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Composition and multiplication operators between Orlicz function spaces

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Abstract

Composition operators and multiplication operators between two Orlicz function spaces are investigated. First, necessary and sufficient conditions for their continuity are presented in several forms. It is shown that, in general, the Radon-Nikodým derivative $\frac{d(\mu\circ\tau^{-1})}{d\mu}(s)$ need not belong to $L^\infty(\Omega)$ to guarantee the continuity of the composition operator $c_\tau x(t) = x(\tau(t))$ from $L^\Phi(\Omega)$ into $L^\Psi(\Omega)$. Next, the problem of compactness of these operators is considered. We apply a compactness criterion in Orlicz spaces which involves compactness with respect to the topology of local convergence in measure and equi-absolute continuity in norm of all the elements of the set under consideration. In connection with this, we state some sufficient conditions for equi-absolute continuity of the composition operator c_τ and the multiplication operator M_w from one Orlicz space into another. Also the problem of necessary conditions is discussed.

MSC: Primary 47B33; secondary 46E30

Keywords: composition operator; multiplication operator; Orlicz spaces

1 Introduction

Since the early 1930s composition operators have been a subject of study of many mathematicians or physicists. At the beginning they were used to solve problems in mathematical physics and classical mechanics [1, 2], or to study ergodic transformations [3, 4]. Up until now many Ph.D. theses have been defended (for instance: Boyd [5], Gupta [6], Ridge [7], Schwartz [8], Singh [9], Swanton [10], Veluchamy [11]), numerous books published [12, 13], and innumerable papers printed on the composition operator or the weighted composition operator in various function spaces, *e.g.*, L^p spaces ([13–23], and others), Orlicz spaces ([24–27], and others), Musielak-Orlicz spaces [28, 29], Musielak-Orlicz spaces of Bochner type ([30] or [31]), Orlicz-Lorentz spaces [32–37], Hilbert spaces [38–40] and many other types of spaces (for instance: [41–43]). The multiplication operator has also been a subject of research of many mathematicians (see for instance: [44–49]). For more details as regards the historical background we refer the reader to [13].

Let (Ω, Σ, μ) be a non-atomic, σ-finite and complete measure space and let $\tau : \Omega \to \Omega$ be a measurable function, *i.e.*, a mapping such that $\tau^{-1}(A) \in \Sigma$ if and only if $A \in \Sigma$ for any $A \subset \Omega$, where $\tau^{-1}(A)$ is the counterimage of A. In the whole paper we will assume that τ is non-singular, that is, $\mu(\tau^{-1}(A)) = 0$ provided $\mu(A) = 0$. The last assumption guarantees that the measure $\mu \circ \tau^{-1}$ defined for any $A \in \Sigma$ by the formula

$$\mu \circ \tau^{-1}(A) = \mu\big(\tau^{-1}(A)\big)$$

is absolutely continuous with respect to the measure μ (what is usually denoted by $\mu \circ \tau^{-1} \ll \mu$). Then the Radon-Nikodým theorem implies the existence of a non-negative locally integrable function h_τ on Ω such that $\mu \circ \tau^{-1}(A) = \int_A h_\tau(s)\,d\mu(s)$ for any $A \in \Sigma$. The function $h_\tau(s) = \frac{d(\mu\circ\tau^{-1})}{d\mu}(s)$ is called the Radon-Nikodým derivative of the measure $\mu \circ \tau^{-1}$ with respect to the measure μ. Let us remark that even if the Radon-Nikodým derivative is unbounded, the corresponding composition operator acting from an Orlicz space onto itself can still be continuous (see [50]).

Let $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ be the space of all (abstract classes of) Σ-measurable functions from Ω into $\mathbb{R}$ with respect to the equivalence relation: $x \sim y$ if and only if $x(t) = y(t)$ for μ-a.e. $t \in \Omega$.

With the help of a non-singular and measurable mapping $\tau : \Omega \to \Omega$ one defines on L^0 the composition operator

$$(c_\tau x)(t) := x\big(\tau(t)\big)$$

for any $t \in \Omega$ and any $x \in L^0(\Omega)$.

Let $w \in L^0(\Omega, \Sigma, \mu)$ be a strictly positive function. We define the multiplication operator $M_w : L^0 \to L^0$ by the formula

$$(M_w x)(t) := w(t)x(t)$$

for any $t \in \Omega$ and any $x \in L^0(\Omega)$.

It is obvious that $c_\tau x \in L^0(\Omega)$ and $M_w x \in L^0(\Omega)$ if $x \in L^0(\Omega)$.

Remark 1.1 We do not assume, if not specifically stated otherwise, that the mapping τ is a surjection, *i.e.*, $\tau(\Omega) = \Omega$.

A function $\Phi : \mathbb{R} \to \mathbb{R}_+ = [0, \infty)$ is said to be an Orlicz function if Φ is convex, even, continuous, vanishing only at 0.

The complementary function in the sense of Young to an Orlicz function Φ is defined to be the function $\Phi^* : [0, \infty) \to [0, \infty]$ such that $\Phi^*(u) = \sup_{v>0}\{uv - \Phi(v)\}$.

A function φ from $\Omega \times \mathbb{R}$ into $\mathbb{R}_+$ such that $\varphi(t, \cdot)$ is an Orlicz function for μ-a.e. $t \in \Omega$ and $\varphi(\cdot, u)$ is a Σ-measurable function for every $u \in \mathbb{R}$ is called a generalized Orlicz function or a Musielak-Orlicz function. The Musielak-Orlicz space $L^\varphi = L^\varphi(\Omega, \Sigma, \mu)$ is the space of all (equivalence classes of) Σ-measurable functions $x : \Omega \to \mathbb{R}$ such that

$$I_\varphi(\lambda x) = \int_\Omega \varphi\big(t, \lambda x(t)\big)\,d\mu < \infty$$

for some $\lambda > 0$ depending on x. The Musielak-Orlicz space equipped with the Luxemburg norm

$$\|x\|_\varphi = \inf\left\{\lambda > 0 : I_\varphi\left(\frac{x}{\lambda}\right) \le 1\right\}$$

is a Banach space (*cf.* [51, 52], and in the case of Orlicz spaces also [53–55]). It is obvious that Orlicz functions are Musielak-Orlicz functions, and consequently, Orlicz spaces are Musielak-Orlicz spaces. For instance, an Orlicz weighted space $L^{\Phi}_h(\Omega)$ is the Musielak-Orlicz space $L^{\varphi}(\Omega)$ generated by $\varphi(t,u) = \Phi(u)h(t)$ for any $t \in \Omega$ and any $u \in \mathbb{R}$.

Throughout the paper, we will make use of Ishii's theorem from [56].

Theorem 1.1 *For Musielak-Orlicz spaces $L^{\varphi}(\Omega)$ and $L^{\psi}(\Omega)$ over a non-atomic measure space $\Omega = (\Omega, \Sigma, \mu)$, the inclusion $L^{\varphi}(\Omega) \subset L^{\psi}(\Omega)$ holds if and only if there exist $k, K > 0$ and $c(\cdot) \in L^1(\Omega)$ such that*

$$\psi(t, k\xi) \leq K\varphi(t,\xi) + c(t)$$

for all $\xi \geq 0$ and a.e. $t \in \Omega$.

For any $A \in \Sigma$, by $I_\Phi(x, A)$ we mean the value of the modular I_Φ at x in the Orlicz space $L^\Phi(A)$ generated by the Orlicz function Φ over the measure space $(A, \Sigma \cap A, \mu|_A)$. In the case when $A = \Omega$, we will write $I_\Phi(x)$ instead of $I_\Phi(x, \Omega)$.

2 Continuity of the composition operator c_τ from L^Φ into L^Ψ and from L^Φ onto L^Ψ

We are interested in finding necessary and sufficient conditions for the continuity of the composition operator c_τ from the Orlicz space $L^\Phi(\Omega) = L^\Phi(\Omega, \Sigma, \mu)$ equipped with the Luxemburg norm into the Orlicz space $L^\Psi(\Omega) = L^\Psi(\Omega, \Sigma, \mu)$ endowed with the corresponding Luxemburg norm. The following fact will be very helpful.

Fact 2.1 For an arbitrary function $x \in L^0(\Omega)$, we have $c_\tau x \in L^\Psi(\Omega)$ if and only if $x \in L^\Psi_h(\tau(\Omega))$, where $L^\Psi_h(\tau(\Omega)) = L^\Psi_h(\tau(\Omega), \Sigma \cap \tau(\Omega), \mu|_{\Sigma \cap \tau(\Omega)})$ is a weighted Orlicz space with the weight function $h(s) = \frac{d\mu \circ \tau^{-1}}{d\mu}(s)$.

