Iida *Journal of Inequalities and Applications* (2016) 2016:4
DOI 10.1186/s13660-015-0953-4

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Weighted estimates of higher order commutators generated by BMO-functions and the fractional integral operator on Morrey spaces

Takeshi Iida*

*Correspondence:
tiida@fukushima-nct.ac.jp
Department of General Education, National Institute of Technology, Fukushima College, Fukushima, 970-8034, Japan

Abstract

The purpose of this paper is to investigate the weighted estimates of commutators generated by BMO-functions and the fractional integral operator on Morrey spaces. The main result generalizes the Sawano, Sugano, and Tanaka result to a weighted setting.

MSC: 26A33; 42B25; 42B35

Keywords: commutator; fractional integral operator; Morrey space; BMO space; weight

1 Introduction

The aim of this paper is to investigate the weighted inequalities of commutators generated by BMO-functions and the fractional integral operator on Morrey spaces. The main results particularly is related to [1] and [2]. The authors introduced the condition of weights in [1]. Under a certain condition of the weights, we investigate the weighted estimates of commutators generated by BMO-functions and the fractional integral operator on Morrey spaces. The results recover the inequality in [2].

For $1 < p < \infty$, we define $p' := \frac{p}{p-1}$. In this paper, a symbol C is a positive constant. Whenever we evaluate the operator, the constant C may be change from one constant to another. Let $|E|$ denote the Lebesgue measure of E. Let $\mathcal{D}(\mathbb{R}^n)$ be the collection of all dyadic cubes on $\mathbb{R}^n$. All cubes are assumed to have their sides parallel to the coordinate axes. For a cube $Q \subset \mathbb{R}^n$, we use $l(Q)$ to denote the side-length $l(Q)$ and cQ to denote the cube with the same center as Q but with side-length $cl(Q)$. The integral average of a measurable function f over Q is written

$$m_Q(f) = \fint_Q f(x)\,dx = \frac{1}{|Q|}\int_Q f(x)\,dx.$$

By a 'weight' we will mean a non-negative function w that is positive measure a.e. on $\mathbb{R}^n$. Given a weight w and a measurable set E, let

$$w(E) := \int_E w(x)\,dx.$$

First we define the Morrey spaces.

Definition 1 Let $1 < p \le p_0 < \infty$. We define the Morrey space $\mathcal{M}_p^{p_0}(\mathbb{R}^n)$ by

$$\mathcal{M}_p^{p_0}(\mathbb{R}^n) := \{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n); \|f\|_{\mathcal{M}_p^{p_0}} < \infty \},$$

where for all measurable functions f, we define

$$\|f\|_{\mathcal{M}_p^{p_0}} := \sup_{Q \in \mathcal{D}(\mathbb{R}^n)} |Q|^{\frac{1}{p_0}} \left(\fint_Q |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Remark 1

(a) The ordinary Morrey norm is equivalent to the Morrey norm in this paper (see [1]):

$$\sup_{\substack{Q \subset \mathbb{R}^n, \\ Q\text{: cubes}}} |Q|^{\frac{1}{p_0}} \left(\fint_Q |f(x)|^p \, dx \right)^{\frac{1}{p}} \cong \|f\|_{\mathcal{M}_p^{p_0}(\mathbb{R}^n)}.$$

(b) Hölder's inequality gives us the following inequality: If $1 < p \le q \le p_0 < \infty$, then we have

$$\|f\|_{\mathcal{M}_p^{p_0}} \le \|f\|_{\mathcal{M}_q^{p_0}}.$$

We define the BMO space (see [3, 4]) as follows.

Definition 2 For an $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-function b, define

$$\|b\|_{\mathrm{BMO}} := \sup_{Q \subset \mathbb{R}^n} \fint_Q |b(x) - m_Q(b)| \, dx,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. Define

$$\mathrm{BMO}(\mathbb{R}^n) := \{ b \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|b\|_{\mathrm{BMO}} < \infty \}.$$

We define the fractional maximal and integral operators.

Definition 3

(1) Let $0 \le \alpha < n$,

$$M_\alpha f(x) := \sup_{Q \ni x} l(Q)^\alpha \fint_Q |f(y)| \, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ such that $x \in Q$.

(2) Let $0 < \alpha < n$,

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy.$$

The point-wise inequality holds:

$$M_\alpha f(x) \le C I_\alpha f(x),$$

for all positive measurable function f.

