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A note on Hardy-Littlewood maximal operators

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Abstract

In this paper, we will prove that, for $1 < p < \infty$, the L^p norm of the truncated centered Hardy-Littlewood maximal operator M^c_γ equals the norm of the centered Hardy-Littlewood maximal operator for all $0 < \gamma < \infty$. When $p = 1$, we also find that the weak (1,1) norm of the truncated centered Hardy-Littlewood maximal operator M^c_γ equals the weak (1,1) norm of the centered Hardy-Littlewood maximal operator for $0 < \gamma < \infty$. Moreover, the same is true for the truncated uncentered Hardy-Littlewood maximal operator. Finally, we investigate the properties of the iterated Hardy-Littlewood maximal function.

Keywords: Hardy-Littlewood maximal function; truncated operator; L^p norm; weak (1,1) norm; iterated Hardy-Littlewood maximal function

1 Introduction

Define the centered Hardy-Littlewood maximal function by

$$M^c f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|f(y)\right| dy, \tag{1.1}$$

and the uncentered Hardy-Littlewood maximal function by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B \left|f(y)\right| dy. \tag{1.2}$$

The basic real-variable construct was introduced by Hardy and Littlewood [1] for $n = 1$, and by Wiener [2] for $n \geq 2$. It is well known that the Hardy-Littlewood maximal function plays an important role in many parts of analysis. It is a classical mean operator, and it is frequently used to majorize other important operators in harmonic analysis.

It is clear that

$$M^c f(x) \leq Mf(x) \leq 2^n M^c f(x) \tag{1.3}$$

holds for all $x \in \mathbb{R}^n$. Both M and M^c are sublinear operators. Although the study of the boundedness for M or M^c is fairly completed, it is very hard to calculate the precise norm about M or M^c.

As is well known, the truncated operator has some important properties. In fact, in most situations, L^p boundedness of the truncated operator and the corresponding oscillatory operator is equivalent. There are many works in this regard and the reader can refer to [3] and [4].

Now we define the truncated centered Hardy-Littlewood maximal operator and the truncated uncentered Hardy-Littlewood maximal operator.

Define

$$M_\gamma^c f(x) := \sup_{0<r<\gamma} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy \tag{1.4}$$

and

$$M_\gamma f(x) := \sup_{0<r<\gamma, |y-x|<r} \frac{1}{|B(y,r)|} \int_{B(y,r)} |f(t)|\, dt, \tag{1.5}$$

for $x \in \mathbb{R}^n$ and some real positive number γ.

Obviously, like the inequality (1.3), in the pointwise sense, we immediately deduce from the definition (1.4) and (1.5) that

$$M_\gamma^c f(x) \le M_\rho^c f(x) \le M^c f(x)$$

and

$$M_\gamma f(x) \le M_\rho f(x) \le M f(x),$$

for all $x \in \mathbb{R}^n$, as long as $\gamma \le \rho$. Consequently, referring to the two truncated operators M_γ^c and M_γ, as the sublinear operators, we naturally obtain

$$\|M_\gamma^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \le \|M_\rho^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \le \|M^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)},$$

and

$$\|M_\gamma\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \le \|M_\rho\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \le \|M\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)},$$

if $\gamma \le \rho$, for $1 < p \le \infty$. Clearly, when γ is fixed, for example $\gamma = 1$, $\|M_1^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}$ and $\|M^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}$ are two fixed numbers. We think that it is very significant to make certain the precise relation of the two numbers. In the paper, we will consider the question. Surprisingly, the two numbers are equal whenever $\gamma > 0$. The same is true for $p = 1$.

Now we formulate our main theorems.

Theorem 1.1 *Let M_γ^c be defined by* (1.4) *and $\gamma > 0$. Then*

$$\|M_\gamma^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \|M^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}$$

holds for $1 < p \le \infty$.

Theorem 1.2 *Let M_γ^c be defined by* (1.4) *and* $\gamma > 0$. *Then*

$$\|M_\gamma^c\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)} = \|M^c\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)}$$

holds.