Proof For any $x \in L^0(\Omega)$, we have

$$\begin{aligned} I_\Psi(c_\tau x, \Omega) &= \int_\Omega \Psi\bigl(c_\tau x(t)\bigr)\, d\mu(t) \\ &= \int_\Omega \Psi\bigl(x\bigl(\tau(t)\bigr)\bigr)\, d\mu(t) \\ &= \int_{\tau(\Omega)} \Psi\bigl(x(s)\bigr)\, d\mu \circ \tau^{-1}(s) \\ &= \int_{\tau(\Omega)} \Psi\bigl(x(s)\bigr) h(s)\, d\mu(s) \\ &= I_{\Psi,h}\bigl(x, \tau(\Omega)\bigr). \end{aligned}$$

□

Theorem 2.1 *If the quadruple Φ, Ψ, h, τ satisfies the condition*

$$\underset{K>1}{\exists} \underset{\substack{A \in \Sigma \cap \tau(\Omega) \\ \mu(A)=0}}{\exists} \underset{g \in L^1_+(\tau(\Omega))}{\exists} \underset{s \in \tau(\Omega)\setminus A}{\forall} \underset{u \geq 0}{\forall} \Psi(u)h(s) \leq \Phi(Ku) + g(s), \tag{1}$$

or, equivalently, $L^\Phi(\tau(\Omega)) \subset L^\Psi_h(\tau(\Omega))$, then the composition operator c_τ acts continuously from $L^\Phi(\Omega)$ into $L^\Psi(\Omega)$.

Moreover, if $\mu(\Omega\setminus\tau(\Omega)) = 0$, *then condition* (1) *is necessary for the continuity of the composition operator* c_τ *from* $L^\Phi(\Omega)$ *into* $L^\Psi(\Omega)$.

Proof We will show that if condition (1) is satisfied then $c_\tau x \in L^\Psi(\Omega)$ whenever $x \in L^\Phi(\Omega)$, *i.e.*, $x \in L^\Psi_h(\tau(\Omega))$ whenever $x \in L^\Phi(\Omega)$ (see Fact 2.1).

Assume that $x \in L^\Phi(\Omega)$. Applying condition (1) and Fact 2.1, we obtain

$$\begin{aligned} I_\Psi\left(\frac{c_\tau x}{K(1+\|g\|_{L^1(\tau(\Omega))})\|x\|_\Phi},\Omega\right) &\le \frac{1}{1+\|g\|_{L^1(\tau(\Omega))}} I_\Psi\left(\frac{c_\tau x}{K\|x\|_\Phi},\Omega\right) \\ &= \frac{1}{1+\|g\|_{L^1(\tau(\Omega))}} I_{\Psi,h}\left(\frac{x}{K\|x\|_\Phi},\tau(\Omega)\right) \\ &\le \frac{I_\Phi(\frac{x}{\|x\|_\Phi},\tau(\Omega)) + \|g\|_{L^1(\tau(\Omega))}}{1+\|g\|_{L^1(\tau(\Omega))}} \\ &\le \frac{1}{1+\|g\|_{L^1(\tau(\Omega))}}\bigl(1+\|g\|_{L^1(\tau(\Omega))}\bigr) = 1, \end{aligned}$$

which shows that c_τ acts from $L^\Phi(\Omega)$ into $L^\Psi(\Omega)$ and

$$\|c_\tau x\|_\Psi \le K\bigl(1+\|g\|_{L^1(\tau(\Omega))}\bigr)\|x\|_\Phi,$$

which finishes the proof of the first part of the theorem.

Now assume that $\mu(\Omega\setminus\tau(\Omega)) = 0$. If the inclusion in the assumption of the theorem fails to hold, then there exists a function x belonging to $L^\Phi(\tau(\Omega)) = L^\Phi(\Omega)$ but not belonging to $L^\Psi_h(\tau(\Omega))$. In virtue of Fact 2.1, we obtain $x \in L^\Phi(\Omega)$ and, simultaneously, $c_\tau x \notin L^\Psi(\Omega)$, hence c_τ does not even act from $L^\Phi(\Omega)$ into $L^\Psi(\Omega)$. □

The preceding theorem can be formulated in a different language, which in some situations might be more useful.

Theorem 2.2 *The composition operator* $c_\tau : L^\Phi(\Omega) \to L^\Psi(\Omega)$ *is continuous if the following simple condition is satisfied*:

$$\int_{\tau(\Omega)} \chi\bigl(h(s)\bigr)\,d\mu(s) < \infty, \tag{2}$$

where $h = \frac{d\mu\circ\tau^{-1}}{d\mu}$ *and* χ *is the function complementary in the sense of Young to the function* $\Phi\circ K\Psi^{-1}$, *with* K *being the constant from condition* (1) *of Theorem* 2.1. *If* $\mu(\Omega\setminus\tau(\Omega)) = 0$, *then condition* (2) *is necessary for the continuity of* c_τ.

Proof It is not too difficult to see that Φ, Ψ, and h satisfy condition (1) if and only if

$$\sup_{u\ge 0}\bigl[\Psi(u)h(s) - \Phi(Ku)\bigr] \in L^1_+\bigl(\tau(\Omega)\bigr).$$

But

$$\begin{aligned} \sup_{u\ge 0}\bigl[\Psi(u)h(s) - \Phi(Ku)\bigr] &= \sup_{v\ge 0}\bigl[vh(s) - \Phi\bigl(K\Psi^{-1}(v)\bigr)\bigr] \\ &= \chi\bigl(h(s)\bigr). \end{aligned}$$

□

Remark 2.1 Notice that if $b(\chi) := \sup\{u \geq 0 : \chi(u) < \infty\} < \infty$ then condition (2) implies that $\|h\|_{L^\infty(\tau(\Omega))} \leq b(\chi) < \infty$, that is, the Radon-Nikodým derivative $h = \frac{d\mu\circ\tau^{-1}}{}d\mu$ is essentially bounded. Moreover, it is easy to see that the integral (2) can be finite for some $h \notin L^\infty(\tau(\Omega))$ if and only if the function χ has only finite values (for the case when $\Phi = \Psi$ see [50]; the proof in the case when $\Phi \neq \Psi$ is similar).

The next theorem states a necessary and sufficient condition in order that χ is such a function.

Theorem 2.3 *The function* $\chi = (\Phi \circ K\Psi^{-1})^*$, *with* $K > 1$, *assumes only finite values* (*i.e.*, $b(\chi) = \infty$) *if and only if* $\liminf_{t\to\infty} \frac{\Phi(Kt)}{\Psi(t)} = \infty$.

Proof Sufficiency. Let $\liminf_{t\to\infty} \frac{\Phi(Kt)}{\Psi(t)} = \infty$. Then $\lim_{t\to\infty} \frac{\Phi(Kt)}{\Psi(t)} = \infty$ and so $\lim_{t\to\infty} \frac{\Phi(K\Psi^{-1}(t))}{t} = \infty$. Take an arbitrary $\nu > 0$. In virtue of the last equality, there exists $u_\nu > 0$ such that for all $u \geq u_\nu$ the inequality

$$\frac{\Phi(K\Psi^{-1}(u))}{u} \geq \nu$$

holds. Hence

$$\begin{aligned}\chi(\nu) := \sup_{u\geq 0}\bigl\{u\nu - \Phi\bigl(K\Psi^{-1}(u)\bigr)\bigr\} &= \sup_{u\geq 0}\left\{u\left(\nu - \frac{\Phi(K\Psi^{-1}(u))}{u}\right)\right\} \\ &= \sup_{u\in[0,u_\nu]}\bigl\{u\nu - \Phi\bigl(K\Psi^{-1}(u)\bigr)\bigr\} < \infty\end{aligned}$$

as a supremum of a continuous function on a compact interval.

Necessity. Assume that $\liminf_{t\to\infty} \frac{\Phi(Kt)}{\Psi(t)} < \infty$ for $K > 1$. Then $\liminf_{t\to\infty} \frac{\Phi(K\Psi^{-1}(t))}{t} < \infty$ for $K > 1$, and so there exist $c > 0$ and a sequence of positive numbers $(u_n)_{n=1}^\infty$ such that $u_n \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} \frac{\Phi(K\Psi^{-1}(u_n))}{u_n} = c$. Consequently, $c' \in (c,\infty)$ can be found such that for $n \in \mathbb{N}$ large enough we have

$$\Phi\bigl(K\bigl(\Psi^{-1}(u_n)\bigr)\bigr) \leq c' u_n.$$

Therefore, taking $\nu > c'$, we get

$$u_n\nu - \Phi\bigl(K\Psi^{-1}(u_n)\bigr) \geq u_n\nu - c'u_n = u_n\bigl(\nu - c'\bigr)$$

for $n \in \mathbb{N}$ large enough, whence

$$\begin{aligned}\sup_{u\geq 0}\bigl\{u\nu - \Phi\bigl(K\Psi^{-1}(u)\bigr)\bigr\} &\geq \sup_{n\in\mathbb{N}}\bigl\{u_n\nu - \Phi\bigl(K\Psi^{-1}(u_n)\bigr)\bigr\} \\ &\geq \bigl(\nu - c'\bigr)\sup_{n\in\mathbb{N}} u_n = \infty,\end{aligned}$$

which finishes the proof of the theorem. □

Now we show that if the function χ from Theorem 2.2 assumes only finite values, *i.e.*, $b(\chi) = \infty$, then it may happen that the composition operator c_τ from $L^\Phi(\Omega)$ into $L^\Psi(\Omega)$ is continuous despite the fact that $h = \frac{d\mu\circ\tau^{-1}}{d\mu} \notin L^\infty(\tau(\Omega))$.

