}{2}\|h_1 + q_1\|_1^2 - \frac{1}{2}\|q_1\|_1^2 + a_1.$$

It follows that

$$J_1(h_1) \to \infty \quad \bigl(\text{as } \|h_1\|_1 \to \infty\bigr). \tag{2.2}$$

Now let $\{h_n^1\}$ be a minimizing sequence of J_1 on $K_1(x)$, that is,

$$\lim_{n\to\infty} J_1\bigl(h_n^1\bigr) = \inf_{h_1\in K_1(x)} J_1(h_1).$$

We claim that $\{h_n^1\}$ is bounded. If it is not true, then there exists its subsequence $\{h_{n_k}^1\}$ such that $\|h_{n_k}^1\|_1 \geq k$, $k = 1, 2, \ldots$. By (2.2) we have $J_1(h_{n_k}^1) \to \infty$, which contradicts the fact that $\{h_n^1\}$ is a minimizing sequence of J_1 on $K_1(x)$. By the Weierstrass theorem (see [14]) there exists $p_1 \in K_1(x)$ such that

$$J_1(p_1) = \min_{h_1\in K_1(x)} J_1(h_1).$$

Again from the strict convexity of J_1 we have that J_1 has a unique minimum point p_1 in $K_1(x)$. Using a similar argument, we get that J_2 has a unique minimum point p_2 in $K_2(y)$.

Now suppose that J_1 has a unique minimum point p_1 in $K_1(x)$ and J_2 has a unique minimum point p_2 in $K_2(y)$. Let us show that (p_1, p_2) is a unique solution of the system of auxiliary variational inequalities (2.1). For any $h_1 \in K_1(x)$ and $t \in [0, 1]$, since j_1 is convex and J_1 is lower semicontinuous on $K_1(x)$, we have

$$\begin{aligned} J_1(p_1) &= \frac{1}{2}\langle p_1, p_1\rangle_1 + j_1(p_1) \leq J_1\bigl(p_1 + t(h_1 - p_1)\bigr) \\ &= 12\bigl\langle p_1 + t(h_1 - p_1), p_1 + t(h_1 - p_1)\bigr\rangle_1 + j_1\bigl(p_1 + t(h_1 - p_1)\bigr) \\ &\leq \frac{1}{2}\langle p_1, p_1\rangle_1 + \frac{t^2}{2}\langle h_1 - p_1, h_1 - p_1\rangle_1 + t\langle p_1, h_1 - p_1\rangle_1 + j_1(p_1) + t\bigl(j_1(h_1) - j_1(p_1)\bigr) \\ &= J_1(p_1) + \frac{t^2}{2}\langle h_1 - p_1, h_1 - p_1\rangle_1 + t\langle p_1, h_1 - p_1\rangle_1 + t\bigl(j_1(h_1) - j_1(p_1)\bigr). \end{aligned}$$

It follows that

$$\frac{t}{2}\langle h_1 - p_1, h_1 - p_1\rangle_1 + \langle p_1, h_1 - p_1\rangle_1 + j_1(h_1) - j_1(p_1) \geq 0.$$

Letting $t \to 0$ in this inequality, we obtain

$$\begin{aligned} &\langle p_1, h_1 - p_1\rangle_1 + j_1(h_1) - j_1(p_1) \\ &\quad= \langle p_1, h_1 - p_1\rangle_1 + \rho_1\bigl\langle N_1(u_1, v_1) + w_1, \eta_1(h_1, x)\bigr\rangle_1 + \rho_1 b_1(x, h_1) - \langle x, h_1\rangle_1 \\ &\qquad - \rho_1\bigl\langle N_1(u_1, v_1) + w_1, \eta_1(p_1, x)\bigr\rangle_1 - \rho_1 b_1(x, p_1) + \langle x, p_1\rangle_1 \\ &\quad\geq 0. \end{aligned}$$

By Assumption 1.1(1) we observe that $\eta_1(p_1, x) - \eta_1(h_1, x) = \eta_1(h_1, p_1)$, so

$$\begin{aligned} \langle p_1, h_1 - p_1\rangle_1 \geq{}& \langle x, h_1 - p_1\rangle_1 - \rho_1\bigl\langle N_1(u_1, v_1) + w_1, \eta_1(h_1, p_1)\bigr\rangle_1 \\ &+ \rho_1\bigl[b_1(x, p_1) - b_1(x, h_1)\bigr], \quad \forall h_1 \in K_1(x). \end{aligned}$$

In a similar way, we have

$$\begin{aligned}\langle p_2, h_2 - p_2\rangle_2 \geq{}& \langle y, h_2 - p_2\rangle_2 - \rho_2\bigl\langle N_2(u_2, v_2) + w_2, \eta_2(h_2, p_2)\bigr\rangle_2 \\ &+ \rho_2\bigl[b_2(y, p_2) - b_2(y, h_2)\bigr], \quad \forall h_2 \in K_2(y).\end{aligned}$$

So, (p_1, p_2) is a unique solution of the system of auxiliary variational inequalities (2.1).