For the truncated uncentered Hardy-Littlewood Maximal operator, we have similar conclusions.

Theorem 1.3 *Let M_γ be defined by* (1.5) *and* $\gamma > 0$. *Then*

$$\|M_\gamma\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \|M\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}$$

holds for $1 < p \le \infty$.

Theorem 1.4 *Let M_γ be defined by* (1.5) *and* $\gamma > 0$. *Then*

$$\|M_\gamma\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)} = \|M\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)}$$

holds.

In Section 4, we will investigate the properties of the iterated Hardy-Littlewood maximal function.

2 Auxiliary and some lemmas

To prove our main theorems, we first provide some definitions and lemmas which will be used in the follows. Some lemmas can be found in the classic literature and here we omit their proofs.

Definition 2.1 Let f be a measurable function on $\mathbb{R}^n$. The distribution function of f is the function d_f defined on $[0, +\infty)$ as follows:

$$d_f(\alpha) = \left|\left\{x \in \mathbb{R}^n : \left|f(x)\right| > \alpha\right\}\right|, \tag{2.1}$$

where $|A|$ is the Lebesgue measure of the measurable set A.

Lemma 2.1 *For $f \in L^p(\mathbb{R}^n)$ with $0 < p < \infty$, we have*

$$\|f\|_{L^p(\mathbb{R}^n)}^p = p\int_0^\infty \alpha^{p-1} d_f(\alpha)\,d\alpha. \tag{2.2}$$

It is easy for us to verify the lemma by Fubini's theorem. For more details as regards this lemma, one can refer to [5].

Lemma 2.2 *Suppose that μ is a positive measure on a σ-algebra $\mathbb{M}$. If $A_1 \subset A_2 \subset A_3 \cdots$, $A_n \in \mathbb{M}$, and $A = \bigcup_{n=1}^\infty A_n$, then*

$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$

Lemma 2.2 can be found in the book [6]. Using Lemma 2.2, we can formulate the following conclusions.

Lemma 2.3 *Suppose that the operators M^c and M^c_γ are defined as in* (1.1) *and* (1.4). *The equality*

$$d_{M^c f}(\lambda) = \lim_{\gamma\to\infty} d_{M^c_\gamma f}(\lambda) \tag{2.3}$$

holds for all $f \in L^p(\mathbb{R}^n)$ and $\lambda > 0$.

Proof For a fixed $x \in \mathbb{R}^n$, by the definition of M^c in (1.1), associate to each ε a ball $B(x, r_\varepsilon)$ which satisfies

$$\frac{1}{|B(x,r_\varepsilon)|}\int_{B(x,r_\varepsilon)} |f(y)|\,dy > M^c f(x) - \varepsilon. \tag{2.4}$$

Now taking $\gamma > r_\varepsilon$, it follows from the definition of M^c_γ that

$$M^c_\gamma f(x) \geq \frac{1}{|B(x,r_\varepsilon)|}\int_{B(x,r_\varepsilon)} |f(y)|\,dy > M^c f(x) - \varepsilon. \tag{2.5}$$

Note that $M^c_\gamma f(x)$ increases as $\gamma \to \infty$. Thus we have

$$\lim_{\gamma\to\infty} M^c_\gamma f \geq Mf. \tag{2.6}$$

Clearly, we have

$$M^c_\gamma f \leq Mf. \tag{2.7}$$

Hence combining (2.6) with (2.7) yields

$$\lim_{\gamma\to\infty} M^c_\gamma f = M^c f. \tag{2.8}$$

Obviously it implies from (2.8) that

$$\lim_{n\to\infty} M^c_n f = M^c f. \tag{2.9}$$

Set

$$A_n = \{x \in \mathbb{R}^n : M^c_n f(x) > \lambda\},$$

and

$$A = \{x \in \mathbb{R}^n : M^c f(x) > \lambda\}.$$

We have $A_n \subset A_{n+1}$ for $n = 1, 2, \ldots,$ and $A = \bigcup_{n=1}^\infty A_n$. It follows from Lemma 2.2 and the definition of the distribution function that

$$d_{M^c f}(\lambda) = |A| = \lim_{n\to\infty} |A_n| = \lim_{n\to\infty} d_{M^c_n f}(\lambda) = \lim_{\gamma\to\infty} d_{M^c_\gamma f}(\lambda).$$

This is our desired result. □

Using the same method as in the proof of Lemma 2.3, we obtain Lemma 2.4.