Example 2.1 Let $\Omega = (0,1]$, (Ω, Σ, μ) be the Lebesgue measure space and $A_n = (\frac{1}{n+1}, \frac{1}{n}]$ for every $n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} A_n = \Omega$. Let χ be the function from Theorem 2.2, let $b(\chi) = \infty$ and $u_n > 0$ be such that $\chi(u_n) = \sqrt{n}$ for every $n \in \mathbb{N}$. Since $u_n \nearrow \infty$ as $n \to \infty$, there exists $m \in \mathbb{N}$ such that $u_n \geq 1$ for any $n > m$. Define $\tau : (0,1] \to (0,1]$ by the formula

$$\tau(t) = \begin{cases} t, & \text{for } t \in A_n \text{ with } n \leq m, \\ \frac{t}{u_n}, & \text{for } t \in A_n, \text{ where } n > m. \end{cases}$$

Let us denote $v_n = 1$ if $n \leq m$ and $v_n = u_n$ if $n > m$. Then $\tau^{-1} : \tau(\Omega) \to \Omega$ is defined by the formula $\tau^{-1}(s) = v_n s$ for any $s \in \tau(A_n)$. Consequently, for any $A \in \Sigma \cap \tau(\Omega)$, we have

$$\begin{aligned} \mu \circ \tau^{-1}(A) := \mu\bigl(\tau^{-1}(A)\bigr) &= \mu\left(\bigcup_{n=1}^{\infty} \tau^{-1}\bigl(A \cap \tau(A_n)\bigr)\right) \\ &= \sum_{n=1}^{\infty} \mu\bigl(\tau^{-1}\bigl(A \cap \tau(A_n)\bigr)\bigr) \\ &= \sum_{n=1}^{\infty} \mu\left(v_n\left(A \cap \frac{1}{v_n}(A_n)\right)\right) \\ &= \sum_{n=1}^{\infty} \mu(v_n A \cap A_n). \end{aligned} \tag{3}$$

Let us define

$$h(s) = \sum_{n=1}^{\infty} v_n \mathbb{1}_{\frac{1}{u_n} A_n}(s). \tag{4}$$

Then, for any $A \in \Sigma \cap \tau(\Omega)$,

$$\begin{aligned} \int_A h(s)\, d\mu(s) = \int_{\bigcup_{n=1}^{\infty} A \cap \tau(A_n)} h(s)\, d\mu(s) &= \sum_{n=1}^{\infty} \int_{A \cap \frac{1}{v_n}(A_n)} v_n \, d\mu(s) \\ &= \sum_{n=1}^{\infty} v_n \mu\left(A \cap \frac{1}{v_n} A_n\right) = \sum_{n=1}^{\infty} \mu(v_n A \cap A_n). \end{aligned} \tag{5}$$

By inequalities (3) and (5), we get

$$\mu \circ \tau^{-1}(A) = \int_A h(s)\, d\mu(s), \quad \forall A \in \Sigma \cap \tau(\Omega),$$

which means that $h = \frac{d(\mu \circ \tau^{-1})}{d\mu}$. By formula (4) and the definition of v_n, we have

$$\begin{aligned} \int_{\tau(\Omega)} \chi\bigl(h(s)\bigr) &= \sum_{n=1}^{\infty} \chi(v_n) \mu\left(\frac{1}{v_n} A_n\right) \\ &= \sum_{n=1}^{m} \mu(A_n) + \sum_{n=m+1}^{\infty} \sqrt{n} \mu\left(\frac{1}{v_n} A_n\right) \end{aligned}$$

$$\leq 1 + \sum_{n=m+1}^{\infty} \sqrt{n}\mu(A_n)$$
$$\leq 1 + \sum_{n=m+1}^{\infty} \frac{\sqrt{n}}{n(n+1)}$$
$$< 1 + \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < \infty.$$

In virtue of Theorem 2.2, we conclude that the composition operator c_τ from $L^\Phi(\Omega)$ into $L^\Psi(\Omega)$ is continuous. However, since $u_n \to \infty$ as $n \to \infty$, we have $\|h\|_{L^\infty(\tau(\Omega))} = \infty$.

Theorem 2.4 *The composition operator c_τ acts continuously from $L^\Phi(\Omega,\Sigma,\mu)$ onto $L^\Psi(\Omega,\Sigma,\mu)$ if and only if $\mu(\Omega\setminus\tau(\Omega)) = 0$ and the following two conditions are jointly satisfied*:

(i) $\exists_{K>1} \exists_{\substack{A\in\Sigma\cap\Omega \\ \mu(A)=0}} \exists_{g\in L^1_+(\Omega)} \forall_{s\in\Omega\setminus A} \forall_{u\geq 0} \Psi(u)h(s) \leq \Phi(Ku) + g(s)$;

(ii) $\exists_{K>1} \exists_{\substack{A\in\Sigma\cap\Omega \\ \mu(A)=0}} \exists_{p\in L^1_+(\Omega)} \forall_{s\in\Omega\setminus A} \forall_{u\geq 0} \Phi(u) \leq \Psi(Ku)h(s) + p(s)$,

where $h(s) = \frac{d\mu\circ\tau^{-1}}{d\mu}(s)$ for μ-a.e. $s \in \Omega$.

Proof Obviously, the condition $\mu(\Omega\setminus\tau(\Omega)) = 0$ is necessary for $c_\tau(L^\Phi(\Omega)) = L^\Psi(\Omega)$, so in the further part of the proof we assume this condition holds. Therefore, by Fact 2.1, we know that c_τ acts from $L^\Phi(\Omega)$ onto $L^\Psi(\Omega)$ if and only if $L^\Phi(\Omega) = L^\Psi_h(\Omega)$. Equivalently, this holds if and only if we have two inclusions: $L^\Phi(\Omega) \subset L^\Psi_h(\Omega)$ and $L^\Psi_h(\Omega) \subset L^\Phi(\Omega)$. The first inclusion holds if and only if condition (i) is satisfied and the reverse inclusion holds if and only if condition (ii) is satisfied (see [51] and [56]), and this finishes the proof. □

It is interesting and profitable to observe that the preceding theorem can be written in the following form.

Theorem 2.5 *The composition operator c_τ acts continuously from $L^\Phi(\Omega)$ onto $L^\Psi(\Omega)$ if and only if $\mu(\Omega\setminus\tau(\Omega)) = 0$ and for some $K > 1$ the following two conditions are jointly satisfied*:

(1) $\int_\Omega \chi(h(s))\,d\mu(s) < \infty$;

(2) $\int_\Omega h(s)q(\frac{1}{h(s)})\,d\mu(s) < \infty$,

where χ is the function complementary in the sense of Young to the function $\Phi \circ K\Psi^{-1}$, q is the function complementary in the sense of Young to the function $\Psi \circ K\Phi^{-1}$, and $h(s) = \frac{d\mu\circ\tau^{-1}}{d\mu}(s)$ for μ-a.e. $s \in \Omega$.

Proof In the proof of Theorem 2.2 we already showed that condition (i) from Theorem 2.4 is equivalent to condition (1). So, the proof will be finished if we show that condition (2) is equivalent to condition (ii) from Theorem 2.4.

It is easy to see that condition (ii) is equivalent to the fact that $q \in L^1_+(\Omega)$, where

$$q(s) = \sup_{u\geq 0}\bigl[\Phi(u) - \Psi(Ku)h(s)\bigr]$$

for all $s \in \Omega$. Since $h(s) \neq 0$ for μ-a.e. $s \in \Omega$, we have

$$\begin{aligned} q(s) &= h(s) \sup_{u \geq 0} \left[\frac{\Phi(u)}{h(s)} - \Psi(Ku) \right] \\ &= h(s) \sup_{v \geq 0} \left[\frac{v}{h(s)} - \Psi\big(K\Phi^{-1}(v)\big) \right] \\ &= h(s) \big(\Psi \circ K\Phi^{-1}\big)^* \left(\frac{1}{h(s)} \right) \end{aligned}$$

for μ-a.e. $s \in \Omega$, where $(\Psi \circ K\Phi^{-1})^*$ is the complementary function to the function $\Psi \circ K\Phi^{-1}$, which finishes the proof. □

3 Continuity of the multiplication operator M_w from L^Φ into L^Ψ and from L^Φ onto L^Ψ

We will state criteria in order that M_w map $L^\Phi(\Omega, \Sigma, \mu)$ into $L^\Psi(\Omega, \Sigma, \mu)$, where Φ and Ψ are distinct Orlicz functions. Note that $M_w x \in L^\Psi(\Omega, \Sigma, \mu)$ means that there is $\lambda > 0$ such that $I_\Psi(\lambda w(t)x(t)) = \int_\Omega \Psi(\lambda w(t)x(t))\, d\mu(t) < \infty$. This is equivalent to the fact that $x \in L^{\Psi_w}(\Omega, \Sigma, \mu)$, where $L^{\Psi_w}(\Omega, \Sigma, \mu)$ is a Musielak-Orlicz space generated by the Musielak-Orlicz function $\Psi_w(t,u) := \Psi(w(t)u)$. Let us begin with the following.