Conversely, let (p_1, p_2) be a unique solution of the system of auxiliary variational inequalities (2.1). By the first inequality of (2.1) we have

$$\begin{aligned}&\frac{1}{2}\bigl(\langle h_1, h_1\rangle_1 - \langle p_1, p_1\rangle_1\bigr) \\ &\quad= \frac{1}{2}\bigl(\bigl\langle p_1 + (h_1 - p_1), p_1 + (h_1 - p_1)\bigr\rangle_1 - \langle p_1, p_1\rangle_1\bigr) \\ &\quad= \langle p_1, h_1 - p_1\rangle_1 + \frac{1}{2}\|h_1 - p_1\|_1^2 \\ &\quad\geq \langle p_1, h_1 - p_1\rangle_1 \\ &\quad\geq \langle x, h_1 - p_1\rangle_1 - \rho_1\bigl\langle N_1(u_1, v_1) + w_1, \eta_1(h_1, p_1)\bigr\rangle_1 \\ &\qquad+ \rho_1\bigl[b_1(x, p_1) - b_1(x, h_1)\bigr].\end{aligned}$$

By Assumption 1.1(1) this implies that

$$\begin{aligned}&\frac{1}{2}\langle h_1, h_1\rangle_1 - \langle x, h_1\rangle_1 + \rho_1\bigl\langle N_1(u_1, v_1) + w_1, \eta_1(h_1, x)\bigr\rangle_1 + \rho_1 b_1(x, h_1) \\ &\quad\geq \frac{1}{2}\langle p_1, p_1\rangle_1 - \langle x, p_1\rangle_1 + \rho_1\bigl\langle N_1(u_1, v_1) + w_1, \eta_1(p_1, x)\bigr\rangle_1 + \rho_1 b_1(x, p_1),\end{aligned}$$

that is,

$$J_1(h_1) \geq J_1(p_1), \quad \forall h_1 \in K_1(x).$$

From the strict convexity of J_1 we conclude that J_1 has a unique minimum point p_1 in $K_1(x)$. Using a similar argument, we obtain that J_2 has a unique minimum point p_2 in $K_2(y)$. This completes the proof. □

By virtue of Lemma 1.1 and Theorem 2.1 we now construct an iterative algorithm for solving problem (1.1).

Algorithm 2.1 *For given* $(x_0, y_0) \in H_1 \times H_2$, $u_0^1 \in A_1 x_0$, $v_0^1 \in B_1 x_0$, $u_0^2 \in A_2 y_0$, $v_0^2 \in B_2 y_0$, *and* $w_0^i \in F_i(x_0, y_0)$, *let the sequences* $\{(x_n, y_n)\} \subset K_1(x_{n-1}) \times K_2(y_{n-1})$, $\{u_n^1\}$, $\{v_n^1\}$, $\{u_n^2\}$, $\{v_n^2\}$, $\{w_n^1\}$, *and* $\{w_n^2\}$ *satisfy the following conditions*:

$$\begin{aligned}\langle x_{n+1}, h_1 - x_{n+1}\rangle_1 \geq{}& \langle x_n, h_1 - x_{n+1}\rangle_1 - \rho_1\bigl\langle N_1\bigl(u_n^1, v_n^1\bigr) + w_n^1, \eta_1(h_1, x_{n+1})\bigr\rangle_1 \\ &+ \rho_1\bigl[b_1(x_n, x_{n+1}) - b_1(x_n, h_1)\bigr], \quad \forall h_1 \in K_1(x_{n+1}); \qquad (2.3)\\ \langle y_{n+1}, h_2 - y_{n+1}\rangle_2 \geq{}& \langle y_n, h_2 - y_{n+1}\rangle_2 - \rho_2\bigl\langle N_2\bigl(u_n^2, v_n^2\bigr) + w_n^2, \eta_2(h_2, y_{n+1})\bigr\rangle_2 \\ &+ \rho_2\bigl[b_2(y_n, y_{n+1}) - b_2(y_n, h_2)\bigr], \quad \forall h_2 \in K_2(y_{n+1}); \qquad (2.4)\end{aligned}$$

$$\begin{aligned}
&u_n^1 \in A_1x_n, \quad \|u_{n+1}^1 - u_n^1\|_1 \le \left(1 + \frac{1}{n+1}\right)H(A_1x_{n+1}, A_1x_n);\\
&v_n^1 \in B_1x_n, \quad \|v_{n+1}^1 - v_n^1\|_1 \le \left(1 + \frac{1}{n+1}\right)H(B_1x_{n+1}, B_1x_n);\\
&u_n^2 \in A_2y_n, \quad \|u_{n+1}^2 - u_n^2\|_2 \le \left(1 + \frac{1}{n+1}\right)H(A_2y_{n+1}, A_2y_n);\\
&v_n^2 \in B_2y_n, \quad \|v_{n+1}^2 - v_n^2\|_2 \le \left(1 + \frac{1}{n+1}\right)H(B_2y_{n+1}, B_2y_n);\\
&w_n^i \in F_i(x_n, y_n), \quad \|w_{n+1}^i - w_n^i\|_1 \le \left(1 + \frac{1}{n+1}\right)H\big(F_i(x_{n+1}, y_{n+1}), F_i(x_n, y_n)\big),
\end{aligned} \tag{2.5}$$

for every $n = 0, 1, 2, \ldots$, where $\rho_1, \rho_2 > 0$ are constants.