Lemma 2.4 *Suppose that the operators M and M_γ are defined as in* (1.2) *and* (1.5). *For a given $\lambda > 0$, the equality*

$$d_{Mf}(\lambda) = \lim_{\gamma\to\infty} d_{M_\gamma f}(\lambda) \tag{2.10}$$

holds for all $f \in L^p(\mathbb{R}^n)$.

Lemma 2.5 *Let $1 < p < \infty$. For $\varepsilon > 0$, there exists a function $g \in C_c^\infty(\mathbb{R}^n)$ such that*

$$\frac{\|M^c g\|_{L^p(\mathbb{R}^n)}}{\|g\|_{L^p(\mathbb{R}^n)}} \geq \|M^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} - \varepsilon, \tag{2.11}$$

where

$$\|M^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \sup_{\|f\|_{L^p(\mathbb{R}^n)}\neq 0} \frac{\|M^c f\|_{L^p(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)}}.$$

Proof By the definition of the operator norm of M^c, we can find a function $f \in L^p(\mathbb{R}^n)$ such that

$$\frac{\|M^c f\|_{L^p(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)}} \geq \|M^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} - \frac{\varepsilon}{2}. \tag{2.12}$$

Since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, for $\delta > 0$, there exists a function $g \in C_c^\infty(\mathbb{R}^n)$ which satisfies

$$\|f - g\|_{L^p(\mathbb{R}^n)} < \delta. \tag{2.13}$$

Thus it implies from (2.13) that

$$\|M^c(f - g)\|_{L^p(\mathbb{R}^n)} \leq A\|f - g\|_{L^p(\mathbb{R}^n)} < A\delta, \tag{2.14}$$

where the constant A is a bound of the operator M^c.

Combining (2.13) with (2.14) yields

$$\frac{\|M^c g\|_{L^p(\mathbb{R}^n)}}{\|g\|_{L^p(\mathbb{R}^n)}} \geq \frac{\|M^c f\|_{L^p(\mathbb{R}^n)} - \|M^c(f-g)\|_{L^p(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)} + \|f-g\|_{L^p(\mathbb{R}^n)}} \geq \frac{\|M^c f\|_{L^p(\mathbb{R}^n)} - A\delta}{\|f\|_{L^p(\mathbb{R}^n)} + \delta}. \tag{2.15}$$

If the number δ is small enough, we can immediately deduce that

$$\frac{\|M^c f\|_{L^p(\mathbb{R}^n)} - A\delta}{\|f\|_{L^p(\mathbb{R}^n)} + \delta} \geq \frac{\|M^c f\|_{L^p(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)}} - \frac{\varepsilon}{2} \geq \|M^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} - \varepsilon. \tag{2.16}$$

It implies from (2.15) and (2.16) that the inequality (2.11) holds. □

3 Proof of main theorems

Now we shall prove our main theorems. We first consider the case $1 < p < \infty$.

Proof of Theorem 1.1 For convenience, we first prove

$$\|M_\gamma^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \|M_1^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}$$

for all $0 < \gamma < \infty$.

From the definition of the operator M_γ^c in (1.4), we have

$$M_\gamma^c f(x) = \sup_{0<r<\gamma} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy = \sup_{0<r<\gamma} \frac{1}{\nu_n r^n} \int_{|y|\le r} |f(x-y)|\,dy, \tag{3.1}$$

for $x \in \mathbb{R}^n$ and $0 < \gamma < \infty$, where ν_n is the volume of the unit ball in $\mathbb{R}^n$.