Theorem 3.1 *The multiplication operator M_w maps $L^\Phi(\Omega, \Sigma, \mu)$ into $L^\Psi(\Omega, \Sigma, \mu)$ if and only if $\int_\Omega \chi_K(t,1)\, d\mu(t) < \infty$ for some $K > 1$, where $\chi_K(t,u)$ is, for fixed $t \in \Omega$, the function complementary in the sense of Young to the function $\Phi \circ \frac{K}{w(t)} \Psi^{-1}$ with respect to u.*

Proof The fact that $M_w : L^\Phi(\Omega, \Sigma, \mu) \to L^\Psi(\Omega, \Sigma, \mu)$ means that $L^\Phi(\Omega, \Sigma, \mu) \subset L^{\Psi_w}(\Omega, \Sigma, \mu)$, which, by Theorem 1.1, holds if and only if there are $A \in \Sigma$ with $\mu(A) = 0$, a constant $K > 1$, and a function $h \in L^1_+(\Omega, \Sigma, \mu)$ such that the inequality

$$\Psi\big(w(t)u\big) \leq \Phi(Ku) + h(t) \tag{6}$$

holds for all $t \in \Omega \setminus A$ and all $u \in \mathbb{R}$. It is easy to see that this is equivalent to the fact that for some $K > 1$ the function $\tilde{h}_K$ defined by the formula

$$\tilde{h}_K(t) = \sup_{u \in \mathbb{R}} \big[\Psi\big(w(t)u\big) - \Phi(Ku)\big] = \sup_{u \geq 0} \big[\Psi\big(w(t)u\big) - \Phi(Ku)\big] \tag{7}$$

is integrable over Ω. In fact, $\tilde{h}_K$ is the smallest function such that condition (6) holds with $\tilde{h}_K$ in place of h. Setting in (7) $u = \frac{\Psi^{-1}(v)}{w(t)}$, we get

$$\tilde{h}_K(t) = \sup_{v > 0} \left[v - \Phi\left(\frac{K}{w(t)} \Psi^{-1}(v) \right) \right] = \chi_K(t,1),$$

where $\chi_K(t,u)$ is the function complementary in the sense of Young to $\Phi \circ \frac{K}{w(t)} \Psi^{-1}$. Therefore, the fact that M_w maps continuously $L^\Phi(\Omega, \Sigma, \mu)$ into $L^\Psi(\Omega, \Sigma, \mu)$ is equivalent to the fact that $\int_\Omega \chi_K(t,1)\, d\mu(t) < \infty$. □

Theorem 3.2 *The multiplication operator M_w maps $L^\Phi(\Omega, \Sigma, \mu)$ onto the whole of $L^\Psi(\Omega, \Sigma, \mu)$ if and only if the following two conditions are jointly satisfied*:

(i) $\int_\Omega \chi_K(t,1)\,d\mu(t) < \infty$ *for some* $K > 1$, *where* $\chi_K(t,u)$ *is the function defined in Theorem* 3.1;
(ii) $\int_\Omega U_K(t,1)\,d\mu(t) < \infty$ *for some* $K > 1$, *where* $U_K(t,u)$ *is, for fixed* $t \in \Omega$, *the function complementary in the sense of Young, with respect to* u, *to the function* $(\Psi \circ Kw(\cdot)\Phi^{-1})(t,u) = \Psi(Kw(t)\Phi^{-1}(u))$.

Proof It is obvious that M_w maps $L^\Phi(\Omega,\Sigma,\mu)$ onto $L^\Psi(\Omega,\Sigma,\mu)$ if and only if $\mu(\Omega\setminus \operatorname{supp} w) = 0$ and $\{w(t)x(t) : x \in L^\Phi\} = L^\Psi(\Omega,\Sigma,\mu)$. The space on the left-hand side of the last equality is the Musielak-Orlicz space $L^{\Phi_w}(\Omega)$, where $\Phi_w(t,u) = \Phi(\frac{u}{w(t)})$. Observe that $L^{\Phi_w}(\Omega,\Sigma,\mu) = L^\Psi(\Omega,\Sigma,\mu)$ is equivalent to $L^\Phi(\Omega,\Sigma,\mu) = L^{\Psi_w}(\Omega,\Sigma,\mu)$, where $\Psi_w(t,u) = \Psi(w(t)u)$. Therefore, the assumption that M_w maps $L^\Phi(\Omega,\Sigma,\mu)$ onto $L^\Psi(\Omega,\Sigma,\mu)$ means that $L^\Phi(\Omega,\Sigma,\mu) = L^{\Psi_w}(\Omega,\Sigma,\mu)$, that is, the inclusions $L^\Phi(\Omega) \subset L^{\Psi_w}(\Omega)$ and $L^{\Psi_w}(\Omega) \subset L^\Phi(\Omega)$ hold. The preceding theorem established that the first inclusion is equivalent to condition (i). We need only to prove that the reverse inclusion is equivalent to (ii). However, by Ishii's theorem, the inclusion $L^{\Psi_w}(\Omega) \subset L^\Phi(\Omega)$ holds if and only if there is a constant $K > 1$, a set $A \in \Sigma$ with $\mu(A) = 0$, and $g \in L^1_+(\Omega,\Sigma,\mu)$, such that

$$\Phi(u) \le \Psi\bigl(Kw(t)u\bigr) + g(t)$$

for all $t \in \Omega \setminus A$ and $u \ge 0$. The last condition is equivalent to the condition

$$\sup_{u\ge 0}\bigl[\Phi(u) - \Psi\bigl(Kw(t)u\bigr)\bigr] \in L^1_+(\Omega,\Sigma,\mu).$$

But note that

$$\begin{aligned}\sup_{u\ge 0}\bigl[\Phi(u) - \Psi\bigl(Kw(t)u\bigr)\bigr] &= \sup_{v\ge 0}\bigl[v - \Psi\bigl(Kw(t)\Phi^{-1}(v)\bigr)\bigr] \\ &= U_K(t,1),\end{aligned}$$

where $U_K(t,u)$ is, for fixed $t \in \Omega$, the function complementary in the sense of Young, with respect to u, to the function $M(t,u) = \Psi(Kw(t)\Phi^{-1}(u))$. This finishes the proof. □

4 Compactness of the composition operator c_τ from one Orlicz space into another

We begin with some notions that will be useful in the following. Let (Ω,Σ,μ) be a nonatomic, complete and σ-finite measure space. We say that functions in a set A contained in the Musielak-Orlicz space $L^\Phi(\Omega)$ have equi-absolutely continuous norms if for any real number $\varepsilon > 0$ there exist a set $B_\varepsilon \in \Sigma$ with $\mu(B_\varepsilon) < \infty$ and a real number $\delta = \delta(\varepsilon) > 0$ such that for any function $x \in A$ we have $\|x\chi_{\Omega\setminus B_\varepsilon}\|_\Phi < \varepsilon$ and $\|x\chi_B\|_\Phi < \varepsilon$ whenever $B \in \Sigma \cap B_\varepsilon$ and $\mu(B) < \delta$.

Let $L^\Phi(\Omega) = L^\Phi(\Omega,\Sigma,\mu)$ and $L^\Psi(\Omega) = L^\Psi(\Omega,\Sigma,\mu)$ be distinct Orlicz spaces. We say that the operator $T : L^\Phi(\Omega) \to L^\Psi(\Omega)$ is equi-absolutely continuous if for any bounded set $A \subset L^\Phi(\Omega)$ all functions of the set $T(A) \subset L^\Psi(\Omega)$ have equi-absolutely continuous norms.

We will make use of the following theorem which gives necessary and sufficient conditions for the relative compactness of a set of functions in a Musielak-Orlicz space.

Theorem 4.1 (Theorem 1.2 in [57]) *Let (Ω, Σ, μ) be a non-atomic σ-finite measure space and let φ be a Musielak-Orlicz function. If the functions in a set $A \subset L^{\varphi}(\Omega)$ all have equi-absolutely continuous norms and A is relatively compact with respect to local convergence in measure, then A is relatively compact in $E^{\varphi}(\Omega)$, the subspace of absolutely continuous functions in $L^{\varphi}(\Omega)$.*

Conversely, if a set $A \subset E^{\varphi}(\Omega)$ is relatively compact, then all the functions in A have equi-absolutely continuous norms and A is relatively compact with respect to local convergence in measure.

In the proof of the forthcoming theorem we will need the following.

Lemma 4.1 (Lemma 8.3 in [51]) *Let the measure μ be atomless and let a sequence $\{\alpha_i\}$ of positive numbers and a sequence $\{a_i\}$ of measurable, finite, non-negative functions in Ω be given, satisfying the inequalities*

$$\int_{\Omega} a_i(t)\, d\mu \geq 2^i \alpha_i \quad \text{for } i = 1, 2, \ldots.$$

Then there exist an increasing sequence $\{i_k\}$ of positive integers and a sequence $\{A_k\}$ of pairwise disjoint sets from Σ such that

$$\int_{A_k} a_{i_k}(t)\, d\mu = \alpha_{i_k} \quad \text{for } k = 1, 2 \ldots.$$

The following theorem will be of great importance in proving necessary and sufficient conditions for the compactness of the composition operator $c_\tau : L^{\Phi}(\Omega) \to L^{\Psi}(\Omega)$.

