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Stability of numerical method for semi-linear stochastic pantograph differential equations

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Abstract

As a particular expression of stochastic delay differential equations, stochastic pantograph differential equations have been widely used in nonlinear dynamics, quantum mechanics, and electrodynamics. In this paper, we mainly study the stability of analytical solutions and numerical solutions of semi-linear stochastic pantograph differential equations. Some suitable conditions for the mean-square stability of an analytical solution are obtained. Then we proved the general mean-square stability of the exponential Euler method for a numerical solution of semi-linear stochastic pantograph differential equations, that is, if an analytical solution is stable, then the exponential Euler method applied to the system is mean-square stable for arbitrary step-size $h > 0$. Numerical examples further illustrate the obtained theoretical results.

Keywords: semi-linear stochastic pantograph differential equations; exponential Euler method; mean-square stability; general mean-square stability

1 Introduction

Stochastic delay differential equations played an important role in application areas, such as physics, biology, economics, and finance [1–4]. Stochastic pantograph differential equations are particular cases of stochastic unbounded delay differential equations, Ockendon and Tayler [5] found how the electric current is collected by the pantograph of an electric locomotive, therefore one speaks of stochastic pantograph differential equations.

In recent years, as one of the most important characteristics of stochastic systems, the stability analysis caused much more attention [6–10]. Generally speaking, due to the characteristics of stochastic differential equations themselves, it is difficult for us to get analytical solution of equations, therefore, researching the proper numerical methods for a numerical solution has certain theoretical value and practical significance. However, the research for the numerical solution of stochastic pantograph differential equations is still rare. Fan [11] investigated mean-square asymptotic stability of the θ method for linear stochastic pantograph differential equations. Hua [12, 13] developed an almost surely asymptotic stability analytical solution and numerical solution for neutral stochastic pantograph differential equations. Xiao [14] proved mean-square stability of the Milstein method for stochastic pantograph differential equations under suitable conditions. Zhou [15] showed that the Euler-Maruyama method can preserve almost surely exponential stability of stochastic pantograph differential equations under the linear growth conditions, and the backward Euler-Maruyama method can reproduce almost surely exponential stability for highly nonlinear stochastic pantograph differential equations. The numerical research for stochastic pantograph differential equations has just begun, and the stability analysis of the numerical solution of the equations needs further perfection and development.

Unfortunately, some conditions of stability are somewhat restrictive as applied to practical applications. This paper mainly proves that if an analytical solution is stable, then so is the exponential Euler method applied to the system for any step-size $h > 0$. Namely, the exponential Euler method for semi-linear stochastic pantograph differential equations is general mean-square stable.

2 Exponential Euler scheme for stochastic pantograph differential equations

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$, which is increasing and right continuous, and $\mathcal{F}_0$ contains all P-null sets. $W(t)$ is Wiener process defined on the probability space, which may be $\mathcal{F}_t$-adapted and independent of $\mathcal{F}_0$. Let $|\cdot|$ be the Euclidean norm. The inner product of x, y in $\mathbb{R}^n$ is denoted by $\langle x, y\rangle$ or x^Ty, $\mathbb{R}^n$ is n-dimensional Euclidean space. If A is a vector or matrix, its transpose is denoted by A^T, if A is a matrix, the trace norm of the matrix A is $|A| = \sqrt{\operatorname{trace}(A^TA)}$. We use $a \vee b$ and $a \wedge b$ to denote $\max\{a, b\}$ and $\min\{a, b\}$.

We first introduce the exponential Euler method [16, 17] for a semi-linear ordinary differential equation,

$$\begin{cases} u'(t) + Au(t) = g(t, u), \\ u(t_0) = u_0. \end{cases} \tag{1}$$

Making use of method of variation of constant, the expression of the solution is

$$u(t_n + h) = e^{-Ah}u(t_n) + \int_0^t e^{-A(h-\tau)} g\bigl(t_n + \tau, u(t_n + \tau)\bigr)\, d\tau.$$

Applying the exponential Runge-Kutta method to equation (1),

$$\begin{aligned} u_{n+1} &= e^{-Ah}u_n + h\sum_{i=1}^{p} \lambda_i(-Ah) g(t_n + c_ih, u_{n,i}), \\ u_{n,i} &= e^{-c_iAh}u_n + h\sum_{j=1}^{p} \mu_{ij}(-Ah) g(t_n + c_jh, u_{n,j}), \end{aligned}$$

where

$$\lambda_i(-Ah) = \frac{1}{h}\int_0^h e^{-A(h-\tau)} L_i(\tau)\, d\tau, \qquad \mu_{ij}(-Ah) = \frac{1}{h}\int_0^{c_ih} e^{-A(c_ih-\tau)} L_j(\tau)\, d\tau,$$