A simple computation implies that

$$\begin{aligned} M_\gamma^c f(\gamma x) &= \sup_{0<r<\gamma} \frac{1}{\nu_n r^n} \int_{|y|\le r} |f(\gamma x-y)|\,dy \\ &= \sup_{0<r<\gamma} \frac{\gamma^n}{\nu_n r^n} \int_{|y|\le \frac{r}{\gamma}} |f(\gamma x-\gamma y)|\,dy \\ &= \sup_{0<\frac{r}{\gamma}<1} \frac{1}{\nu_n (\frac{r}{\gamma})^n} \int_{|y|\le \frac{r}{\gamma}} |(\tau_\gamma f)(x-y)|\,dy \\ &= \sup_{0<r<1} \frac{1}{\nu_n r^n} \int_{|y|\le r} |(\tau_\gamma f)(x-y)|\,dy \\ &= M_1^c(\tau_\gamma f)(x), \end{aligned} \tag{3.2}$$

where the dilation operator τ_γ is defined as follows:

$$(\tau_\gamma f)(x) = f(\gamma x), \tag{3.3}$$

for $\gamma > 0$ and $x \in \mathbb{R}^n$.

It follows from (3.2) that

$$\frac{\|M_\gamma^c f\|_{L^p(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)}} = \frac{\|M_\gamma^c f(\gamma\cdot)\|_{L^p(\mathbb{R}^n)}}{\|f(\gamma\cdot)\|_{L^p(\mathbb{R}^n)}} = \frac{\|M_1^c(\tau_\gamma f)\|_{L^p(\mathbb{R}^n)}}{\|\tau_\gamma f\|_{L^p(\mathbb{R}^n)}}. \tag{3.4}$$

Taking the supremum over all $f \in L^p(\mathbb{R}^n)$ with $\|f\|_{L^p(\mathbb{R}^n)} \ne 0$ for the two sides of equation (3.4), we have

$$\|M_\gamma^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \|M_1^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}. \tag{3.5}$$

Next, we will use equation (3.5) to prove

$$\|M_\gamma^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \|M^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}$$

for all $\gamma > 0$.

Since

$$\|M_\gamma^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \le \|M^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)},$$

we merely need to prove

$$\|M_\gamma^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \ge \|M^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}.$$

By Lemma 2.5, for $\varepsilon > 0$, there exists a function $g \in C_c^\infty(\mathbb{R}^n)$ such that

$$\frac{\|M^c g\|_{L^p(\mathbb{R}^n)}}{\|g\|_{L^p(\mathbb{R}^n)}} \ge \|M^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} - \varepsilon. \tag{3.6}$$

We may assume that the support of g is contained in the ball $B(0,R)$, where R is a positive number. Since $g \in C_c^\infty(\mathbb{R}^n)$ implies $g \in L^p(\mathbb{R}^n)$, naturally we have $M^c g \in L^p(\mathbb{R}^n)$ by the L^p boundedness of the operator M^c. It is not hard to find a positive number S such that

$$\|(M^c g)\chi_{\{|\cdot|\ge S\}}\|_{L^p(\mathbb{R}^n)} \le \varepsilon \|g\|_{L^p(\mathbb{R}^n)}. \tag{3.7}$$

Now we set $\gamma_0 = R + S$. Then it can be deduced from the definition of M_γ^c that

$$M^c g(x) = M_{\gamma_0}^c g(x) \tag{3.8}$$

holds for $|x| < S$.

It follows from (3.6), (3.7), and (3.8) that

$$\begin{aligned}
\|M_{\gamma_0}^c g\|_{L^p(\mathbb{R}^n)} &\ge \|(M_{\gamma_0}^c g)\chi_{\{|\cdot|<S\}}\|_{L^p(\mathbb{R}^n)} \\
&= \|(M^c g)\chi_{\{|\cdot|<S\}}\|_{L^p(\mathbb{R}^n)} \\
&\ge \|M^c g\|_{L^p(\mathbb{R}^n)} - \|(M^c g)\chi_{\{|\cdot|\ge S\}}\|_{L^p(\mathbb{R}^n)} \\
&\ge \|M^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)} - 2\varepsilon \|g\|_{L^p(\mathbb{R}^n)}.
\end{aligned} \tag{3.9}$$