3 Existence and convergence theorem

In this section, we prove the existence of the solution of problem (1.1) and the convergence of the iterative sequences generated by Algorithm 2.1.

Theorem 3.1 *For each $i \in I$, let H_i be a Hilbert space, and let $K_1 : H_1 \to 2^{H_1}$ and $K_2 : H_2 \to 2^{H_2}$ be two set-valued mappings such that for each $x \in H_1$, $K_1(x)$ is a nonempty closed convex subset of H_1 and for each $y \in H_2$, $K_2(y)$ is also a nonempty closed convex subset of H_2. Let $N_i, \eta_i : H_i \times H_i \to H_i$, $F_i : H_1 \times H_2 \to CB(H_i)$, $A_i, B_i : H_i \to CB(H_i)$ be mappings, let mappings $m_i : H_i \to H_i$ satisfy* (1.2), *and let $b_i : H_i \times H_i \to R$ be real-valued functionals satisfying the properties in Theorem* 2.1 *and properties* (i)-(iv). *Assume that the following conditions are satisfied*:

(1) *N_i is (α_i, β_i)-Lipschitz continuous and ξ_i-strongly mixed monotone with respect to A_i and B_i*;
(2) *F_i is (l_i, k_i)-H-Lipschitz continuous*;
(3) *A_i is τ_i-H-Lipschitz continuous*;
(4) *B_i is ω_i-H-Lipschitz continuous*;
(5) *m_i is δ_i-Lipschitz continuous*;
(6) *η_i is ε_i-Lipschitz continuous.*

If Assumption 1.1 *holds and there exist constants $\rho_1, \rho_2 > 0$ such that*

$$\begin{cases}
\frac{1}{1-2\delta_1}[1 + \rho_1\gamma_1 + \varepsilon_1(1 + \sqrt{1 - 2\rho_1\xi_1 + \rho_1^2(\alpha_1\tau_1 + \beta_1\omega_1)^2} + \rho_1 l_1)]\\
\quad + \frac{1}{1-2\delta_2}\varepsilon_2\rho_2 l_2 < 1,\\
\frac{1}{1-2\delta_2}[1 + \rho_2\gamma_2 + \varepsilon_2(1 + \sqrt{1 - 2\rho_2\xi_2 + \rho_2^2(\alpha_2\tau_2 + \beta_2\omega_2)^2} + \rho_2 k_2)]\\
\quad + \frac{1}{1-2\delta_1}\varepsilon_1\rho_1 k_1 < 1,
\end{cases} \tag{3.1}$$

then there exist $(x, y) \in K_1(x) \times K_2(y)$, $u_1 \in A_1x$, $v_1 \in B_1x$, $u_2 \in A_2y$, $v_2 \in B_2y$, and $w_i \in F_i(x, y)$ satisfying problem (1.1), *and*

$$x_n \to x, \qquad y_n \to y, \qquad u_n^i \to u_i, \qquad v_n^i \to v_i, \qquad w_n^i \to w_i \quad \text{as } n \to \infty,$$

where the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n^i\}$, $\{v_n^i\}$, and $\{w_n^i\}$ are generated by Algorithm 2.1.

Proof First, it follows from (2.3) in Algorithm 2.1 that, for any $h_1 \in K_1(x_n)$,

$$\begin{aligned}\langle x_n, h_1 - x_n\rangle_1 \geq{}& \langle x_{n-1}, h_1 - x_n\rangle_1 - \rho_1\bigl\langle N_1\bigl(u^1_{n-1}, v^1_{n-1}\bigr) + w^1_{n-1}, \eta_1(h_1, x_n)\bigr\rangle_1 \\ &+ \rho_1\bigl[b_1(x_{n-1}, x_n) - b_1(x_{n-1}, h_1)\bigr]\end{aligned} \tag{3.2}$$

and, for any $h_1 \in K_1(x_{n+1})$,

$$\begin{aligned}\langle x_{n+1}, h_1 - x_{n+1}\rangle_1 \geq{}& \langle x_n, h_1 - x_{n+1}\rangle_1 - \rho_1\bigl\langle N_1\bigl(u^1_n, v^1_n\bigr) + w^1_n, \eta_1(h_1, x_{n+1})\bigr\rangle_1 \\ &+ \rho_1\bigl[b_1(x_n, x_{n+1}) - b_1(x_n, h_1)\bigr].\end{aligned} \tag{3.3}$$