Theorem 4.2 *Let (Ω, Σ, μ) be a finite or infinite but σ-finite non-atomic and complete measure space and τ be such that $\mu(\tau^{-1}(A)) < \infty$ whenever $\mu(A) < \infty$ for any $A \in \Sigma \cap \tau(\Omega)$. Then the composition operator $c_\tau : L^{\Phi}(\Omega) \to L^{\Psi}(\Omega)$ is equi-absolutely continuous whenever the following condition is satisfied*:

$$\underset{\sigma>0}{\forall} \ \underset{\substack{A\in\Sigma\cap\tau(\Omega)\\ \mu(A)=0}}{\exists} \ \underset{g_\sigma \in L^1_+(\tau(\Omega))}{\exists} \ \underset{s\in\tau(\Omega)\setminus A}{\forall} \ \underset{u\geq 0}{\forall} \ \Psi(u)h(s) \leq \Phi(\sigma u) + g_\sigma(s). \tag{8}$$

Condition (8) *is necessary for the equi-absolute continuity of c_τ if $\mu(\Omega) < \infty$.*

Proof Sufficiency. First we prove that for any $\varepsilon > 0$ there exists a set $D \in \Sigma$ with $\mu(\Omega \setminus D) < \infty$ such that all the functions in the set $\{c_\tau x : x \in S(L^{\Phi})\}$, where $S(L^{\Phi})$ is the unit sphere of L^{Φ}, satisfy the condition $\|(c_\tau x)\chi_D\|_\Psi < \varepsilon$.

Let $\sigma > 0$ be a number such that $(1 + \sigma)\sigma < \varepsilon$ and let g_σ be a function from condition (8) corresponding to σ. Since $g_\sigma \in L^1_+(\tau(\Omega))$, there exists a set $C \in \Sigma \cap \tau(\Omega)$ such that $\|g_\sigma \chi_C\|_\Psi < \sigma$ and $\mu(\tau(\Omega) \setminus C) < \infty$. Defining $D = \tau^{-1}(C)$, we have for any function $x \in S(L^{\Phi})$

$$\begin{aligned} I_\Psi\left(\frac{c_\tau x}{\sigma}\chi_D\right) &= \int_\Omega \Psi\left(\frac{c_\tau x(t)}{\sigma}\chi_D(t)\right) d\mu(t) \\ &= \int_D \Psi\left(\frac{x(\tau(t))}{\sigma}\right) d\mu(t) \end{aligned}$$

$$\begin{aligned} &= \int_{\tau(D)} \Psi\left(\frac{x(s)}{\sigma}\right) d\mu \circ \tau^{-1}(s) \\ &= \int_C \Psi\left(\frac{x(s)}{\sigma}\right) h(s)\, d\mu(s) \\ &\leq \int_C \Phi\bigl(x(s)\bigr)\, d\mu(s) + \int_C g_\sigma(s)\, d\mu(s) \\ &\leq 1+\sigma. \end{aligned} \tag{9}$$

By (9) we get

$$I_\Psi\left(\frac{(c_\tau x)\chi_D}{(1+\sigma)\sigma}\right) \leq \frac{1}{1+\sigma} I_\Psi\left(\frac{c_\tau x}{\sigma}\chi_D\right) \leq \frac{1}{1+\sigma}(1+\sigma) = 1$$

for any $x \in S(L^\Phi)$. Consequently, $\|(c_\tau x)\chi_D\|_\Psi \leq (1+\sigma)\sigma < \varepsilon$ for any $x \in S(L^\Phi)$, which finishes the first part of the proof.

Now we show the following implication:

$$\mathop{\forall}_{\sigma>0} \mathop{\exists}_{\delta=\delta(\varepsilon)} \mathop{\forall}_{B\subset\tau(\Omega)\setminus D} \mathop{\forall}_{x\in S(L^\Phi)} \mu(B) < \delta \quad \Longrightarrow \quad \bigl\|(c_\tau x)\chi_B\bigr\|_\Psi < \varepsilon,$$

where D is the set from the first part of this proof.

Take any $x \in S(L^\Phi)$. Since $g_\sigma \in L^1_+(\tau(\Omega))$, it is obvious that there is $\delta = \delta(\sigma) > 0$ such that, if $C \in \Sigma \cap (\tau(\Omega) \setminus D)$ and $\mu(C) < \delta$, then $\int_C g_\sigma(s)\, d\mu(s) < \sigma$. Let $B = \tau^{-1}(C)$. Then applying condition (8), we get

$$\begin{aligned} I_\Psi\left(\frac{c_\tau x}{\sigma}\chi_B\right) &= \int_\Omega \Psi\left(\frac{c_\tau x(t)}{\sigma}\chi_B(t)\right) d\mu(t) \\ &= \int_B \Psi\left(\frac{x(\tau(t))}{\sigma}\right) d\mu(t) \\ &= \int_{\tau(B)} \Psi\left(\frac{x(s)}{\sigma}\right) d\mu \circ \tau^{-1}(s) \\ &= \int_C \Psi\left(\frac{x(s)}{\sigma}\right) h(s)\, d\mu(s) \\ &\leq \int_C \Phi\bigl(x(s)\bigr)\, d\mu(s) + \int_C g_\sigma(s)\, d\mu(s) \\ &\leq 1+\sigma. \end{aligned} \tag{10}$$

Now, by convexity of the modular I_Ψ and the fact that I_Ψ vanishes at zero we have $I_\Psi(\lambda x) \leq \lambda I_\Psi(x)$ for any $x \in L^\Psi(\Omega)$ and $\lambda \in [0,1]$. From this fact and from (10), we get

$$I_\Psi\left(\frac{(c_\tau x)\chi_B}{(1+\sigma)\sigma}\right) \leq \frac{1}{1+\sigma} I_\Psi\left(\frac{c_\tau x}{\sigma}\chi_B\right) \leq \frac{1}{1+\sigma}(1+\sigma) = 1.$$

Hence $\|(c_\tau x)\chi_B\|_\Psi \leq (1+\sigma)\sigma < \varepsilon$. Since σ depends only on ε, $\delta = \delta(\sigma)$ depends only on ε, and so the proof of sufficiency is finished.

Necessity. Assume that $\mu(\Omega) < \infty$ and for any $\sigma > 0$ define the function

$$h_\sigma(s) = \sup_{u\geq 0}\bigl\{\Psi(u)h(s) - \Phi(\sigma u)\bigr\}.$$

$h_\sigma(s)$ is a non-negative (since for $u = 0$, we have $\Psi(0)h(s) - \Phi(\sigma 0) = 0$) measurable function.

Suppose condition (8) is not satisfied. Then there is $\sigma_0 > 0$ such that

$$\int_{\tau(\Omega)} h_{\sigma_0}(s)\,d\mu(s) = +\infty.$$

Let $\{r_i\}_{i=0}^{\infty}$ be a sequence of all non-negative rational numbers with $r_0 = 0$. By the continuity of the Orlicz functions Φ and Ψ, we have

$$h_{\sigma_0}(s) = \sup_{i=0,1,2\ldots} \bigl\{\Psi(r_i)h(s) - \Phi(\sigma_0 r_i)\bigr\}.$$

Let us write

$$h_{\sigma_0,n}(s) = \max_{0\le i\le n} \bigl\{\Psi(r_i)h(s) - \Phi(\sigma_0 r_i)\bigr\}.$$

It is obvious that $h_{\sigma_0,n} \ge 0$ and $h_{\sigma_0,n}$ are measurable functions such that $h_{\sigma_0,n} \nearrow h_{\sigma_0}$ as $n \to +\infty$ μ-a.e. in $\tau(\Omega)$. By Beppo Levi's theorem, we have

$$\int_{\tau(\Omega)} h_{\sigma_0,n}(s)\,d\mu(s) \nearrow \int_{\tau(\Omega)} h_{\sigma_0}(s)\,d\mu(s).$$

Hence there exists a subsequence $\{h_{\sigma_0,n_k}\} \subset \{h_{\sigma_0,n}\}$ satisfying

$$\int_{\tau(\Omega)} h_{\sigma_0,n_k}(s)\,d\mu(s) \ge 2^k.$$

Without loss of generality we may assume that $\int_{\tau(\Omega)} h_{\sigma_0,n}(s)\,d\mu(s) \ge 2^n$ for each $n \in \mathbb{N} \cup \{0\}$. It is clear that for each $s \in \tau(\Omega)$ and each $n \in \mathbb{N} \cup \{0\}$ there exists $\widetilde{r}_n(s) \in \{r_0, r_1, r_2, \ldots, r_n\}$ such that

$$h_{\sigma_0,n}(s) = \Psi\bigl(\widetilde{r}_n(s)\bigr)h(s) - \Phi\bigl(\sigma_0 \widetilde{r}_n(s)\bigr).$$

Hence

$$\int_{\tau(\Omega)} \Psi\bigl(\widetilde{r}_n(s)\bigr)h(s)\,d\mu(s) = \int_{\tau(\Omega)} h_{\sigma_o,n}(s)\,d\mu(s) + \int_{\tau(\Omega)} \Phi\bigl(\sigma_0 \widetilde{r}_n(s)\bigr)\,d\mu(s) \ge 2^n.$$

Applying Lemma 4.1, we conclude that there is a sequence of sets $\{\Omega_k\} \subset \tau(\Omega)$ with $\Omega_i \cap \Omega_j = \emptyset$ for $i \ne j$ such that

$$\int_{\Omega_n} \Psi\bigl(\widetilde{r}_n(s)\bigr)h(s)\,d\mu(s) = 1.$$

Let $\bar{r}_n(s) = \widetilde{r}_n(s)\chi_{\Omega_n}(s)$. Since $\bar{r}_n$ are bounded measurable functions and $h \in L^1(\operatorname{supp} \bar{r}_n)$, we get $\bar{r}_n \in E_h^{\Psi}(\Omega)$ with $\|\bar{r}_n\|_{\Psi,h} = 1$ and $\bar{r}_n \in L^{\Phi}(\Omega)$ for any $n \in \mathbb{N} \cup \{0\}$.