Obviously, (3.9) implies that

$$\frac{\|M_{\gamma_0}^c g\|_{L^p(\mathbb{R}^n)}}{\|g\|_{L^p(\mathbb{R}^n)}} \ge \|M^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} - 2\varepsilon. \tag{3.10}$$

Consequently, the inequality (3.10) yields

$$\|M_{\gamma_0}^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \ge \|M^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} - 2\varepsilon. \tag{3.11}$$

By (3.5) and (3.11), we can derive from the arbitrariness property of ε that

$$\|M_\gamma^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \|M^c\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \tag{3.12}$$

for all $\gamma > 0$.

This finishes the proof of Theorem 1.1. □

Next we will pay attention to proving the weak (1,1) boundedness for the truncated centered Hardy-Littlewood maximal operator.

Proof of Theorem 1.2 First, we prove that

$$\|M_\gamma^c\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)} = \|M_1^c\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)}$$

holds for all $0 < \gamma < \infty$.

From the identity (3.2), we have

$$M_\gamma^c f(\gamma x) = M_1^c(\tau_\gamma f)(x). \tag{3.13}$$

For any $\lambda > 0$, we derive from (3.13) that

$$\begin{aligned} \left|\left\{x : M_1^c(\tau_\gamma f)(x) > \lambda\right\}\right| &= \left|\left\{x : M_\gamma^c f(\gamma x) > \lambda\right\}\right| \\ &= \left|\left\{\frac{x}{\gamma} : M_\gamma^c f(x) > \lambda\right\}\right| \\ &= \frac{1}{\gamma^n}\left|\left\{x : M_\gamma^c f(x) > \lambda\right\}\right|. \end{aligned} \tag{3.14}$$

Thus (3.14) implies that

$$\sup_{\lambda>0} \lambda\left|\left\{x : M_1^c(\tau_\gamma f)(x) > \lambda\right\}\right| = \frac{1}{\gamma^n}\sup_{\lambda>0}\lambda\left|\left\{x : M_\gamma^c f(x) > \lambda\right\}\right|. \tag{3.15}$$

If $\|f\|_{L^1(\mathbb{R}^n)} \neq 0$, then it follows from (3.15) that

$$\begin{aligned} \frac{1}{\gamma^n}\frac{\sup_{\lambda>0}\lambda|\{x : M_\gamma^c f(x) > \lambda\}|}{\|f\|_{L^1(\mathbb{R}^n)}} &= \frac{\sup_{\lambda>0}\lambda|\{x : M_1^c(\tau_\gamma f)(x) > \lambda\}|}{\|f\|_{L^1(\mathbb{R}^n)}} \\ &= \frac{1}{\gamma^n}\frac{\sup_{\lambda>0}\lambda|\{x : M_1^c(\tau_\gamma f)(x) > \lambda\}|}{\|\tau_\gamma f\|_{L^1(\mathbb{R}^n)}}. \end{aligned} \tag{3.16}$$

Now taking the supremum over all $f \in L^1(\mathbb{R}^n)$ with $\|f\|_{L^1(\mathbb{R}^n)} \neq 0$ for the two sides of (3.16), we have

$$\|M_\gamma^c\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)} = \|M_1^c\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)}. \tag{3.17}$$

Next we will use (3.17) to prove that

$$\|M_\gamma^c\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)} = \|M^c\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)}$$

holds for all $\gamma > 0$.

We assert the following equation:

$$\sup_{\lambda>0}\lambda d_{M^c f}(\lambda) = \lim_{\gamma\to\infty}\sup_{\lambda>0}\lambda d_{M_\gamma^c f}(\lambda) \tag{3.18}$$

holds for any $f \in L^1(\mathbb{R}^n)$ with $\|f\|_{L^1} \neq 0$.

Clearly the left side of (3.18) is not smaller than the right side, so it suffices to prove the opposite inequality.