It is well known that the following inequality holds (see [5]). The celebrated result is called the Adams inequality.

Theorem A *Let $0 < \alpha < n$, $1 < p \le p_0 < \infty$ and $1 < q \le q_0 < \infty$. Assume that*

$$\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \text{and} \quad \frac{q}{q_0} = \frac{p}{p_0}.$$

Then we have

$$\|I_\alpha f\|_{\mathcal{M}^{q_0}_q} \le C\|f\|_{\mathcal{M}^{p_0}_p},$$

for all $f \in \mathcal{M}^{p_0}_p(\mathbb{R}^n)$.

Let $m \in \mathbb{Z}_+$. The m-fold commutator $[b, I_\alpha]^{(m)}$ is given by the following definition.

Definition 4 Let $0 < \alpha < n$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then we define

$$[b, I_\alpha]^{(m)} f(x) := \int_{\mathbb{R}^n} \frac{(b(x) - b(y))^m}{|x - y|^{n-\alpha}} f(y)\,dy,$$

as long as the integral in the right-hand side makes sense.

Remark 2 The following inequality holds:

$$\bigl|[b, I_\alpha]^{(m)} f(x)\bigr| \le \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} \bigl|f(y)\bigr|\,dy. \tag{1}$$

As shall be verified in the proof of Theorem 1, we virtually consider the operator

$$x \mapsto \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} f(y)\,dy$$

and hence we may assume that the integral defining $[b, I_\alpha]^{(m)} f(x)$ converges for a.e. $x \in \mathbb{R}^n$.

Di-Fazio and Ragusa [6] obtained the next theorem.

Theorem B *Let $0 < \alpha < n$, $1 < p \le p_0 < \infty$ and $1 < q \le q_0 < \infty$. Assume that*

$$\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \text{and} \quad \frac{q}{q_0} = \frac{p}{p_0}.$$

If $b \in \mathrm{BMO}(\mathbb{R}^n)$, then we have

$$\bigl\|[b, I_\alpha]^{(1)} f\bigr\|_{\mathcal{M}^{q_0}_q} \le C\|f\|_{\mathcal{M}^{p_0}_p}.$$

Conversely if $n-\alpha$ is an even integer and

$$\big\|[b,I_\alpha]^{(1)}f\big\|_{\mathcal{M}^{q_0}_q} \le C\|f\|_{\mathcal{M}^{p_0}_p},$$

then $b \in \mathrm{BMO}(\mathbb{R}^n)$.

Komori and Mizuhara [7] removed the restriction '$n-\alpha$ is an even integer'.

Theorem C *Let $0<\alpha<n$, $1<p\le p_0<\infty$ and $0<q\le q_0<\infty$. Assume that*

$$\frac{1}{q_0}=\frac{1}{p_0}-\frac{\alpha}{n} \quad \textit{and} \quad \frac{q}{q_0}=\frac{p}{p_0}.$$

If $b \in \mathrm{BMO}(\mathbb{R}^n)$, then we have

$$\big\|[b,I_\alpha]^{(1)}f\big\|_{\mathcal{M}^{q_0}_q} \le C\|f\|_{\mathcal{M}^{p_0}_p}.$$

Conversely if

$$\big\|[b,I_\alpha]^{(1)}f\big\|_{\mathcal{M}^{q_0}_q} \le C\|f\|_{\mathcal{M}^{p_0}_p},$$

then $b \in \mathrm{BMO}(\mathbb{R}^n)$.

Sawano *et al.* [2] proved the following inequality.

Theorem D *Let $0<\alpha<n$, $1<p\le p_0<\infty$, $1<q\le q_0<\infty$ and $1<r\le r_0<\infty$. Assume that*

$$q<r, \qquad \frac{1}{p_0}>\frac{\alpha}{n}\ge\frac{1}{r_0}, \qquad \frac{1}{q_0}=\frac{1}{p_0}+\frac{1}{r_0}-\frac{\alpha}{n} \quad \textit{and} \quad \frac{q}{q_0}=\frac{p}{p_0}.$$

Suppose that $v \in \mathcal{M}^{r_0}_r(\mathbb{R}^n)$. Then, for $b \in \mathrm{BMO}(\mathbb{R}^n)$, we have

$$\big\|\big([b,I_\alpha]^{(m)}f\big)v\big\|_{\mathcal{M}^{q_0}_q} \le C\|b\|^m_{\mathrm{BMO}}\|v\|_{\mathcal{M}^{r_0}_r}\|f\|_{\mathcal{M}^{p_0}_p}.$$

In the case of $m=0$, we refer to [1, 8, 9]. In this paper, we generalize Theorem D to a weighted setting. On the other hand, in [1], the following theorem is proved.