$L_j(\tau)$ is the Lagrange interpolating polynomial, $c_1, c_2, \ldots, c_p$ are nodes, u_n, $u_{n,i}$ are approximate values of $u(t_n)$, and $u(t_n + c_ih)$, letting $B_{n,i} = g(t_n + c_ih, u_{n,i})$, then the numerical scheme can be written as

$$u_{n+1} = e^{-Ah}u_n + h\sum_{i=1}^{p}\lambda_i(-Ah)B_{n,i}.$$

When $i = 1$, the numerical scheme of the first-order exponential Runge-Kutta method is

$$u_{n+1} = e^{-Ah}u_n + \int_0^h e^{-A(h-\tau)}\,d\tau g(t_n, u_n).$$

That is,

$$u_{n+1} = e^{-Ah}u_n + e^{-Ah}g(t_n, u_n)\left(\frac{1-e^{Ah}}{A}\right).$$

Hence, $u_{n+1} = e^{-Ah}u_n + e^{-Ah}g(t_n, u_n)h$ is called the numerical scheme of the exponential Euler method.

Then, consider the following semi-linear stochastic pantograph differential equations:

$$\begin{cases} dx(t) = (Ax(t) + f(t, x(t), x(pt)))\,dt + g(t, x(t), x(pt))\,dW(t), \\ x(0) = \xi, \end{cases} \tag{2}$$

where $t > 0$, $0 < p < 1$, ξ is the initial function, $W(t)$ is a Wiener process, $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are two given Borel-measurable functions, and f and g are called drift coefficient and diffusion coefficient, respectively. $A \in \mathbb{R}^{n\times n}$ is the generator of a strongly continuous analytical semi-group $S = (S(t))_{t\geq 0}$ [18]. By the definition of the stochastic differential equations, equation (2) can be rewritten as the following stochastic integral equation:

$$\begin{aligned} x(t) = e^{At}\xi &+ \int_0^t e^{A(t-s)}f\bigl(s, x(s), x(ps)\bigr)\,ds \\ &+ \int_0^t e^{A(t-s)}g\bigl(s, x(s), x(ps)\bigr)\,dW(s). \end{aligned} \tag{3}$$

We can derive numerical schemes by [19]. From this, we have

$$\begin{aligned} x(t_{n+1}) = e^{At}x_n &+ \int_{t_n}^{t_{n+1}} e^{A(t_{n+1}-s)}f\bigl(s, x(s), x(ps)\bigr)\,ds \\ &+ \int_{t_n}^{t_{n+1}} e^{A(t_{n+1}-s)}g\bigl(s, x(s), x(ps)\bigr)\,dW(s) \end{aligned} \tag{4}$$

if we choose the interval to approximate the integrals in the drift and diffusion terms, we obtain

$$x_{n+1} = e^{Ah}x_n + e^{Ah}f(t_n, x_n, x_{[pn]})h + e^{Ah}g(t_n, x_n, x_{[pn]})\triangle W_n, \tag{5}$$

where the initial value $\xi = x_0$, x_n is an approximation to analytical solution $x(t_n)$, which is $\mathcal{F}_{t_n}$-measurable, $h > 0$ is the given step-size, and $h = t_{n+1} - t_n$, $\triangle W_n = W(t_{n+1}) - W(t_n)$ are independent $N(0, h)$ distributed stochastic variables. So equation (5) is called the exponential Euler scheme for semi-linear stochastic pantograph differential equations.

3 Mean-square stability of analytical solution

In this part, we illustrate the mean-square stability of the analytical solution for semi-linear stochastic pantograph differential equations under some suitable conditions. First of all, in order to consider the existence and uniqueness of the solution for equation (2), we impose the following assumption.

Assumption 3.1 [20] We assume that f, g are sufficiently smooth and satisfy the Lipschitz condition and the linear growth condition, that is,

(1) (Lipschitz condition) for all $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$, there exist a positive constant K, and $t \in [0, T]$, such that

$$\begin{aligned} &\left|f(t,x_1,y_1) - f(t,x_2,y_2)\right|^2 \vee \left|g(t,x_1,y_1) - g(t,x_2,y_2)\right|^2 \\ &\quad \le K\left(|x_1 - x_2|^2 + |y_1 - y_2|^2\right); \end{aligned} \tag{6}$$

(2) (linear growth condition) for all $(t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$, and assuming there exists a positive constant L

$$\left|f(t,x,y)\right|^2 \vee \left|g(t,x,y)\right|^2 \le L\left(1 + |x|^2 + |y|^2\right), \tag{7}$$

there exists a unique solution $x(t)$ to equation (2) and the solution belongs to $\mathcal{M}^2([0,T];\mathbb{R})$, namely $x(t)$ satisfies $E\int_0^t |x(t)|^2 < \infty$.