It follows from Lemma 2.3 that

$$\sup_{\lambda>0} \lambda d_{M^c f}(\lambda) = \sup_{\lambda>0} \lambda \Big(\lim_{\gamma\to\infty} d_{M^c_\gamma f}(\lambda) \Big).$$

Set

$$A = \sup_{\lambda>0} \lambda d_{M^c f}(\lambda).$$

For $\varepsilon > 0$, there must be a positive number λ_0 such that

$$A - \varepsilon \le \lambda_0 d_{M^c f}(\lambda_0) \le A.$$

We conclude that

$$\sup_{\lambda>0} \lambda \Big(\lim_{\gamma\to\infty} d_{M^c_\gamma f}(\lambda) \Big) \ge \lim_{\gamma\to\infty} \lambda_0 d_{M^c_\gamma f}(\lambda_0) = \lambda_0 d_{M^c f}(\lambda_0) \ge A - \varepsilon.$$

This is equivalent to

$$\sup_{\lambda>0} \lambda \Big(\lim_{\gamma\to\infty} d_{M^c_\gamma f}(\lambda) \Big) \ge A.$$

Consequently, (3.18) holds.

Using equation (3.18), we deduce that

$$\begin{aligned}
\|M^c\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)} &= \sup_{\|f\|_{L^1(\mathbb{R}^n)}\neq 0} \frac{\sup_{\lambda>0} \lambda d_{M^c f}(\lambda)}{\|f\|_{L^1(\mathbb{R}^n)}} \\
&= \sup_{\|f\|_{L^1(\mathbb{R}^n)}\neq 0} \lim_{\gamma\to\infty} \frac{\sup_{\lambda>0} \lambda d_{M^c_\gamma f}(\lambda)}{\|f\|_{L^1(\mathbb{R}^n)}} \\
&= \lim_{\gamma\to\infty} \sup_{\|f\|_{L^1(\mathbb{R}^n)}\neq 0} \frac{\sup_{\lambda>0} \lambda d_{M^c_\gamma f}(\lambda)}{\|f\|_{L^1(\mathbb{R}^n)}} \\
&= \lim_{\gamma\to\infty} \|M^c_\gamma\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)}. \qquad (3.19)
\end{aligned}$$

Consequently, we immediately obtain our desired conclusion by the two identities (3.17) and (3.19). □

Proof of Theorem 1.3 We conclude from the definition of the operator M_γ in (1.5) that

$$\begin{aligned}
M_\gamma f(\gamma x) &= \sup_{0<r<\gamma, |y-\gamma x|<r} \frac{1}{|B(y,r)|} \int_{B(y,r)} |f(t)|\, dt \\
&= \sup_{0<r<\gamma, |\gamma y-\gamma x|<r} \frac{1}{v_n r^n} \int_{|t|<r} |f(\gamma y - t)|\, dt \\
&= \sup_{0<r<\gamma, |y-x|<\frac{r}{\gamma}} \frac{\gamma^n}{v_n r^n} \int_{|t|<\frac{r}{\gamma}} |f(\gamma y - \gamma t)|\, dt
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{0<\frac{r}{\gamma}<1, |y-x|<\frac{r}{\gamma}} \frac{1}{\nu_n(\frac{r}{\gamma})^n} \int_{|t|<\frac{r}{\gamma}} \left|(\tau_\gamma f)(y-t)\right| dt \\
&= \sup_{0<r<1, |y-x|<r} \frac{1}{|B(y,r)|} \int_{|t|<r} \left|(\tau_\gamma f)(x-t)\right| dt \\
&= M_1(\tau_\gamma f)(x).
\end{aligned} \tag{3.20}
$$

Thus we have

$$
\|M_\gamma\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \|M_1\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \tag{3.21}
$$

for all $\gamma > 0$ and $1 < p < \infty$.