Adding $\langle -m_1(x_n), h_1 - x_n\rangle_1$ to the two sides of inequality (3.2) and then taking $h_1 = m_1(x_n) + x_{n+1} - m_1(x_{n+1}) \in K_1(x_n)$, we get

$$\begin{aligned}&\bigl\langle x_n - m_1(x_n), m_1(x_n) + x_{n+1} - m_1(x_{n+1}) - x_n\bigr\rangle_1 \\ &\quad\geq \bigl\langle x_{n-1} - m_1(x_n), m_1(x_n) + x_{n+1} - m_1(x_{n+1}) - x_n\bigr\rangle_1 \\ &\qquad - \rho_1\bigl\langle N_1\bigl(u^1_{n-1}, v^1_{n-1}\bigr) + w^1_{n-1}, \eta_1\bigl(m_1(x_n) + x_{n+1} - m_1(x_{n+1}), x_n\bigr)\bigr\rangle_1 \\ &\qquad + \rho_1\bigl[b_1(x_{n-1}, x_n) - b_1\bigl(x_{n-1}, m_1(x_n) + x_{n+1} - m_1(x_{n+1})\bigr)\bigr].\end{aligned} \tag{3.4}$$

Adding $\langle -m_1(x_{n+1}), h_1 - x_{n+1}\rangle_1$ to the two sides of inequality (3.3) and then taking $h_1 = m_1(x_{n+1}) + x_n - m_1(x_n) \in K_1(x_{n+1})$, we get

$$\begin{aligned}&\bigl\langle x_{n+1} - m_1(x_{n+1}), m_1(x_{n+1}) + x_n - m_1(x_n) - x_{n+1}\bigr\rangle_1 \\ &\quad\geq \bigl\langle x_n - m_1(x_{n+1}), m_1(x_{n+1}) + x_n - m_1(x_n) - x_{n+1})\bigr\rangle_1 \\ &\qquad - \rho_1\bigl\langle N_1\bigl(u^1_n, v^1_n\bigr) + w^1_n, \eta_1\bigl(m_1(x_{n+1}) + x_n - m_1(x_n), x_{n+1}\bigr)\bigr\rangle_1 \\ &\qquad + \rho_1\bigl[b_1(x_n, x_{n+1}) - b_1\bigl(x_n, m_1(x_{n+1}) + x_n - m_1(x_n)\bigr)\bigr].\end{aligned} \tag{3.5}$$

Adding (3.4) and (3.5), by properties (i) and (iii) of $b(\cdot,\cdot)$ and Assumption 1.1(2), we obtain

$$\begin{aligned}&\bigl\langle x_n - x_{n+1} - m_1(x_n) + m_1(x_{n+1}), x_n - x_{n+1} - m_1(x_n) + m_1(x_{n+1})\bigr\rangle_1 \\ &\quad\leq \bigl\langle x_{n-1} - m_1(x_n), x_n - x_{n+1} - m_1(x_n) + m_1(x_{n+1})\bigr\rangle_1 \\ &\qquad + \bigl\langle -x_n + m_1(x_{n+1}), m_1(x_{n+1}) + x_n - m_1(x_n) - x_{n+1})\bigr\rangle_1 \\ &\qquad + \rho_1\bigl\langle N_1\bigl(u^1_{n-1}, v^1_{n-1}\bigr) + w^1_{n-1}, \eta_1\bigl(m_1(x_n) + x_{n+1} - m_1(x_{n+1}), x_n\bigr)\bigr\rangle_1 \\ &\qquad + \rho_1\bigl\langle N_1\bigl(u^1_n, v^1_n\bigr) + w^1_n, \eta_1\bigl(m_1(x_{n+1}) + x_n - m_1(x_n), x_{n+1}\bigr)\bigr\rangle_1 \\ &\qquad - \rho_1\bigl[b_1(x_{n-1}, x_n) - b_1\bigl(x_{n-1}, m_1(x_n) + x_{n+1} - m_1(x_{n+1})\bigr)\bigr] \\ &\qquad - \rho_1\bigl[b_1(x_n, x_{n+1}) - b_1\bigl(x_n, m_1(x_{n+1}) + x_n - m_1(x_n)\bigr)\bigr] \\ &\quad\leq \bigl\langle x_{n-1} - x_n - m_1(x_n) + m_1(x_{n+1}), x_n - x_{n+1} - m_1(x_n) + m_1(x_{n+1})\bigr\rangle_1 \\ &\qquad + \rho_1\bigl\langle N_1\bigl(u^1_{n-1}, v^1_{n-1}\bigr) + w^1_{n-1} - \bigl(N_1\bigl(u^1_n, v^1_n\bigr) + w^1_n\bigr), \eta_1\bigl(m_1(x_n) + x_{n+1} \\ &\qquad - m_1(x_{n+1}), x_n\bigr)\bigr\rangle_1 + \rho_1\bigl[b_1\bigl(-x_{n-1}, x_n - m_1(x_n) - x_{n+1} + m_1(x_{n+1})\bigr) \\ &\qquad + b_1\bigl(x_n, m_1(x_{n+1}) + x_n - m_1(x_n) - x_{n+1}\bigr)\bigr]\end{aligned}$$