Definition 3.1 The solution of equation (2) is said to be mean-square stable if

$$\lim_{t\to\infty} E\left|x(t)\right|^2 = 0. \tag{8}$$

Definition 3.2 [21] $\mu[A]$ is a logarithmic norm of the matrix A, the definition is as follows:

$$\mu[A] = \lim_{\triangle \to 0^+} \frac{\|I + \triangle A\| - 1}{\triangle}.$$

Particularly, if $\|\cdot\|$ denotes the inner norm, $\mu[A]$ can be written

$$\mu[A] = \max_{\xi \neq 0} \frac{\langle A\xi, \xi\rangle}{\|\xi\|^2}.$$

Theorem 3.1 *Assume that the condition* (6) *holds, assume* $\mu[A]$ *and* K *satisfy*

$$1 + 2\mu[A] + 2K + \frac{2K}{p} < 0. \tag{9}$$

Then the analytical solution of equation (2) *is mean-square stable.*

Proof By the Itô formula [22], we have

$$\begin{aligned} d\left|x(t)\right|^2 &= \left[2\left\langle x(t), Ax(t) + f\left(t, x(t), x(pt)\right)\right\rangle + \left|g\left(t, x(t), x(pt)\right)\right|^2\right] dt \\ &\quad + 2\left\langle x(t), g\left(t, x(t), x(pt)\right)\right\rangle dW(t) \end{aligned}$$

$$
\begin{aligned}
&= \left[2\langle x(t), Ax(t)\rangle + 2\langle x(t), f\left(t, x(t), x(pt)\right)\rangle + \left|g\left(t, x(t), x(pt)\right)\right|^2\right] dt \\
&\quad + 2\langle x(t), g\left(t, x(t), x(pt)\right)\rangle \, dW(t).
\end{aligned}
$$

According to condition (6) and the inequality $2ab \leq a^2 + b^2$, we have

$$
\begin{aligned}
2\langle x(t), f\left(t, x(t), x(pt)\right)\rangle &\leq \left|x(t)\right|^2 + \left|f\left(t, x(t), x(pt)\right)\right|^2 \\
&\leq (1 + K)\left|x(t)\right|^2 + K\left|x(pt)\right|^2.
\end{aligned}
$$

Combining with Definition 3.2, we can obtain

$$
\begin{aligned}
d\left|x(t)\right|^2 &\leq \left[2\mu[A]\left|x(t)\right|^2 + (1 + K)\left|x(t)\right|^2 + K\left|x(pt)\right|^2 + K\left|x(t)\right|^2\right. \\
&\quad \left. + K\left|x(pt)\right|^2\right] dt + 2\langle x(t), g\left(t, x(t), x(pt)\right)\rangle \, dW(t) \\
&= \left[\left(1 + 2\mu[A] + 2K\right)\left|x(t)\right|^2 + 2K\left|x(pt)\right|^2\right] dt \\
&\quad + 2\langle x(t), g\left(t, x(t), x(pt)\right)\rangle \, dW(t).
\end{aligned}
$$

Integrating from 0 to t on both sides of the above inequality, it turns into

$$
\begin{aligned}
\left|x(t)\right|^2 &\leq |\xi|^2 + \int_0^t \left[\left(1 + 2\mu[A] + 2K\right)\left|x(s)\right|^2 + 2K\left|x(ps)\right|^2\right] ds \\
&\quad + 2\int_0^t x(s) g\left(s, x(s), x(ps)\right) dW(s).
\end{aligned}
$$

Taking the expectation,

$$
\begin{aligned}
E\left|x(t)\right|^2 &\leq E|\xi|^2 + E\int_0^t \left[\left(1 + 2\mu[A] + 2K\right)\left|x(s)\right|^2 + 2K\left|x(ps)\right|^2\right] ds \\
&\leq E|\xi|^2 + \left(1 + 2\mu[A] + 2K\right) E\int_0^t \left|x(s)\right|^2 ds + \frac{2K}{p} E\int_0^{pt} \left|x(s)\right|^2 ds \\
&\leq E|\xi|^2 + \left(1 + 2\mu[A] + 2K + \frac{2K}{p}\right) E\int_0^t \left|x(s)\right|^2 ds.
\end{aligned}
$$

Together with condition $1 + 2\mu[A] + 2K + \frac{2K}{p} < 0$, we have

$$
\lim_{t\to\infty} E\left|x(t)\right|^2 = 0.
$$

The analytical solution is mean-square stable. Therefore, the theorem is proven. □

4 General mean-square stability of numerical solution of the exponential Euler method

We introduce the exponential Euler method for semi-linear stochastic pantograph differential equations in this section.