Next we will prove that

$$
\|M_\gamma\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \|M\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}.
$$

If $f \in L^p(\mathbb{R}^n)$, then we have $Mf \in L^p(\mathbb{R}^n)$. It follows from Lemma 2.1, Lemma 2.4, and equation (3.21) that

$$
\begin{aligned}
\|Mf\|_{L^p(\mathbb{R}^n)}^p &= p\int_0^\infty \lambda^{p-1} d_{Mf}(\lambda)\, d\lambda \\
&= p\int_0^\infty \lambda^{p-1} \lim_{\gamma\to\infty} d_{M_\gamma f}(\lambda)\, d\lambda \\
&= \lim_{\gamma\to\infty} p\int_0^\infty \lambda^{p-1} d_{M_\gamma f}(\lambda)\, d\lambda \\
&= \lim_{\gamma\to\infty} \|M_\gamma f\|_{L^p(\mathbb{R}^n)}^p \\
&\le \lim_{\gamma\to\infty} \|M_\gamma\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}^p \|f\|_{L^p(\mathbb{R}^n)}^p \\
&= \|M_1\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}^p \|f\|_{L^p(\mathbb{R}^n)}^p.
\end{aligned} \tag{3.22}
$$

Since we have the obvious inequality

$$
\|M\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \ge \|M_1\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}, \tag{3.23}
$$

we derive from (3.22) that

$$
\|M\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \|M_1\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}.
$$

This is our desired result. □

Proof of Theorem 1.4 Using the almost same methods of proving Theorem 1.2, we can formulate the proof of Theorem 1.4. □

4 Iterated Hardy-Littlewood maximal function

In this section, we will consider the iterated Hardy-Littlewood maximal function.

Let M be the uncentered Hardy-Littlewood maximal function defined by (1.2). Define the iterated Hardy-Littlewood maximal function denoted by M^{k+1} as follows:

$$M^{k+1}f(x) := M\big(M^k f\big)(x), \tag{4.1}$$

for $k = 1, 2, \dots$, and $x \in \mathbb{R}^n$. Set $M^1 f(x) := Mf(x)$.

In order to study the properties of the iterated Hardy-Littlewood maximal function, we first introduce the following lemma.

Lemma 4.1 *Suppose that a sequence* $\{c_i\}_{i=1}^{\infty}$ *satisfies the following two conditions simultaneously*:

(i) $c_1 = r \in (0, 1)$;

(ii) *for any* $k \geq 1$, $c_{k+1} = (1 - r)c_k + r$.

Then $\{c_i\}_{i=1}^{\infty}$ *is strictly monotone increasing and we have*

$$\lim_{k\to\infty} c_k = 1.$$

Proof By the mathematical induction and the two conditions (i) and (ii), we can easily obtain $0 < c_k < 1$ for each $k \in \mathbb{N}$. Moreover, the condition (ii) implies

$$c_{k+1} - c_k = (1 - r)c_k + r - c_k = r(1 - c_k) > 0.$$

This shows that $\{c_i\}_{i=1}^{\infty}$ is strictly monotone increasing. Since $\{c_i\}_{i=1}^{\infty}$ is monotone increasing and has the upper bound, the limit of $\{c_i\}_{i=1}^{\infty}$ exists, and we can easily get

$$\lim_{k\to\infty} c_k = 1. \qquad \square$$

By Lemma 4.1, we have the following theorem.

Theorem 4.2 *For any* $f \in L^{\infty}(\mathbb{R}^n)$, *the equation*

$$\lim_{k\to\infty} M^k f(x) = \|f\|_{\infty} \tag{4.2}$$

holds for any $x \in \mathbb{R}^n$.

Proof If $\|f\|_{\infty} = 0$, the proof is trivial. If $\|f\|_{\infty} > 0$, for any $\varepsilon \in (0, \|f\|_{\infty})$, define a set

$$E_{\varepsilon} := \big\{x \in \mathbb{R}^n : \big|f(x)\big| \geq \|f\|_{\infty} - \varepsilon\big\}. \tag{4.3}$$

Then we have $|E_{\varepsilon}| > 0$, where $|E_{\varepsilon}|$ denotes the Lebesgue measure of E_{ε}. For any fixed point $a \in \mathbb{R}^n$, there exists a number $R > 0$ such that

$$\big|E_{\varepsilon} \cap B(a, R)\big| \geq \frac{1}{2}|E_{\varepsilon}|. \tag{4.4}$$

Denote $\tilde{E}_{\varepsilon} = E_{\varepsilon} \cap B(a, R)$.