Theorem E *Let $0<\alpha<n$, $1<p\le p_0<\infty$ and $1<q\le q_0<r_0<\infty$. Assume that*

$$\frac{1}{p_0}>\frac{\alpha}{n}\ge\frac{1}{r_0}, \qquad \frac{1}{q_0}=\frac{1}{p_0}+\frac{1}{r_0}-\frac{\alpha}{n}, \qquad \frac{q}{q_0}=\frac{p}{p_0}$$

and $1<a<\frac{r_0}{q_0}$. Suppose that the weights v and w satisfy the following condition:

$$\begin{aligned}[v,w]_{aq_0,r_0,aq,p/a} &:= \sup_{Q\subset Q'}\left(\frac{|Q|}{|Q'|}\right)^{\frac{1}{aq_0}}\big|Q'\big|^{\frac{1}{r_0}}\left(⨍_Q v(x)^{aq}\,dx\right)^{\frac{1}{aq}}\left(⨍_{Q'} w(x)^{-(p/a)'}\,dx\right)^{\frac{1}{(p/a)'}} \\ &< \infty. \end{aligned} \tag{2}$$

Then we have

$$\left\|(I_\alpha f)v\right\|_{\mathcal{M}_q^{q_0}} \le C[v,w]_{aq_0,r_0,aq,p/a}\|fw\|_{\mathcal{M}_p^{p_0}}.$$

In this paper, we investigate the boundedness of higher order commutators generated by BMO-functions and the fractional integral operator on Morrey spaces corresponding to Theorem E.

2 Main results and their corollaries

In this paper, we obtain two main theorems.

2.1 One of the main results

Theorem 1 *Let* $0<\alpha<n$, $1<p\le p_0<\infty$ *and* $1<q\le q_0<r_0<\infty$. *Assume that*

$$\frac{1}{p_0}>\frac{\alpha}{n}\ge\frac{1}{r_0},\qquad \frac{1}{q_0}=\frac{1}{p_0}+\frac{1}{r_0}-\frac{\alpha}{n},\qquad \frac{q}{q_0}=\frac{p}{p_0}$$

and $1<a<\frac{r_0}{q_0}$. *Suppose that the weights* v *and* w *satisfy the condition* (2). *Then, for* $b\in\mathrm{BMO}(\mathbb{R}^n)$, *we have*

$$\left\|\left([b,I_\alpha]^{(m)}f\right)v\right\|_{\mathcal{M}_q^{q_0}} \le C\|b\|_{\mathrm{BMO}}^m[v,w]_{aq_0,r_0,aq,p/a}\|fw\|_{\mathcal{M}_p^{p_0}}.$$

Remark 3 The condition of Theorem 1 corresponds with the condition of Theorem E. This implies that Theorem 1 gives us the same type of corollaries as in Theorem E.

Taking $w(x)=M_{\frac{aq}{r_0}n}(v^{aq})(x)^{\frac{1}{aq}}$, we have the following corollary.

Corollary 1 *Let* $0<\alpha<n$, $1<p\le p_0<\infty$ *and* $1<q\le q_0<r_0<\infty$. *Assume that*

$$\frac{1}{p_0}>\frac{\alpha}{n}\ge\frac{1}{r_0},\qquad \frac{1}{q_0}=\frac{1}{p_0}+\frac{1}{r_0}-\frac{\alpha}{n},\qquad \frac{q}{q_0}=\frac{p}{p_0}$$

and $1<a<\frac{r_0}{q_0}$. *Let* v *be a weight. Suppose that* $b\in\mathrm{BMO}(\mathbb{R}^n)$, *then we have*

$$\left\|\left([b,I_\alpha]^{(m)}f\right)v\right\|_{\mathcal{M}_q^{q_0}} \le C\|b\|_{\mathrm{BMO}}^m\left\|fM_{\frac{aq}{r_0}n}\left(v^{aq}\right)^{\frac{1}{aq}}\right\|_{\mathcal{M}_p^{p_0}}.$$

Taking $w(x)\equiv 1$, we obtain the following corollary.