Definition 4.1 For any step-size $h > 0$, if the exponential Euler method to equation (2) generates a numerical approximation that satisfies

$$
\lim_{n\to\infty} E|x_n|^2 = 0 \tag{10}
$$

then the numerical method applied to equation (2) is said to be general mean-square stable.

Theorem 4.1 *Suppose that the conditions* (6) *and* (9) *hold, for arbitrary* $h > 0$, *then the numerical solution of the exponential Euler method is general mean-square stable.*

Proof According to equation (5) and taking squares on both sides, we can get

$$\begin{aligned}
|x_{n+1}|^2 &= \left|e^{Ah}x_n + e^{Ah}f(t_n,x_n,x_{[pn]})h + e^{Ah}g(t_n,x_n,x_{[pn]})\triangle W_n\right|^2 \\
&= e^{2\mu[A]h}\left(|x_n|^2 + \left|f(t_n,x_n,x_{[pn]})\right|^2h^2 + \left|g(t_n,x_n,x_{[pn]})\triangle W_n\right|^2\right) \\
&\quad + 2e^{2\mu[A]h}\langle x_n, f(t_n,x_n,X_{[pn]})h\rangle + 2e^{2\mu[A]h}\langle x_n, g(t_n,x_n,x_{[pn]})\triangle W_n\rangle \\
&\quad + 2e^{2\mu[A]h}\langle f(t_n,x_n,x_{[pn]})h, g(t_n,x_n,x_{[pn]})\triangle W_n\rangle.
\end{aligned}$$

Taking the expectation and substituting condition (6), we obtain

$$\begin{aligned}
E|x_{n+1}|^2 &\le e^{2\mu[A]h}E\left[|x_n|^2 + K\left(|x_n|^2 + |x_{[pn]}|^2\right)h^2 + K\left(|x_n|^2 + |x_{[pn]}|^2\right)|\triangle W_n|^2\right] \\
&\quad + e^{2\mu[A]h}E\left[(1+K)|x_n|^2 + K|x_{[pn]}|^2\right]h + 2e^{2\mu[A]h}E\langle x_n, g(t_n,x_n,x_{[pn]})\triangle W_n\rangle \\
&\quad + 2e^{2\mu[A]h}E\langle f(t_n,x_n,x_{[pn]})h, g(t_n,x_n,x_{[pn]})\triangle W_n\rangle. \qquad (11)
\end{aligned}$$

We still note that $E(\triangle W_n) = 0$, $E[(\triangle W_n)^2] = h$, and x_n, $x_{[pn]}$ are $\mathcal{F}_{t_n}$ measurable, then

$$E\langle x_n, g(t_n,x_n,x_{[pn]})\triangle W_n\rangle = E\left(x_n^T g(t_n,x_n,x_{[pn]})\right)E(\triangle W_n|\mathcal{F}_{t_n}) = 0.$$

Similarly

$$\begin{aligned}
&E\langle f(t_n,x_n,x_{[pn]})h, g(t_n,x_n,x_{[pn]})\triangle W_n\rangle = 0, \\
&E\left|g(t_n,x_n,x_{[pn]})\triangle W_n\right|^2 = E\left|g(t_n,x_n,x_{[pn]})\right|^2 E\left(\triangle W_n^2|\mathcal{F}_{t_n}\right) \\
&\qquad = E\left|g(t_n,x_n,x_{[pn]})\right|^2 h \le K\left(E|x_n|^2 + E|x_{[pn]}|^2\right)h.
\end{aligned}$$

Equation (11) turns into

$$\begin{aligned}
E|x_{n+1}|^2 &\le e^{2\mu[A]h}\left[\left(1 + Kh^2 + 2Kh + h\right)E|x_n|^2 + \left(Kh^2 + 2Kh\right)E|x_{[pn]}|^2\right] \\
&= B_1E|x_n|^2 + B_2E|x_{[pn]}|^2,
\end{aligned}$$

where

$$B_1 = e^{2\mu[A]h}\left(1 + Kh^2 + 2Kh + h\right), \qquad B_2 = e^{2\mu[A]h}\left(Kh^2 + 2Kh\right).$$