$$
\begin{aligned}
&\le \langle x_{n-1} - x_n - m_1(x_n) + m_1(x_{n+1}), x_n - x_{n+1} - m_1(x_n) + m_1(x_{n+1})\rangle_1 \\
&\quad + \rho_1 \langle N_1(u_{n-1}^1, v_{n-1}^1) + w_{n-1}^1 - (N_1(u_n^1, v_n^1) + w_n^1), \eta_1(m_1(x_n) + x_{n+1} \\
&\quad - m_1(x_{n+1}), x_n)\rangle_1 + \rho_1 b_1(x_n - x_{n-1}, x_n - m_1(x_n) - x_{n+1} + m_1(x_{n+1})).
\end{aligned} \tag{3.6}
$$

By properties (i) and (iii) of $b(\cdot,\cdot)$ this implies

$$
\begin{aligned}
&\|x_n - x_{n+1} - m_1(x_n) + m_1(x_{n+1})\|_1^2 \\
&\quad \le \|x_{n-1} - x_n - m_1(x_n) + m_1(x_{n+1})\|_1 \cdot \|x_n - x_{n+1} - m_1(x_n) + m_1(x_{n+1})\|_1 \\
&\qquad + [\|x_{n-1} - x_n - \rho_1(N_1(u_{n-1}^1, v_{n-1}^1) - N_1(u_n^1, v_n^1))\|_1 + \|x_{n-1} - x_n\|_1 \\
&\qquad + \rho_1 \|w_{n-1}^1 - w_n^1\|_1] \cdot \|\eta_1(m_1(x_n) + x_{n+1} - m_1(x_{n+1}), x_n)\|_1 \\
&\qquad + \rho_1\gamma_1 \|x_{n-1} - x_n\|_1 \|x_n - x_{n+1} - m_1(x_n) + m_1(x_{n+1})\|_1.
\end{aligned} \tag{3.7}
$$

So, by Algorithm 2.1 and condition (6) we have

$$
\begin{aligned}
&\|x_n - x_{n+1}\|_1 \\
&\quad \le \|x_{n-1} - x_n\|_1 + 2\|m_1(x_n) - m_1(x_{n+1})\|_1 \\
&\qquad + \varepsilon_1 \Bigl[\|x_{n-1} - x_n - \rho_1(N_1(u_{n-1}^1, v_{n-1}^1) - N_1(u_n^1, v_n^1))\|_1 + \|x_{n-1} - x_n\|_1 \\
&\qquad + \rho_1 \Bigl(1 + \frac{1}{n}\Bigr) H(F_1(x_{n-1}, y_{n-1}), F_1(x_n, y_n)) \Bigr] + \rho_1\gamma_1 \|x_{n-1} - x_n\|_1.
\end{aligned} \tag{3.8}
$$

By conditions (1), (3), and (4) and Algorithm 2.1 we have

$$
\begin{aligned}
&\|x_{n-1} - x_n - \rho_1(N_1(u_{n-1}^1, v_{n-1}^1) - N_1(u_n^1, v_n^1))\|_1^2 \\
&\quad = \|x_{n-1} - x_n\|_1^2 - 2\rho_1 \langle N_1(u_{n-1}^1, v_{n-1}^1) - N_1(u_n^1, v_n^1), x_{n-1} - x_n\rangle_1 \\
&\qquad + \rho_1^2 \|N_1(u_{n-1}^1, v_{n-1}^1) - N_1(u_n^1, v_n^1)\|_1^2 \\
&\quad \le (1 - 2\rho_1\xi_1)\|x_{n-1} - x_n\|_1^2 + \rho_1^2 [\alpha_1 \|u_{n-1}^1 - v_n^1\|_1 + \beta_1 \|v_{n-1}^1 - v_n^1\|_1]^2 \\
&\quad \le \Bigl[1 - 2\rho_1\xi_1 + \rho_1^2(\alpha_1\tau_1 + \beta_1\omega_1)^2 \Bigl(1 + \frac{1}{n}\Bigr)^2\Bigr] \|x_{n-1} - x_n\|_1^2.
\end{aligned} \tag{3.9}
$$

It follows from (3.8) and (3.9) and from conditions (2) and (5) that

$$
\begin{aligned}
\|x_n - x_{n+1}\|_1 \le \frac{1}{1 - 2\delta_1} \Biggl\{ \Biggl[1 + \rho_1\gamma_1 + \varepsilon_1 \Biggl(1 + \sqrt{1 - 2\rho_1\xi_1 + \rho_1^2(\alpha_1\tau_1 + \beta_1\omega_1)^2 \Bigl(1 + \frac{1}{n}\Bigr)^2} \\
+ \rho_1 l_1 \Bigl(1 + \frac{1}{n}\Bigr) \Biggr) \Biggr] \|x_{n-1} - x_n\|_1 + \varepsilon_1\rho_1 k_1 \Bigl(1 + \frac{1}{n}\Bigr) \|y_{n-1} - y_n\|_2 \Biggr\}.
\end{aligned} \tag{3.10}
$$