Corollary 2 *Let* $0<\alpha<n$, $1<p\le p_0<\infty$ *and* $1<q\le q_0<r_0<\infty$. *Assume that*

$$\frac{1}{p_0}>\frac{\alpha}{n}\ge\frac{1}{r_0},\qquad \frac{1}{q_0}=\frac{1}{p_0}+\frac{1}{r_0}-\frac{\alpha}{n},\qquad \frac{q}{q_0}=\frac{p}{p_0}$$

and $1<a<\frac{r_0}{q_0}$. *Suppose that* $v\in\mathcal{M}_{aq}^{r_0}(\mathbb{R}^n)$. *Then, for* $b\in\mathrm{BMO}(\mathbb{R}^n)$, *we have*

$$\left\|\left([b,I_\alpha]^{(m)}f\right)v\right\|_{\mathcal{M}_q^{q_0}} \le C\|b\|_{\mathrm{BMO}}^m\|v\|_{\mathcal{M}_{aq}^{r_0}}\|f\|_{\mathcal{M}_p^{p_0}}.$$

On the other hand, letting $r_0\to\infty$, we obtain the weighted Adams type inequality for the m-fold commutator $[b,I_\alpha]^{(m)}$.

Corollary 3 *Let* $0 < \alpha < n$, $1 < p \le p_0 < \infty$ *and* $1 < q \le q_0 < \infty$. *Assume that*

$$\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}, \qquad \frac{q}{q_0} = \frac{p}{p_0}$$

and $1 < a < \frac{r_0}{q_0}$. *Suppose that the weights* v *and* w *satisfy the following condition*:

$$[v,w]_{aq_0,aq,p/a} := \sup_{Q\subset Q'} \left(\frac{|Q|}{|Q'|}\right)^{\frac{1}{aq_0}} \left(\fint_Q v(x)^{aq}\,dx\right)^{\frac{1}{aq}} \left(\fint_{Q'} w(x)^{-(p/a)'}\,dx\right)^{\frac{1}{(p/a)'}} < \infty. \tag{3}$$

Then, for $b \in \mathrm{BMO}(\mathbb{R}^n)$, *we have*

$$\left\| \left([b,I_\alpha]^{(m)} f\right) v \right\|_{\mathcal{M}^{q_0}_q} \le C\|b\|^m_{\mathrm{BMO}} [v,w]_{aq_0,aq,p/a} \|fw\|_{\mathcal{M}^{p_0}_p}.$$

Corollary 3 gives us the following inequality in letting $p = p_0$, $q = q_0$ and $v = w$.

Corollary 4 *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ *and* $1 < q < \infty$. *Assume that*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}.$$

Suppose that $w \in A_{p,q}(\mathbb{R}^n)$, *i.e.*

$$[w]_{A_{p,q}(\mathbb{R}^n)} := \sup_{Q\subset\mathbb{R}^n} \left(\fint_Q w(x)^q\,dx\right)^{\frac{1}{q}} \left(\fint_Q w(x)^{-p'}\,dx\right)^{\frac{1}{p'}} < \infty. \tag{4}$$

Then, for $b \in \mathrm{BMO}(\mathbb{R}^n)$, *we have*

$$\left\| \left([b,I_\alpha]^{(m)} f\right) \right\|_{L^q(w^q)} \le C\|b\|^m_{\mathrm{BMO}} [w]_{A_{p,q}(\mathbb{R}^n)} \|f\|_{L^p(w^p)} \quad (m = 0,1,2,\ldots).$$

Corollary 3 and Theorem C give us the following corollary.

Corollary 5 *Let* $0 < \alpha < n$, $1 < p \le p_0 < \infty$ *and* $1 < q \le q_0 < \infty$. *Assume that*

$$\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}, \qquad \frac{q}{q_0} = \frac{p}{p_0}$$

and $a > 1$. *Suppose that the weights* v *and* w *satisfy the condition* (3). *If*

$$\left\| [b,I_\alpha]^{(1)} f \right\|_{\mathcal{M}^{q_0}_q} \le C\|f\|_{\mathcal{M}^{p_0}_p}$$

holds, then we have for $b \in \mathrm{BMO}(\mathbb{R}^n)$,

$$\left\| \left([b,I_\alpha]^{(m)} f\right) v \right\|_{\mathcal{M}^{q_0}_q} \le C\|b\|^m_{\mathrm{BMO}} [v,w]_{aq_0,aq,p/a} \|fw\|_{\mathcal{M}^{p_0}_p}.$$

According to Theorem 1.8 in [2], we can pass our result to the operator given by

$$[\vec{b},I_\alpha]f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \prod_{j=1}^m \left(b_j(x) - b_j(y)\right)dy,$$

where $\vec{b} = (b_1, \dots, b_m)$. By a similar argument to [2], as a consequence of Theorem 1 in this paper, we can obtain the following estimate.