Then

$$E|x_{n+1}|^2 \le (B_1 + B_2)\max\left\{E|x_n|^2, E|x_{[pn]}|^2\right\},$$

and if

$$2\mu[A]h + \ln\bigl(1 + 2Kh^2 + 4Kh + h\bigr) < 0 \tag{12}$$

the exponential Euler method is mean-square stable. Then we verify that (12) holds under the conditions (9) and the following inequality. We all know that

$$e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

if

$$\bigl(1 + 2Kh^2 + 4Kh + h\bigr) < 1 - 2\mu[A]h + \frac{(-2\mu[A]h)^2}{2!} + \frac{(-2\mu[A]h)^3}{3!}. \tag{13}$$

Simplifying equation (13), we obtain

$$\frac{4}{3}\mu[A]^3h^2 + \bigl(2K - 2\mu[A]\bigr)h + \bigl(1 + 2\mu[A] + 4K\bigr) < 0. \tag{14}$$

Let

$$m(h) = \frac{4}{3}\mu[A]^3h^2 + \bigl(2K - 2\mu[A]\bigr)h + \bigl(1 + 2\mu[A] + 4K\bigr).$$

Due to $0 < p < 1$ and condition (9),

$$1 + 2\mu[A] + 4K < 1 + 2\mu[A] + 2K + \frac{2K}{P} < 0.$$

We can see $\mu[A] < 0$, it is easy to know that

$$m'(h) = \frac{8}{3}\mu[A]^3h + 2K - 2\mu[A] < 0$$

when $h > 0$. We have the monotonicity of the function, namely, $m(h) < m(0)$ and $m(0) = 1 + 2\mu[A] + 4K < 0$. Hence, equation (13) holds and this implies that

$$\bigl(1 + 2Kh^2 + 4Kh + h\bigr) < e^{-2\mu[A]h}$$

and (12) holds.

So

$$B_1 + B_2 = e^{2\mu[A]h}\bigl(1 + 2Kh^2 + 4Kh + h\bigr) < 1.$$

Because of $B_1 + B_2 < 1$, it is not difficult to see that $E|x_n|^2 \le E|x_{[pn]}|^2$, therefore

$$\begin{aligned} E|x_{n+1}|^2 &\le (B_1 + B_2)E|x_{[pn]}|^2 \le (B_1 + B_2)^2E|x_{[p([pn]-1)]}|^2 \\ &\le \cdots \le (B_1 + B_2)^kE|x_0|^2 \end{aligned}$$

as k tends to infinity, $(B_1 + B_2)^k < 1$. Then $\lim_{n\to\infty} E|x_n|^2 = 0$, the exponential Euler method is general mean-square stable. This completes the proof. □

Remark 4.1 When $p = 1$, equation (2) turns into

$$\begin{cases} dx(t) = (Ax(t) + f(t,x(t)))\,dt + g(t,x(t))\,dW(t), \\ x(0) = \xi. \end{cases} \tag{15}$$

For convenience, we consider the scalar semi-linear stochastic pantograph differential equation

$$\begin{cases} dx(t) = (ax(t) + f(t,x(t)))\,dt + g(t,x(t))\,dW(t), \\ x(0) = \xi, \end{cases} \tag{16}$$

where $a < 0$, if conditions (6) and $2a + 2\sqrt{K} + K < 0$ hold, for any step-size $h > 0$, the exponential Euler method is stable. This result was demonstrated by Shi and Xiao [23].

Remark 4.2 Consider the following scalar stochastic pantograph differential equation:

$$\begin{cases} dx(t) = ax(t)\,dt + (bx(t) + bx(pt))\,dW(t), \\ x(0) = \xi. \end{cases} \tag{17}$$

Take $a = -5$, $b = 1$, $c = 2$. It is easy to see the coefficients satisfy the condition $a < -\frac{1}{2}(|b| + |c|)^2$. Using the exponential Euler method for (17), we get

$$x_{n+1} = e^{-5h}x_n + e^{-5h}(x_n + 2x_{[pn]})\triangle W_n. \tag{18}$$

Squaring both sides of (18), taking the expectation, and using the inequality $2ab \leq a^2 + b^2$, we have

$$\begin{aligned} Ex_{n+1}^2 &= e^{-10h}x_n^2 + e^{-10h}\bigl(x_n^2 + 4x_n x_{[pn]} + 4x_{[pn]}^2\bigr)h \\ &\leq e^{-10h}x_n^2 + e^{-10h}\bigl[x_n^2 + 2\bigl(x_n^2 + x_{[pn]}^2\bigr) + 4x_{[pn]}^2\bigr]h \\ &\leq e^{-10h}\bigl[(1+3h)x_n^2 + 6hx_{[pn]}^2\bigr]. \end{aligned}$$