Since, by assumption, $\mu(\Omega) < \infty$, we have $\sum_{n=0}^{\infty} \mu(\Omega_n) = \mu(\bigcup_{n=0}^{\infty} \Omega_n) \leq \mu(\Omega) < \infty$, whence $\lim_{n\to\infty} \mu(\Omega_n) = 0$. This means that $\{\bar{r}_n\}_{n=1}^{+\infty}$ does not have equi-absolutely continuous norms in $E_h^{\Psi}(\Omega)$. Yet, using the fact that $g_{\sigma_0,n} \geq 0$ ($n \in \mathbb{N} \cup \{0\}$), we get the following:

$$\Psi\big(\bar{r}_n(s)\big)h(s) \geq \Phi\big(\sigma_0 \bar{r}_n(s)\big), \quad n \in \mathbb{N} \cup \{0\}.$$

So $I_\Phi(\sigma_0 \bar{r}_n) \leq 1$, that is, $\|\bar{r}_n\|_\Phi \leq \frac{1}{\sigma_0}$, which means that the operator $c_\tau : L^\Phi(\Omega) \to L^\Psi(\Omega)$ is not equi-absolutely continuous. Hence we proved that condition (8) is necessary for the equi-absolutely continuity of c_τ. □

From Theorem 4.2, applying Theorem 4.1 and the definition of a compact operator, we directly get the following.

Theorem 4.3 *If (Ω, Σ, μ) is a non-atomic complete finite or infinite but σ-finite measure space and τ satisfies the assumption from Theorem* 4.2, *then the composition operator c_τ from $L^\Phi(\Omega)$ into $L^\Psi(\Omega)$ is compact whenever the set $c_\tau(S(L^\Phi))$ is relatively compact with respect to local convergence in measure and condition* (8) *from Theorem* 4.2 *is satisfied.*

Under the assumption that $\mu(\Omega) < \infty$, if the composition operator c_τ from $L^\Phi(\Omega)$ into $E^\Psi(\Omega)$ is compact then the set $c_\tau(S(L^\Phi))$ is relatively compact with respect to convergence in measure and condition (8) *is satisfied.*

In the case when Ω has infinite measure, we were unable to show that (8) is a necessary condition for the equi-absolute continuity of the composition operator c_τ. Instead, we can deduce a slightly different (and weaker) condition, as the following theorem states.

Theorem 4.4 *Assume that $\mu(\Omega) = \infty$ and $\mu(\Omega \setminus \tau(\Omega)) = 0$. If the composition operator $c_\tau : L^\Phi(\Omega) \to L^\Psi(\Omega)$ is equi-absolutely continuous, then the condition*

$$\underset{\lambda>0}{\forall} \underset{\substack{A\in\Sigma \\ \mu(A)=0}}{\exists} \underset{K_\lambda>0}{\exists} \underset{g_\lambda \in L_+^1(\Omega)}{\exists} \underset{s\in\Omega\setminus A}{\forall} \underset{u\geq 0}{\forall} \Psi(\lambda u)h(s) \leq K_\lambda \Phi(u) + g_\lambda(s) \tag{11}$$

is satisfied.

Proof We can assume without loss of generality that $\tau(\Omega) = \Omega$. First, notice that if Φ, Ψ, and h satisfy condition (11) then $L^\Phi(\Omega) \subset E_h^\Psi(\Omega)$, where

$$E_h^\Psi(\Omega) = \left\{ x \in L^0(\Omega) : \underset{\lambda>0}{\forall} I_{\Psi,h}(\lambda x) = \int_\Omega \Psi\big(\lambda x(t)\big)h(t)\, d\mu(t) < \infty \right\}$$

is the subspace of absolutely continuous elements of $L_h^\Psi(\Omega)$. The inclusion results from the assumption that c_τ is an equi-absolutely continuous operator, *i.e.*, for any bounded set $A \subset L^\Phi(\Omega)$, the functions of the set $c_\tau(A) \subset L^\Psi(\Omega)$ all have equi-absolutely continuous norms, and the observation that, for any $x \in L^\Phi(\Omega)$, the singleton set $\{x\}$ is bounded, and thus x has an equi-absolutely continuous norm, which means that x is an absolutely continuous element of $L_h^\Psi(\Omega)$, *i.e.*, $x \in E_h^\Psi(\Omega)$.

Further, observe that

$$L^\Phi(\Omega) = \bigcup_{\lambda>0} L^{\Phi_\lambda,*}(\Omega), \qquad E_h^\Psi(\Omega) = \bigcap_{\lambda>0} L_h^{\Psi_\lambda,*}(\Omega),$$

where for any $\lambda > 0$ we define $\Phi_\lambda(u) = \Phi(u)$ and $L^{\Phi,*}(\Omega) = \{x \in L^0(\Omega) : I_\Phi(x) < \infty\}$ is a Musielak-Orlicz class (Ψ_λ and $L_h^{\Psi,*}(\Omega)$ are defined analogously). Hence the inclusion $L^\Phi(\Omega) \subset E_h^\Psi(\Omega)$ can be expressed as

$$\bigcup_{\lambda>0} L^{\Phi_\lambda,*}(\Omega) \subset \bigcap_{\lambda>0} L_h^{\Psi_\lambda,*}(\Omega).$$

This means that, for any $\lambda_1 > 0$, the Musielak-Orlicz class $L^{\Phi_{\lambda_1},*}(\Omega)$ is contained in all of $L_h^{\Psi_\lambda,*}(\Omega)$ ($\lambda > 0$). In particular, taking $\lambda_1 = 1$, we see that $L^{\Phi,*}(\Omega)$ is contained in all the Musielak-Orlicz classes $L_h^{\Psi_\lambda,*}(\Omega)$ ($\lambda > 0$). By Theorem 8.4 in [51], this is equivalent to

$$\underset{\lambda>0}{\forall} \ \underset{\substack{A\in\Sigma \\ \mu(A)=0}}{\exists} \ \underset{K_\lambda>0}{\exists} \ \underset{g_\lambda\in L_+^1(\Omega)}{\exists} \ \underset{s\in\Omega\setminus A}{\forall} \ \underset{u\geq 0}{\forall} \Psi(\lambda u)h(s) \leq K_\lambda \Phi(u) + g_\lambda(s),$$

which finishes the proof. □

From the preceding theorem, applying Theorem 4.1, we can deduce the following necessary condition for the compactness of the composition operator c_τ:

Theorem 4.5 *Assume that* $\mu(\Omega) = \infty$ *and* $\mu(\Omega\setminus\tau(\Omega)) = 0$. *If the composition operator* $c_\tau : L^\Phi(\Omega) \to E^\Psi(\Omega)$ *is compact then the following conditions are jointly satisfied*:

(1) *the set* $c_\tau(S(L^\Phi))$ *is relatively compact with respect to local convergence in measure*;

(2) *the functions* Φ *and* Ψ *satisfy condition* (11).

Remark 4.1 If there exists $\varepsilon > 0$ such that the set

$$B_\varepsilon := \left\{ t \in \tau(\Omega) : \underset{u\geq 0}{\forall} \Psi(u)h(t) > \Phi(\varepsilon u) \right\}$$

has positive measure, then no composition operator $c_\tau : L^\Phi(\Omega) \to L^\Psi(\Omega)$ over a non-atomic measure space (Ω, Σ, μ) is compact.