Second, it follows from (2.4) in Algorithm 2.1 that, for any $h_2 \in K_2(y_n)$,

$$
\begin{aligned}
\langle y_n, h_2 - y_n\rangle_2 \ge \langle y_{n-1}, h_2 - y_n\rangle_2 - \rho_2 \langle N_2(u_{n-1}^2, v_{n-1}^2) + w_{n-1}^2, \eta_2(h_2, y_n)\rangle_2 \\
+ \rho_2 [b_2(y_{n-1}, y_n) - b_2(y_{n-1}, h_2)]
\end{aligned} \tag{3.11}
$$

and, for any $h_2 \in K_2(y_{n+1})$,

$$\begin{aligned}\langle y_{n+1}, h_2 - y_{n+1}\rangle_2 \geq{}& \langle y_n, h_2 - y_{n+1}\rangle_2 - \rho_2\bigl\langle N_2\bigl(u_n^2, v_n^2\bigr) + w_n^2, \eta_2(h_2, y_{n+1})\bigr\rangle_2 \\ &+ \rho_2\bigl[b_2(y_n, y_{n+1}) - b_2(y_n, h_2)\bigr].\end{aligned} \tag{3.12}$$

Adding $\langle -m_2(y_n), h_2 - y_n\rangle_2$ to the two sides of inequality (3.11) and then taking $h_2 = m_2(y_n) + y_{n+1} - m_2(y_{n+1}) \in K_2(y_n)$, we get

$$\begin{aligned}&\bigl\langle y_n - m_2(y_n), m_2(y_n) + y_{n+1} - m_2(y_{n+1}) - y_n\bigr\rangle_2 \\ &\quad\geq \bigl\langle y_{n-1} - m_2(y_n), m_2(y_n) + y_{n+1} - m_2(y_{n+1}) - y_n\bigr\rangle_2 \\ &\qquad - \rho_2\bigl\langle N_2\bigl(u_{n-1}^2, v_{n-1}^2\bigr) + w_{n-1}^2, \eta_2\bigl(m_2(y_n) + y_{n+1} - m_2(y_{n+1}), y_n\bigr)\bigr\rangle_2 \\ &\qquad + \rho_2\bigl[b_2(y_{n-1}, y_n) - b_2\bigl(y_{n-1}, m_2(y_n) + y_{n+1} - m_2(y_{n+1})\bigr)\bigr].\end{aligned} \tag{3.13}$$

Adding $\langle -m_2(y_{n+1}), h_2 - y_{n+1}\rangle_2$ to the two sides of inequality (3.12) and then taking $h_2 = m_2(y_{n+1}) + y_n - m_2(y_n) \in K_2(y_{n+1})$, we get

$$\begin{aligned}&\bigl\langle y_{n+1} - m_2(y_{n+1}), m_2(y_{n+1}) + y_n - m_2(y_n) - y_{n+1}\bigr\rangle_2 \\ &\quad\geq \bigl\langle y_n - m_2(y_{n+1}), m_2(y_{n+1}) + y_n - m_2(y_n) - y_{n+1}\bigr\rangle_2 \\ &\qquad - \rho_2\bigl\langle N_2\bigl(u_n^2, v_n^2\bigr) + w_n^2, \eta_2\bigl(m_2(y_{n+1}) + y_n - m_2(y_n), y_{n+1}\bigr)\bigr\rangle_2 \\ &\qquad + \rho_2\bigl[b_2(y_n, y_{n+1}) - b_2\bigl(y_n, m_2(y_{n+1}) + y_n - m_2(y_n)\bigr)\bigr].\end{aligned} \tag{3.14}$$

Then repeating the method, we have

$$\begin{aligned}\|y_n - y_{n+1}\|_2 \leq{}& \frac{1}{1-2\delta_2}\Biggl\{\Biggl[1 + \rho_2\gamma_2 + \varepsilon_2\Biggl(1 + \sqrt{1 - 2\rho_2\xi_2 + \rho_2^2(\alpha_2\tau_2 + \beta_2\omega_2)^2\biggl(1 + \frac{1}{n}\biggr)^2} \\ &+ \rho_2 k_2\biggl(1 + \frac{1}{n}\biggr)\Biggr)\Biggr]\|y_{n-1} - y_n\|_2 + \varepsilon_2\rho_2 l_2\biggl(1 + \frac{1}{n}\biggr)\|x_{n-1} - x_n\|_1\Biggr\}.\end{aligned} \tag{3.15}$$