Namely

$$e^{10h}Ex_{n+1}^2 \leq (1+3h)x_n^2 + 6hx_{[pn]}^2.$$

Use the inequality $e^{10h} > 1 + 10h$. So

$$Ex_{n+1}^2 \leq \frac{(1+3h)x_n^2 + 6hx_{[pn]}^2}{e^{10h}} \leq \frac{1+9h}{1+10h}\max\bigl\{x_n^2, x_{[pn]}^2\bigr\}.$$

The coefficients $\frac{1+9h}{1+10h} < 1$. According to Theorem 4.1, the numerical solution produced by the exponential Euler method is mean-square stable for any step-size $h > 0$.

5 Numerical example

We will use numerical example to prove the effectiveness of the exponential Euler method. Consider the following stochastic pantograph differential equation:

$$\begin{cases} dx(t) = [a_1 x(t) + a_2 x(pt)]\,dt + [b_1 x(t) + b_2 x(pt)]\,dW(t), \\ x(0) = 1. \end{cases} \tag{19}$$

If the coefficients of equation (19) satisfy

$$a_1 + |a_2| + \frac{1}{2}\bigl(|b_1| + |b_2|\bigr)^2 < 0, \tag{20}$$

then the solution of (19) is mean-square stable.

Case 1. We choose the coefficients of the test equation (19) as $a_1 = -1.5$, $a_2 = 3$, $b_1 = 1$, $b_2 = 0.5$, and $p = 0.5$. Obviously, the coefficients do not satisfy the condition (20). Numerical solutions produced by the exponential Euler method with $h_1 = 0.05$, $h_2 = 0.5$ are shown Figure 1. It is easy to see that numerical solutions are not mean-square stable.

Case 2. Taking the coefficients as $a_1 = -6.5$, $a_2 = 1$, $b_1 = 1$, $b_2 = 0.5$, and $p = 0.5$. The coefficients satisfy the condition (20). Namely, the analytical solution is stable. We used Matlab to randomly generate 50,000 discrete trajectories, getting the mean-square value of 50,000 trajectories at the same time, that is,

$$Y_j = \frac{1}{50{,}000} \sum_{i=1}^{50{,}000} \bigl|y_j^i\bigr|^2,$$

where y_j^i is numerical solution of i trajectories at the time t_j. Apply the exponential Euler method with step-size $h_3 = 0.05$, $h_4 = 0.5$, $h_5 = 1.5$, and $h_6 = 2.5$ as shown Figure 2. We observe that numerical solutions produced by the exponential Euler method with arbitrary step-sizes $h > 0$ are all stable.

Case 3. Considering (17) and taking $p = 0.5$. We can know that numerical solutions produced by the Euler Maruyama method are not stable under $h = 0.2$, $h = 0.5$ (see [24]). While, under the same step-size, the exponential Euler numerical solutions are stable as shown Figure 3. It is proved that the exponential Euler method is more advantageous than the Euler Maruyama method in certain cases.

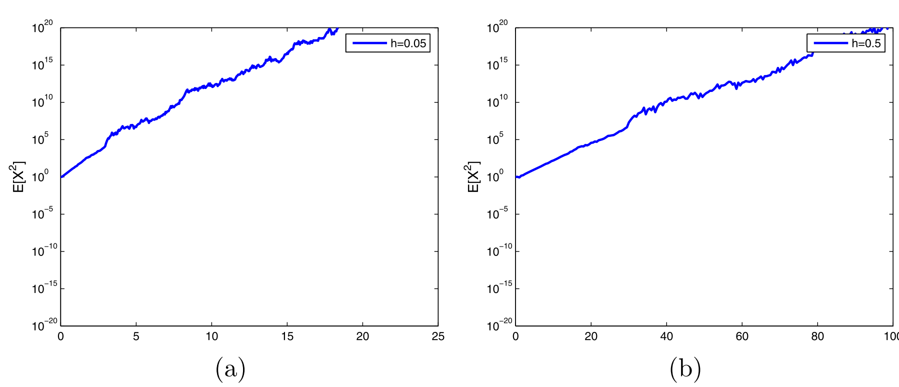

Figure 1 The exponential Euler method with $h_1 = 0.05$, $h_2 = 0.5$.