Define a set as

$$S_L(f) := \{x \in \mathbb{R}^n : x \text{ is the Lebesgue point of } f \text{ and } M_k f, k = 1, 2, \ldots\}.$$

Actually if $f \in L^p(\mathbb{R}^n)$ with $1 \le p \le \infty$, then $|(S_L(f))^c| = 0$, where $(S_L(f))^c$ denotes the complement set of $S_L(f)$. When $x \in \tilde{E}_\varepsilon \cap S_L(f)$, we derive from (4.3) that

$$\left|f(x)\right| \ge \|f\|_\infty - \varepsilon,$$

and

$$M_k f(x) \ge \|f\|_\infty - \varepsilon,$$

for all $k = 1, 2, \ldots$.

When $x \in B(a, R)$, we consider the uncentered Hardy-Littlewood maximal function of f at the point x. It follows that

$$Mf(x) \ge \frac{1}{|B(a,R)|} \int_{B(a,R)} \left|f(y)\right| dy \ge \frac{|\tilde{E}_\varepsilon|}{|B(a,R)|} \bigl(\|f\|_\infty - \varepsilon\bigr). \tag{4.5}$$

Set

$$r = \frac{|\tilde{E}_\varepsilon|}{|B(a,R)|} > 0.$$

It implies from (4.5) that

$$Mf(x) \ge r\bigl(\|f\|_\infty - \varepsilon\bigr). \tag{4.6}$$

A straightforward computation implies from (4.6) that

$$\begin{aligned} M^2 f(x) &\ge \frac{1}{|B(a,R)|} \int_{B(a,R)} \left|Mf(y)\right| dy \\ &= \frac{1}{|B(a,R)|} \int_{\tilde{E}_\varepsilon} \left|Mf(y)\right| dy + \frac{1}{|B(a,R)|} \int_{B(a,R)\setminus\tilde{E}_\varepsilon} \left|Mf(y)\right| dy \\ &\ge \frac{|\tilde{E}_\varepsilon|}{|B(a,R)|} \bigl(\|f\|_\infty - \varepsilon\bigr) + \frac{|B(a,R)| - |\tilde{E}_\varepsilon|}{|B(a,R)|} r\bigl(\|f\|_\infty - \varepsilon\bigr) \\ &= \bigl(r + (1-r)r\bigr)\bigl(\|f\|_\infty - \varepsilon\bigr). \end{aligned} \tag{4.7}$$

Denote

$$c_1 = r$$

and

$$c_{k+1} = c_1 + (1 - c_1)c_k,$$

for $k = 1, 2, \ldots$.

It implies from (4.7) that

$$M^2 f(x) \geq c_2\big(\|f\|_\infty - \varepsilon\big).$$

Using the inductive method, we can easily obtain

$$M^{k+1} f(x) \geq c_{k+1}\big(\|f\|_\infty - \varepsilon\big).$$

Thus Lemma 4.1 implies that

$$\liminf_{k\to\infty} M^k f(x) \geq \|f\|_\infty - \varepsilon.$$

When $\varepsilon \to 0$, we have

$$\liminf_{k\to\infty} M^k f(x) \geq \|f\|_\infty. \tag{4.8}$$

By the definition of the Hardy-Littlewood function, we obviously deduce

$$\limsup_{k\to\infty} M^k f(x) \leq \|f\|_\infty. \tag{4.9}$$

Combining (4.8) with (4.9) yields

$$\lim_{k\to\infty} M^k f(x) = \|f\|_\infty.$$

By the arbitrary choice of a, we obtain

$$\lim_{k\to\infty} M^k f(x) = \|f\|_\infty$$

for all $x \in \mathbb{R}^n$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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