From (3.10) and (3.15) we have

$$\begin{aligned}&\|x_n - x_{n+1}\|_1 + \|y_n - y_{n+1}\|_2 \\ &\quad\leq \Biggl\{\frac{1}{1-2\delta_1}\Biggl[1 + \rho_1\gamma_1 + \varepsilon_1\Biggl(1 + \sqrt{1 - 2\rho_1\xi_1 + \rho_1^2(\alpha_1\tau_1 + \beta_1\omega_1)^2\biggl(1 + \frac{1}{n}\biggr)^2} \\ &\qquad + \rho_1 l_1\biggl(1 + \frac{1}{n}\biggr)\Biggr)\Biggr] + \frac{1}{1-2\delta_2}\varepsilon_2\rho_2 l_2\biggl(1 + \frac{1}{n}\biggr)\Biggr\}\|x_{n-1} - x_n\|_1 \\ &\qquad + \Biggl\{\frac{1}{1-2\delta_2}\Biggl[1 + \rho_2\gamma_2 + \varepsilon_2\Biggl(1 + \sqrt{1 - 2\rho_2\xi_2 + \rho_2^2(\alpha_2\tau_2 + \beta_2\omega_2)^2\biggl(1 + \frac{1}{n}\biggr)^2} \\ &\qquad + \rho_2 k_2\biggl(1 + \frac{1}{n}\biggr)\Biggr)\Biggr] + \frac{1}{1-2\delta_1}\varepsilon_1\rho_1 k_1\biggl(1 + \frac{1}{n}\biggr)\Biggr\}\|y_{n-1} - y_n\|_1 \\ &\quad\leq \max\bigl\{\theta_n^1, \theta_n^2\bigr\}\bigl(\|x_{n-1} - x_n\|_1 + \|y_{n-1} - y_n\|_2\bigr),\end{aligned} \tag{3.16}$$

where

$$\theta_n^1 = \frac{1}{1-2\delta_1}\left[1+\rho_1\gamma_1+\varepsilon_1\left(1+\sqrt{1-2\rho_1\xi_1+\rho_1^2(\alpha_1\tau_1+\beta_1\omega_1)^2\left(1+\frac{1}{n}\right)^2}\right.\right.$$
$$\left.\left.+\rho_1 l_1\left(1+\frac{1}{n}\right)\right)\right]+\frac{1}{1-2\delta_2}\varepsilon_2\rho_2 l_2\left(1+\frac{1}{n}\right),$$
$$\theta_n^2 = \frac{1}{1-2\delta_2}\left[1+\rho_2\gamma_2+\varepsilon_2\left(1+\sqrt{1-2\rho_2\xi_2+\rho_2^2(\alpha_2\tau_2+\beta_2\omega_2)^2\left(1+\frac{1}{n}\right)^2}\right.\right.$$
$$\left.\left.+\rho_2 k_2\left(1+\frac{1}{n}\right)\right)\right]+\frac{1}{1-2\delta_1}\varepsilon_1\rho_1 k_1\left(1+\frac{1}{n}\right).$$

Letting

$$\theta_1 = \frac{1}{1-2\delta_1}\left[1+\rho_1\gamma_1+\varepsilon_1\left(1+\sqrt{1-2\rho_1\xi_1+\rho_1^2(\alpha_1\tau_1+\beta_1\omega_1)^2}+\rho_1 l_1\right)\right]+\frac{1}{1-2\delta_2}\varepsilon_2\rho_2 l_2$$

and

$$\theta_2 = \frac{1}{1-2\delta_2}\left[1+\rho_2\gamma_2+\varepsilon_2\left(1+\sqrt{1-2\rho_2\xi_2+\rho_2^2(\alpha_2\tau_2+\beta_2\omega_2)^2}+\rho_2 k_2\right)\right]+\frac{1}{1-2\delta_1}\varepsilon_1\rho_1 k_1,$$

we can see that $\theta_n^1 \to \theta_1$ and $\theta_n^2 \to \theta_2$ as $n \to \infty$. Now, by condition (3.1) we have $\max\{\theta_1,\theta_2\} < 1$. Therefore, it follows from (3.16) that $\{(x_n,y_n)\}$ is a Cauchy sequence in $H_1 \times H_2$. Let $(x_n,y_n) \to (x,y) \in H_1 \times H_2$ as $n \to \infty$. Since A_i, B_i, and F_i are all H-Lipschitz continuous, by (2.5) and by conditions (2), (3), and (4) we have