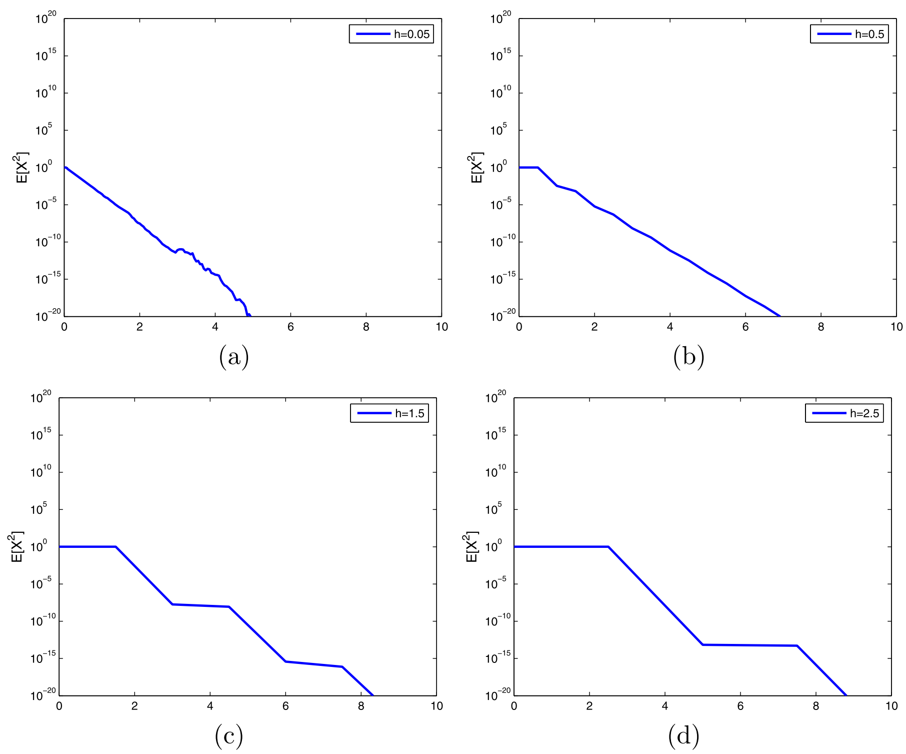

Figure 2 The exponential Euler method with $h_3 = 0.05$, $h_4 = 0.5$, $h_5 = 1.5$, $h_6 = 2.5$.

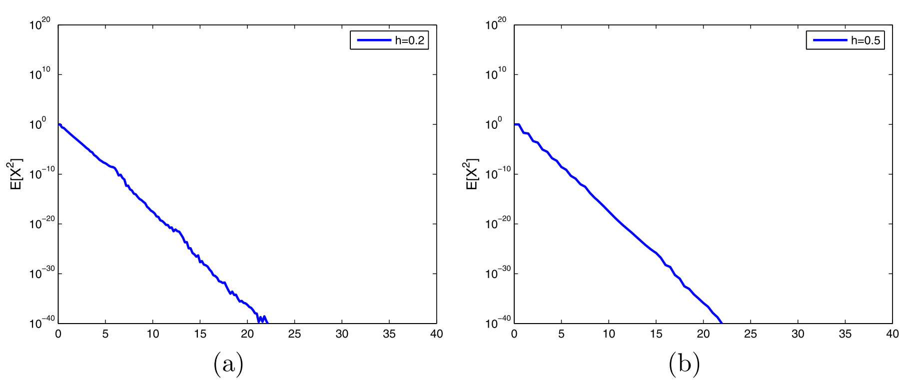

Figure 3 The exponential Euler method with $h_1 = 0.2$, $h_2 = 0.5$.

6 Conclusions

In this paper, we investigate the stability of analytical solutions and numerical solutions for a class of semi-linear stochastic pantograph differential equations. We not only obtain the mean-square stability of the analytical solution under some sufficient conditions but we also prove the general mean-square stability of numerical solution. That is, if the semi-linear stochastic pantograph differential equation is stable, then the exponential Euler method applied to the system is mean-square stable for any step-size $h > 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YZ carried out stochastic differential equations studies, analyzed, and drafted the manuscript. LSL participated in its design and coordination and helped to analyze the manuscript. All authors read and approved the final manuscript.