Proof of Remark 4.1 Assume that there exists $\varepsilon > 0$ such that the measure of the set B_ε is positive. Then in the set B_ε we can find a sequence of measurable and pairwise disjoint sets $\{B_n\}$ in $\tau(\Omega)$ having positive and finite measure. Define

$$x_n = \frac{\chi_{B_n}}{\|\chi_{B_n}\|_\Phi} = \Phi^{-1}\left(\frac{1}{\mu(B_n)}\right)\chi_{B_n}.$$

Obviously, $I_\Phi(x_n) = 1$, so $\|x_n\|_\Phi = 1$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned} I_\Phi\left(\frac{\varepsilon x_n}{\|x_n \circ \tau\|_\Psi}, \Omega\right) &= \int_{\tau(\Omega)} \Phi\left(\frac{\varepsilon x_n(t)}{\|x_n \circ \tau\|_\Psi}\right) d\mu(t) \\ &< \int_{\tau(\Omega)} \Psi\left(\frac{x_n(t)}{\|x_n \circ \tau\|_\Psi}\right) h(t)\, d\mu(t) \\ &= \int_\Omega \Psi\left(\frac{(x_n \circ \tau)(t)}{\|x_n \circ \tau\|_\Psi}\right) d\mu(t) = I_\Psi\left(\frac{c_\tau x_n}{\|c_\tau x_n\|_\Psi}, \Omega\right) \leq 1. \end{aligned}$$

Therefore,

$$\left\| \frac{\varepsilon x_n}{\|x_n \circ \tau\|_\Psi} \right\|_\Phi \le 1,$$

i.e., $\|c_\tau x_n\|_\Psi \ge \varepsilon \|x_n\|_\Phi = \varepsilon$ because $\|x_n\|_\Phi = 1$. Since the supports of x_n are pairwise disjoint, for all $n, m \in \mathbb{N}$, $m \neq n$ we get

$$\|c_\tau x_n - c_\tau x_m\|_\Psi = \left\| c_\tau (x_n - x_m) \right\|_\Psi \ge \max\left\{ \|c_\tau x_n\|_\Psi, \|c_\tau x_m\|_\Psi \right\} \ge \varepsilon.$$

Hence the sequence $\{c_\tau x_n\}_{n=1}^\infty$ has no Cauchy subsequence, that is, $c_\tau(S(L^\Phi(\Omega)))$ is not relatively compact. Consequently, no composition operator c_τ from $L^\Phi(\Omega)$ into $L^\Psi(\Omega)$ over a non-atomic measure space (Ω, Σ, μ) is compact. □

5 Compactness of the multiplication operator M_w from one Orlicz space into another

We state a sufficient condition for the compactness of the multiplication operator $M_w : L^\Phi(\Omega) \to L^\Psi(\Omega)$.

Theorem 5.1 *Let (Ω, Σ, μ) be a non-atomic complete, finite or infinite but σ-finite measure space and let Φ, Ψ be two Orlicz functions and $w \in L^0_+(\Omega, \Sigma, \mu)$. If the triple Φ, Ψ, w satisfies the condition*

$$\underset{\sigma>0}{\forall} \ \underset{\substack{A_\sigma \in \Sigma \\ \mu(A_\sigma)=0}}{\exists} \ \underset{g_\sigma \in L^1_+(\Omega)}{\exists} \ \underset{t \in \Omega \setminus A}{\forall} \ \underset{u \ge 0}{\forall} \ \Psi\bigl(w(t)u\bigr) \le \Phi(\sigma u) + g_\sigma(t) \tag{12}$$

then the multiplication operator M_w from $L^\Phi(\Omega)$ into $L^\Psi(\Omega)$ is equi-absolutely continuous.

Proof First we show that given $\varepsilon > 0$ there exists a set $B \in \Sigma$ with $\mu(B) < \infty$ such that $\|x\chi_{\Omega\setminus B}\|_\Psi < \varepsilon$ for any function $x \in S(L^\Phi(\Omega))$.

Take $\varepsilon > 0$ and let $\sigma > 0$ be such that $\sigma(1+\sigma) < \varepsilon$. Let g_σ be a function from condition (12) corresponding to σ. Since $g_\sigma \in L^1_+(\Omega)$, there exists $B \in \Sigma$ with $\mu(B) < \infty$ such that $\|g_\sigma \chi_{\Omega\setminus B}\|_{L^1(\Omega)} < \sigma$. Then

$$I_\Psi\left(\frac{M_w x}{\sigma}\chi_{\Omega\setminus B}\right) \le I_\Phi(x\chi_{\Omega\setminus B}) + \int_{\Omega\setminus B} g_\sigma(t)\,d\mu(t) \le 1+\sigma,$$

whence

$$I_\Psi\left(\frac{M_w x}{\sigma(1+\sigma)}\chi_{\Omega\setminus B}\right) \le \frac{1}{1+\sigma} I_\Psi\left(\frac{M_w x \chi_{\Omega\setminus B}}{\sigma}\right) \le \frac{1}{1+\sigma}(1+\sigma) = 1,$$

that is, $\|M_w x \chi_{\Omega\setminus B}\|_\Psi \le \sigma(1+\sigma) < \varepsilon$.

Next, we show that for $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any $D \subset B$ and any function $x \in S(L^\Phi)$, if $\mu(D) < \delta$ then $\|M_w x \chi_D\|_\Psi < \varepsilon$.

Let $\varepsilon > 0$ and let $\sigma > 0$ be such that $\sigma(1+\sigma) < \varepsilon$. By the absolute continuity of g_σ in $L_1(\Omega \cap B)$, there exists $\delta = \delta(\sigma)$ such that $\|g_\sigma \chi_C\|_{L^1(\Omega)} < \sigma$ whenever $C \subset B$ and $\mu(C) < \delta$. Then, by condition (12),

$$I_\Psi\left(\frac{M_w x}{\sigma}\chi_C\right) \le I_\Phi(x\chi_C) + \|g_\sigma \chi_C\|_{L^1(\Omega)} \le 1+\sigma,$$

whence

$$I_\Psi\left(\frac{M_w x}{\sigma(1+\sigma)}\chi_C\right) \le \frac{1}{1+\sigma} I_\Psi\left(\frac{M_w x \chi_C}{\sigma}\right) \le 1,$$

and so $\|M_w x\|_\Psi \le \sigma(1+\sigma) < \varepsilon$, which finishes the proof. □

Remark 5.1 Let us note that in the case when $\Phi = \Psi$, Theorem 5.1 can only hold when $\Phi \in \Delta_2(\infty)$.

Indeed, if Theorem 5.1 holds, then the operator M_w acts, in fact, from $L^\Phi(\Omega)$ into $E^\Phi(\Omega)$. However, if we assume that $\Phi \notin \Delta_2(\infty)$ and $w \notin L^\infty(\Omega, \Sigma, \mu)$, then defining the set

$$A = \{t \in \Omega : w(t) \ge 2\},$$

we have $\mu(A) > 0$. Therefore, we can build $x \in L^\Phi(\Omega)$ such that $\operatorname{supp} x \subset A$, $I_\Phi(x) \le 1$, and $I_\Phi(\lambda x) = \infty$ for any $\lambda > 1$ (see [58] and [59]). Hence $I_\Phi(M_w x) \ge I_\Phi(2x) = \infty$, which means that $M_w x \notin E^\Phi(\Omega)$, so M_w does not act from $L^\Phi(\Omega)$ into $E^\Phi(\Omega)$.

Applying Theorem 4.1, we directly get from Theorem 5.1 and the definition of a compact operator the following.

Theorem 5.2 *If (Ω, Σ, μ) is a non-atomic complete finite or infinite but σ-finite measure space, then the multiplication operator c_τ form $L^\Phi(\Omega)$ into $L^\Psi(\Omega)$ is compact whenever the set $M_w(S(L^\Phi))$ is relatively compact with respect to local convergence in measure and condition* (12) *from Theorem* 5.1 *is satisfied.*

Theorems 5.1 and 5.2 resemble closely the sufficiency part of Theorems 4.2 and 4.3 for the composition operator. Similarly, we will formulate necessary conditions for the equi-absolute continuity of the multiplication operator: one in the case when $\mu(\Omega) < \infty$ and the other in the case when $\mu(\Omega) = \infty$. The respective proofs proceed along the lines of the proofs for the composition operator, and therefore will be omitted.

Theorem 5.3 *If (Ω, Σ, μ) is a finite non-atomic and complete measure space and the multiplication operator $M_w : L^\Phi(\Omega) \to L^\Psi(\Omega)$ is equi-absolutely continuous then the following condition is satisfied*:

$$\underset{\sigma>0}{\forall} \; \underset{\substack{A\in\Sigma \\ \mu(A)=0}}{\exists} \; \underset{g_\sigma\in L^1_+(\Omega)}{\exists} \; \underset{t\in\Omega\setminus A}{\forall} \; \underset{u\ge 0}{\forall} \; \Psi\big(w(t)u\big) \le \Phi(\sigma u) + g_\sigma(t).$$

Theorem 5.4 *If (Ω, Σ, μ) is an infinite but σ-finite non-atomic and complete measure space and the multiplication operator $M_w : L^\Phi(\Omega) \to L^\Psi(\Omega)$ is equi-absolutely continuous, then the following condition is satisfied*:

$$\underset{\lambda>0}{\forall} \; \underset{\substack{A\in\Sigma \\ \mu(A)=0}}{\exists} \; \underset{K_\lambda>0}{\exists} \; \underset{g_\lambda\in L^1_+(\Omega)}{\exists} \; \underset{t\in\Omega\setminus A}{\forall} \; \underset{u\ge 0}{\forall} \; \Psi\big(w(t)\lambda u\big) \le K_\lambda \Phi(u) + g_\lambda(t).$$

The respective necessary conditions for the compactness of the multiplication operator are analogous to the ones for the composition operator from Theorems 4.3 and 4.5.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Although the subject of this paper was initiated by HH who was heading and controlling the scientific research of the other authors of the paper, all authors contributed equally to this paper. All authors read and approved the final manuscript.