$$\begin{aligned}
\|u_{n+1}^1-u_n^1\|_1 &\le \left(1+\frac{1}{n+1}\right)H(A_1x_{n+1},A_1x_n) \le \left(1+\frac{1}{n+1}\right)\tau_1\|x_{n+1}-x_n\|_1;\\
\|v_{n+1}^1-v_n^1\|_1 &\le \left(1+\frac{1}{n+1}\right)H(B_1x_{n+1},B_1x_n) \le \left(1+\frac{1}{n+1}\right)\omega_1\|x_{n+1}-x_n\|_1;\\
\|u_{n+1}^2-u_n^2\|_2 &\le \left(1+\frac{1}{n+1}\right)H(A_2y_{n+1},A_2y_n) \le \left(1+\frac{1}{n+1}\right)\tau_2\|y_{n+1}-y_n\|_2;\\
\|v_{n+1}^2-v_n^2\|_2 &\le \left(1+\frac{1}{n+1}\right)H(B_2y_{n+1},B_2y_n) \le \left(1+\frac{1}{n+1}\right)\omega_2\|y_{n+1}-y_n\|_2;\\
\|w_{n+1}^i-w_n^i\|_i &\le \left(1+\frac{1}{n+1}\right)H\big(F_i(x_{n+1},y_{n+1}),F_i(x_n,y_n)\big)\\
&\le \left(1+\frac{1}{n+1}\right)\big(l_i\|x_{n+1}-x_n\|_1+k_i\|y_{n+1}-y_n\|_2\big).
\end{aligned}$$

Therefore, $\{u_n^i\}$, $\{v_n^i\}$, and $\{w_n^i\}$ ($i \in I$) are also Cauchy sequences. Let $u_n^i \to u_i$, $v_n^i \to v_i$, and $w_n^i \to w_i$ ($i \in I$) as $n \to \infty$. Since $u_n^1 \in A_1x_n$, we have

$$\begin{aligned}
d(u_1,A_1x) &\le \|u_1-u_n^1\|_1 + d\big(u_n^1,A_1x\big)\\
&\le \|u_1-u_n^1\|_1 + H(A_1x_n,A_1x)\\
&\le \|u_1-u_n^1\|_1 + \tau_1\|x_n-x\|_1 \to 0 \quad (n\to\infty).
\end{aligned}$$

Hence, we conclude that $u_1 \in A_1x$. Similarly, we can obtain $u_2 \in A_2y$, $v_1 \in B_1x$, $v_2 \in B_2y$, $w_i \in F_i(x,y)$, $\forall i \in I$.

By Theorem 2.1 we may assume that $(p_1,p_2) \in K_1(x) \times K_2(y)$ is the unique solution of the system of auxiliary variational inequalities (2.1), that is,

$$\begin{aligned}\langle p_1,h_1-p_1\rangle_1 \geq{}& \langle x,h_1-p_1\rangle_1 - \rho_1\bigl\langle N_1(u_1,v_1)+w_1,\eta_1(h_1,p_1)\bigr\rangle_1 \\ &+ \rho_1\bigl[b_1(x,p_1)-b_1(x,h_1)\bigr], \quad \forall h_1 \in K_1(x), \end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\langle p_2,h_2-p_2\rangle_2 \geq{}& \langle y,h_2-p_2\rangle_2 - \rho_2\bigl\langle N_2(u_2,v_2)+w_2,\eta_2(h_2,p_2)\bigr\rangle_2 \\ &+ \rho_2\bigl[b_2(y,p_2)-b_2(y,h_2)\bigr], \quad \forall h_2 \in K_2(y). \end{aligned} \tag{3.18}$$

Now, we prove $p_1 = x$ and $p_2 = y$. By applying (3.2), (3.17), and a similar argument as in proving (3.8), we can easily get

$$\begin{aligned}&\|x_n-p_1\|_1 \\ &\quad\leq \|x_{n-1}-x\|_1 + 2\bigl\|m_1(x_n)-m_1(p_1)\bigr\|_1 \\ &\qquad + \rho_1\varepsilon_1\bigl[\bigl\|N_1\bigl(u^1_{n-1},v^1_{n-1}\bigr)-N_1\bigl(u^1_n,v^1_n\bigr)\bigr\|_1 + \bigl\|w^1_{n-1}-w^1_n\bigr\|_1\bigr] + \rho_1\gamma_1\|x_{n-1}-x\|_1. \end{aligned} \tag{3.19}$$

Since $x_n \to x$, $w^1_n \to w_1$, and $N_1(u^1_n,v^1_n) \to N_1(u_1,v_1)$, from (3.19) we have $x_n \to p_1$. Therefore, we have $p_1 = x$.

Using a similar method, we have $p_2 = y$. Finally, taking them into (3.17) and (3.18), we have

$$\bigl\langle N_1(u_1,v_1)+w_1,\eta_1(h_1,x)\bigr\rangle_1 - b_1(x,x) + b_1(x,h_1) \geq 0, \quad \forall h_1 \in K_1(x),$$

and

$$\bigl\langle N_2(u_2,v_2)+w_2,\eta_2(h_2,y)\bigr\rangle_2 - b_2(y,y) + b_2(y,h_2) \geq 0, \quad \forall h_2 \in K_2(y).$$

This completes the proof. □

Remark 3.1 Theorem 3.1 answers positively the open problem raised by Noor [1, 2] in the setting of a more general system of generalized nonlinear mixed quasi-variational inequalities. We emphasize that A and B may not be compact-valued mappings.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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