Corollary 6 *Let $0 < \alpha < n$, $1 < p \le p_0 < \infty$ and $1 < q \le q_0 < r_0 < \infty$. Assume that*

$$\frac{1}{p_0} > \frac{\alpha}{n} \ge \frac{1}{r_0}, \qquad \frac{1}{q_0} = \frac{1}{p_0} + \frac{1}{r_0} - \frac{\alpha}{n}, \qquad \frac{q}{q_0} = \frac{p}{p_0}$$

and $1 < a < \frac{r_0}{q_0}$. Suppose that the weights v and w satisfy the condition (2). Then, for $\vec{b} = (b_1, \dots, b_m) \in \mathrm{BMO}(\mathbb{R}^n) \times \cdots \times \mathrm{BMO}(\mathbb{R}^n)$, we have

$$\left\| \left([\vec{b}, I_\alpha]f\right)v \right\|_{\mathcal{M}^{q_0}_q} \le C \left(\prod_{j=1}^m \|b_j\|_{\mathrm{BMO}} \right) [v,w]_{aq_0, r_0, aq, p/a} \|fw\|_{\mathcal{M}^{p_0}_p}.$$

2.2 Fractional integral operators having rough kernel

We define the following operators (see [10–12] and [4]).

Definition 5 Let $0 < \alpha < n$, a measurable function Ω on $\mathbb{R}^n \setminus \{0\}$ and a measurable function b. Then we define

$$I_{\Omega,\alpha} f(x) := \int_{\mathbb{R}^n} \frac{\Omega(x-y) f(y)}{|x-y|^{n-\alpha}}\, dy$$

and

$$[b, I_{\Omega,\alpha}]^{(m)} f(x) := \int_{\mathbb{R}^n} \frac{\Omega(x-y)(b(x)-b(y))^m f(y)}{|x-y|^{n-\alpha}}\, dy.$$

Remark 4 The following inequality holds:

$$\left| [b, I_{\Omega,\alpha}]^{(m)} f(x) \right| \le \int_{\mathbb{R}^n} \frac{|\Omega(x-y)||b(x)-b(y)|^m}{|x-y|^{n-\alpha}} \left| f(y) \right| dy. \tag{5}$$

As shall be verified in the proof of Theorem 2, we consider the operator

$$x \mapsto \int_{\mathbb{R}^n} \frac{|\Omega(x-y)||b(x)-b(y)|^m}{|x-y|^{n-\alpha}} f(y)\, dy$$

and hence we may assume that the integral defining $[b, I_{\Omega,\alpha}]^{(m)} f(x)$ converges for a.e. $x \in \mathbb{R}^n$.

By a similar argument to the proof of Theorem 1, we have the following estimate.

Theorem 2 *Let $1 < s \le \infty$, $0 < \alpha < n$, $1 \le s' < p \le p_0 < \infty$, $1 < q \le q_0 < \infty$ and $1 < r \le r_0 < \infty$. Assume that*

$$\frac{1}{p_0} > \frac{\alpha}{n} \ge \frac{1}{r_0}, \qquad \frac{1}{q_0} = \frac{1}{p_0} + \frac{1}{r_0} - \frac{\alpha}{n}, \qquad \frac{q}{q_0} = \frac{p}{p_0}$$

and $1 < a < \frac{r_0}{q_0}$. Suppose that the weights v and w satisfy $[v^{s'}, w^{s'}]^{\frac{1}{s'}}_{\frac{aq_0}{s'}, \frac{r_0}{s'}, \frac{aq}{s'}, \frac{p}{s'a}} < \infty$. Moreover, suppose that $\Omega \in L^s(\mathbb{S}^{n-1})$ is homogeneous of order 0: For any $\lambda > 0$, $\Omega(\lambda x) = \Omega(x)$. Then, for

$b \in \mathrm{BMO}(\mathbb{R}^n)$, *we have*

$$\left\| \left([b, I_{\Omega,\alpha}]^{(m)} f\right) v \right\|_{\mathcal{M}^{q_0}_q} \leq C \|b\|^m_{\mathrm{BMO}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \left[v^{s'}, w^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'}, \frac{r_0}{s'}, \frac{aq}{s'}, \frac{p}{s'a}} \|fw\|_{\mathcal{M}^{p_0}_p}.$$

Since $[b, I_{\Omega,\alpha}]^{(0)} = I_{\Omega,\alpha}$, we refer to [12]. Theorem 2 recovers the following result (see [4, 11]).