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References

1. Zhao, GH, Song, MH, Yang, ZW: Mean-square stability of analytic solution and Euler-Maruyama method for impulsive stochastic differential equations. Appl. Math. Comput. **251**, 527-538 (2015)
2. Lu, C, Ding, SH: Persistence and extinction for a stochastic logistic model with infinite delay. Electron. J. Differ. Equ. **2013**, 262 (2013)
3. Baker, C, Buckwar, E: Numerical analysis of explicit one-step methods for stochastic delay differential equations. LMS J. Comput. Math. **3**, 315-335 (2000)
4. Iftikhar, A, Areej, M: Stochastic approach for the solution of multi-pantograph differential equation arising in cell-growth model. Appl. Math. Comput. **261**, 360-372 (2015)
5. Ockendon, JR, Tayler, AB: The dynamics of current collection system for an electric locomotive. Proc. R. Soc. Lond. Ser. A, Math. Phys. Sci. **322**, 447-468 (1971)
6. Mohammed, S: The Lyapunov spectrum and stable manifolds for stochastic linear delay equations. Stoch. Stoch. Rep. **29**, 89-131 (1990)
7. Liu, MZ, Cao, WR, Fan, ZC: Convergence and stability of semi-implicit Euler methods for a linear stochastic delay equations. Appl. Math. Comput. **159**, 127-135 (2004)
8. Baker, C, Buckwar, E: Exponential stability in p-th mean of solutions and of convergence Euler-type solutions of stochastic delay differential equations. J. Comput. Appl. Math. **184**, 404-427 (2005)
9. Mao, XR: Exponential stability of equidistant Euler-Maruyama approximations of stochastic differential delay equations. J. Comput. Appl. Math. **200**, 297-316 (2007)
10. You, SR, Mao, W, Mao, XR, Hu, LJ: Analysis on exponential stability of hybrid pantograph stochastic differential equations with highly nonlinear coefficients. Appl. Math. Comput. **263**, 73-83 (2015)
11. Liu, MZ, Fan, ZC: The asymptotically mean square stability of the linear stochastic pantograph equation. Appl. Math. Comput. **20**, 519-523 (2007)
12. Hua, ZH: Almost surely asymptotic stability of exact and numerical solutions for neutral stochastic pantograph equations. Abstr. Appl. Anal. **2011**, Article ID 143079 (2011)
13. Hua, ZH: Razumikhin-type theorem and mean square asymptotic behavior of the backward Euler method for neutral stochastic pantograph equations. J. Inequal. Appl. **2013**, 299 (2013)
14. Xiao, FY: Mean-square stability of Milstein methods for stochastic pantograph equations. Math. Probl. Eng. **8**, 1024-1231 (2013)
15. Zhou, SB: Almost surely exponential stability of numerical solutions for stochastic pantograph equations. Abstr. Appl. Anal. **2014**, Article ID 751209 (2014)
16. Hochbruck, M, Lubich, C, Selhofer, H: Exponential integrators for large systems of differential equations. SIAM J. Sci. Comput. **19**, 1552-1574 (1998)
17. Hochbruck, M, Ostermann, A: Exponential Runge-Kutta methods for semi-linear parabolic problems. Appl. Numer. Math. **43**, 1069-1090 (2005)
18. Kunze, M, Neerven, N: Approximating the coefficients in semi-linear stochastic partial differential equations. J. Evol. Equ. **11**, 577-604 (2011)
19. Komori, Y, Burrage, K: A stochastic exponential Euler scheme for simulation of stiff biochemical reaction systems. BIT Numer. Math. **54**, 1067-1085 (2014)
20. Fan, ZC, Liu, MZ: Existence and uniqueness of the solutions and convergence of semi-implicit Euler method for stochastic pantograph equations. J. Math. Anal. Appl. **325**, 1142-1159 (2007)
21. Dekker, K, Verwer, JG: Stability of Runge-Kutta Methods for Stiff Nonlinear Differential Equations. CWI Monographs, vol. 2. North-Holland, Amsterdam (1984)
22. Mao, XR: Stochastic Differential Equations and Applications. Harwood, New York (1997)
23. Shi, CM, Xiao, Y, Zhang, CP: The convergence and MS stability of exponential Euler method for semilinear stochastic differential equations. Abstr. Appl. Anal. **2012**, Article ID 350407 (2012)
24. Xiao, Y, Zhang, HY: Convergence and stability of numerical methods with variable step size for stochastic pantograph differential equations. Int. J. Comput. Math. **88**, 2955-2968 (2011)