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References

1. Koopman, BO: Hamiltonian systems and transformations in Hilbert space. Proc. Natl. Acad. Sci. USA **17**, 315-318 (1931)
2. Koopman, BO, Von Neumann, J: Dynamical systems of continuous spectra. Proc. Natl. Acad. Sci. USA **18**, 255-263 (1932)
3. Halmos, PR: Lectures on Ergodic Theory. Chelsea, New York (1965)
4. Halmos, PR, Von Neumann, J: Operator methods in classical mechanics II. Ann. Math. **43**, 332-350 (1942)
5. Boyd, DM: Composition operators on the Bergman spaces and analytic function spaces on annulus. Thesis, University of North Carolina (1974)
6. Gupta, DK: Composition operators on weighted sequence spaces. Thesis, University of Jammu (1979)
7. Ridge, WC: Composition operators. Thesis, Indiana University (1969)
8. Schwartz, HJ: Composition operators on H^p. Thesis, University of Toledo (1969)
9. Singh, RK: Composition operators. Thesis, University of New Hampshire (1972)
10. Swanton, DW: Composition operators on $H^p(D)$. Thesis, Northwestern University (1974)
11. Veluchamy, T: Composition operators on L^p-spaces. Thesis, University of Jammu (1983)
12. Shapiro, JH: Composition Operators and Classical Function Theory. Universitext: Tracts in Mathematics. Springer, New York (1993)
13. Singh, RK, Manhas, JS: Composition Operators on Function Spaces. North-Holland Mathematics Studies, vol. 179. North-Holland, Amsterdam (1993)
14. Nordgren, EA: Composition operators. Can. J. Math. **20**, 442-449 (1968)
15. Parrot, SK: Weighted translation operators. Thesis, University of Michigan (1965)
16. Singh, RK: Compact and quasinormal composition operators. Proc. Am. Math. Soc. **45**, 80-82 (1974)
17. Singh, RK: Composition operators induced by rational functions. Proc. Am. Math. Soc. **59**, 329-333 (1976)
18. Singh, RK, Chandra Kumar, RD: Compact weighted composition operators on $L^2(\lambda)$. Acta Sci. Math. **49**, 339-344 (1985)
19. Singh, RK, Komal, BS: Composition operator on ℓ^p and its adjoint. Proc. Am. Math. Soc. **70**, 21-25 (1978)
20. Singh, RK, Kumar, A: Compact composition operators. J. Aust. Math. Soc. Ser. A **28**, 309-314 (1979)
21. Takagi, H: Compact weighted composition operators on L^p. Proc. Am. Math. Soc. **116**, 505-511 (1992)
22. Takagi, H, Yokouchi, K: Multiplication and composition operators between two L^p-spaces. In: Function Spaces, Edwardsville, IL, 1998. Contemp. Math., vol. 232, pp. 321-338. Am. Math. Soc., Providence (1999)
23. Xu, XM: Compact composition operators on $L^p(X, \mathscr{A}, \mu)$. Adv. Math. **20**(2), 221-225 (1991) (in Chinese)
24. Cui, Y, Hudzik, H, Kumar, R, Maligranda, L: Composition operators in Orlicz spaces. J. Aust. Math. Soc. **76**(2), 189-206 (2004)
25. Gupta, S, Komal, BS, Suri, N: Weighted composition operators on Orlicz spaces. Int. J. Contemp. Math. Sci. **5**(1-4), 11-20 (2010)
26. Komal, BS, Gupta, S: Composition operators on Orlicz spaces. Indian J. Pure Appl. Math. **32**(7), 1117-1122 (2001)
27. Kumar, R: Composition operators on Orlicz spaces. Integral Equ. Oper. Theory **29**(1), 17-22 (1997)
28. Raj, K, Kumar, A, Sharma, SK: Composition operators on Musielak-Orlicz spaces. J. Contemp. Appl. Math. **1**(1), 54-62 (2011)
29. Raj, K, Sharma, AK, Kumar, A: Weighted composition operators on Musielak-Orlicz spaces. Ars Comb. **107**, 431-439 (2012)
30. Raj, K, Jamwal, S, Sharma, SK: Weighted composition operators on Musielak-Orlicz spaces of Bochner-type. Ann. Univ. Buchar. Math. Ser. **4(LXII)**(2), 513-523 (2013)
31. Raj, K, Sharma, SK: Composition operators on Musielak-Orlicz spaces of Bochner type. Math. Bohem. **137**(4), 449-457 (2012)
32. Arora, SC, Datt, G: Compact weighted composition operators on Orlicz-Lorentz spaces. Acta Sci. Math. **77**(3-4), 567-578 (2011)
33. Arora, SC, Datt, G, Verma, S: Multiplication and composition operators on Orlicz-Lorentz spaces. Int. J. Math. Anal. **1**(25-28), 1227-1234 (2007)
34. Castillo, RE, Chaparro, HC, Ramos Fernández, JC: Orlicz-Lorentz spaces and their composition operators. Proyecciones **34**(1), 85-105 (2015)
35. Cui, Y, Hudzik, H, Kumar, R, Kumar, R: Compactness and essential norms of composition operators on Orlicz-Lorentz spaces. Appl. Math. Lett. **25**(11), 1778-1783 (2012)

36. Cui, Y, Hudzik, H, Kumar, R, Kumar, R: Erratum and addendum to the paper 'Compactness and essential norms of composition operators on Orlicz-Lorentz spaces' [Applied Mathematics Letters 25 (2012) 1778-1783]. Comment. Math. **54**(2), 259-260 (2014)
37. Kumar, R, Kumar, R: Composition operators on Orlicz-Lorentz spaces. Integral Equ. Oper. Theory **60**(1), 79-88 (2008)
38. Nordgren, EA: Composition Operators on Hilbert Spaces. Lecture Notes in Mathematics, vol. 693, pp. 37-68. Springer, New York (1978)
39. Singh, RK, Chandra Kumar, RD: Weighted composition operators on functional Hilbert spaces. Bull. Aust. Math. Soc. **31**(1), 117-126 (1985)
40. Al-Rawashdeh, W: Compact composition operators on weighted Hilbert spaces. J. Appl. Funct. Anal. **10**(1-2), 101-108 (2015)
41. Boyd, DM: Composition operators on the Bergman spaces. Colloq. Math. **34**, 127-136 (1975)
42. Boyd, DM: Composition operators on $H^p(A)$. Pac. J. Math. **62**, 55-60 (1976)
43. Charpentier, S: Composition operators on weighted Bergman-Orlicz spaces on the ball. Complex Anal. Oper. Theory **7**(1), 43-68 (2013)
44. Komal, BS, Gupta, S: Multiplication operators between Orlicz spaces. Integral Equ. Oper. Theory **41**(3), 324-330 (2001)
45. Raj, K, Sharma, SK, Kumar, A: Multiplication operators on Musielak-Orlicz spaces of Bochner type. J. Adv. Stud. Topol. **3**(4), 1-7 (2012)
46. Seid, HA: Cyclic multiplication operators on L^p-spaces. Pac. J. Math. **51**, 542-562 (1974)
47. Sharma, SK, Komal, BS: Multiplication operators on weighted Hardy spaces. Lobachevskii J. Math. **33**(2), 165-169 (2012)
48. Singh, RK, Kumar, A: Multiplication operators and composition operators with closed ranges. Bull. Aust. Math. Soc. **16**, 247-252 (1977)
49. Singh, RK, Manhas, JS: Multiplication operators on weighted spaces of vector-valued continuous functions. J. Aust. Math. Soc. Ser. A **50**, 98-107 (1991)
50. Hudzik, H, Krbec, M: On non-effective weights in Orlicz spaces. Indag. Math. **18**(2), 215-231 (2007)
51. Musielak, J: Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics, vol. 1034. Springer, Berlin (1983)
52. Musielak, J, Orlicz, W: On modular spaces. Stud. Math. **18**, 49-65 (1959)
53. Krasnoselskii, MA, Rutickii, YB: Convex Functions and Orlicz Spaces. Translation. Nordhoff, Groningen (1961)
54. Luxemburg, WAJ: Banach function spaces. PhD thesis, Delft (1955)
55. Maligranda, L: Orlicz Spaces and Interpolation. Seminars in Math. 5. Univ. Estadual de Campinas, Campinas SP, Brazil (1989)
56. Ishii, J: On equivalence of modular function spaces. Proc. Jpn. Acad. **35**, 551-556 (1959)
57. Hudzik, H, Musielak, J, Urbański, R: Interpolation of compact sublinear operators in generalized Orlicz space of nonsymmetric type. In: Topology. Vol. I, Proc. Fourth Colloq., Budapest, 1978. Colloq. Math. Soc. János Bolyai, vol. 23, pp. 625-638. North-Holland, Amsterdam (1980)
58. Turett, B: Rotundity of Orlicz spaces. Proc. K. Ned. Akad. Wet., Ser. A, Indag. Math. **79**(5), 462-469 (1981)
59. Hudzik, H: Banach lattices with order isometric copies of l^∞. Indag. Math. **9**(4), 521-527 (1998)