Corollary 7 *Let* $1 < s \leq \infty$, $0 < \alpha < n$, $1 \leq s' < p < \frac{n}{\alpha}$ *and* $1 < q < \infty$. *Assume that*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$$

and $w^{s'} \in A_{\frac{p}{s'}, \frac{q}{s'}}(\mathbb{R}^n)$. *Suppose that* $\Omega \in L^s(\mathbb{S}^{n-1})$ *is homogeneous of order* 0: *For any* $\lambda > 0$, $\Omega(\lambda x) = \Omega(x)$. *Then we have, for* $b \in \mathrm{BMO}(\mathbb{R}^n)$,

$$\left\| \left([b, I_{\Omega,\alpha}]^{(m)} f\right) \right\|_{L^q(w^q)} \leq C \left[w^{s'}\right]^{\frac{1}{s'}}_{A_{\frac{p}{s'}, \frac{q}{s'}}(\mathbb{R}^n)} \|b\|^m_{\mathrm{BMO}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|f\|_{L^p(w^p)}.$$

3 Some lemmas

In this section, we prepare some lemmas for proving main results. We recall the following inequalities (see [3, 13] and [4]).

Lemma 1 (The John-Nirenberg inequality) *Let* $1 \leq p < \infty$ *and let* Q *be a cube. Then there exists a constant* $C > 0$ *such that*

$$\left(\fint_Q \left|b(x) - m_Q(b)\right|^p dx \right)^{\frac{1}{p}} \leq C \|b\|_{\mathrm{BMO}},$$

for all $b \in \mathrm{BMO}(\mathbb{R}^n)$.

We invoke the following decomposition which is derived in [14–16]. We omit the details; see [1, 12] for the proof.

Let $\mathcal{D}(Q_0)$ be the collection of all dyadic subcubes of Q_0, that is, all those cubes obtained by dividing Q_0 into 2^n congruent cubes of half its length, dividing each of those into 2^n congruent cubes. By convention Q_0 itself to $\mathcal{D}(Q_0)$, and so on.

Lemma 2 *Let* $\gamma := m_{3Q_0}(f)$ *and* $A > 2 \cdot 18^n$. *For* $k = 1, 2, \ldots$ *we take*

$$D_k := \bigcup \left\{ Q \in \mathcal{D}(Q_0) : m_{3Q}(f) > \gamma A^k \right\}.$$

For $\theta_1 > 1$, *let*

$$\gamma' := \left(\fint_{3Q_0} \left|f(y)\right|^{\theta_1} dy \right)^{\frac{1}{\theta_1}}$$

and $A' > (2 \cdot 18^n)^{\frac{1}{\theta_1}}$. *For* $k = 1, 2, \ldots$ *we take*

$$D'_k := \bigcup \left\{ Q \in \mathcal{D}(Q_0) : \left(\fint_{3Q} \left|f(y)\right|^{\theta_1} dy \right)^{\frac{1}{\theta_1}} > \gamma' A'^k \right\}.$$

Considering the maximality cube, we have

$$D_k = \bigcup_j Q_{k,j} \quad \textit{and} \quad D'_k = \bigcup_j Q'_{k,j}.$$

Then we have

$$\gamma A^k < m_{3Q_{k,j}}(f) \le 2^n \gamma A^k \quad \textit{and} \quad \gamma' A'^k < \left(\fint_{3Q'_{k,j}} \left|f(y)\right|^{\theta_1} dy \right)^{\frac{1}{\theta_1}} \le 2^{\frac{n}{\theta_1}} \gamma' A^k.$$

Let $E_{k,j} := Q_{k,j} \backslash D_{k+1}$ *and* $E'_{k,j} := Q'_{k,j} \backslash D'_{k+1}$. *Moreover we obtain*

$$|Q_{k,j}| \le 2|E_{k,j}| \quad \textit{and} \quad \left|Q'_{k,j}\right| \le 2\left|E'_{k,j}\right|.$$

Lemma 3 *Under the condition of Theorem* 1, *we can choose auxiliary indices* $\theta_1, \theta_2, \theta_3, \theta_4$ *and* θ_5 *so that the following conditions hold*:

1. $\theta_1, \theta_2, \theta_3, \theta_4$ *and* $\theta_5 \in (1, p)$.
2. $L > 1$ *and* $s \in (q, r)$ *such that* $s\theta_2 < Lq$ *and* $s'\theta_2 < q'$.
3. *For the index* $\theta_1 \in (1, p)$, *we can choose* $a_* > 1$ *such that* $a_*\theta_1 < p$.

Assume in addition that, for these indices,

$$a \ge \max\left\{ \theta_4, L, \frac{p}{(\theta_5(\frac{p}{\theta_5})')'}, \frac{p}{(\theta_1(\frac{p}{\theta_1 a_*})')'}, \theta_3 \right\} > 1.$$

Then we obtain

$$\max\left\{ \theta_5\left(\frac{p}{\theta_5}\right)', \theta_1\left(\frac{p}{\theta_1 a_*}\right)' \right\} \le \left(\frac{p}{a}\right)'.$$

Proof We examine the second item; $s\theta_2 < Lq$ and $s'\theta_2 < q'$. For $0 < \varepsilon < 1$, we take $\delta = \frac{\varepsilon}{q^2} < \varepsilon$. If $s = q + \varepsilon$ and $\theta_2 = 1 + \delta$, then we have the following estimate:

$$\begin{aligned} s\theta_2 &= (q+\varepsilon)(1+\delta) = q + q\delta + \varepsilon + \varepsilon\delta \\ &\le q + q\max\{\varepsilon,\delta\} + \max\{\varepsilon,\delta\} + \max\{\varepsilon,\delta\}^2 \\ &< q + q\max\{\varepsilon,\delta\} + 2\max\{\varepsilon,\delta\} \\ &< q + q\max\{\varepsilon,\delta\} + 2q\max\{\varepsilon,\delta\} \\ &= q\bigl(1 + 3\max\{\varepsilon,\delta\}\bigr) = q(1+3\varepsilon) = Lq. \end{aligned}$$

On the other hand, we check $s'\theta_2 < q'$:

$$q' - s'\theta_2 = \frac{\frac{\varepsilon^2}{q^2} + \frac{\varepsilon}{q}(1-\varepsilon)}{(q-1)(q+\varepsilon-1)} > 0.$$

Next we check $\frac{p}{(\theta_5(\frac{p}{\theta_5})')'} > 1$. Since $\theta_5 > 1$, we obtain

$$\theta_5\left(\frac{p}{\theta_5}\right)' > \left(\frac{p}{\theta_5}\right)' > p'.$$

Therefore we have

$$\left(\theta_5\left(\frac{p}{\theta_5}\right)'\right)' < (p')' = p.$$

This gives us

$$\frac{p}{(\theta_5(\frac{p}{\theta_5})')'} > 1.$$

By a similar argument, we obtain

$$\frac{p}{(\theta_1(\frac{p}{\theta_1 a_*})')'} > 1. \qquad \square$$

Remark 5 The index θ_1 in Lemma 2 corresponds with the index θ_1 in Lemma 3.

4 Proof of Theorem 1

Proof of Theorem 1 Fix a dyadic cube $Q_0 \in \mathcal{D}(\mathbb{R}^n)$. Let $\mathcal{D}_\nu$ be the collection of dyadic cubes. The volume of the elements of $\mathcal{D}_\nu$ is $2^{n\nu}$. For $x \in Q_0$, we have

$$\begin{aligned}
\left|[b, I_\alpha]^{(m)} f(x)\right| &\le C \sum_{\nu\in\mathbb{Z}} \sum_{\substack{Q\in\mathcal{D}_\nu,\\ |Q|=2^{\nu n}}} 2^{-\nu(n-\alpha)} \chi_Q(x) \int_{3Q} \left|b(x) - b(y)\right|^m \left|f(y)\right| \, dy \\
&= C \sum_{\nu\in\mathbb{Z}} \left(\sum_{\substack{Q\in\mathcal{D}_\nu,\\ Q\subseteq Q_0}} + \sum_{\substack{Q\in\mathcal{D}_\nu,\\ Q\supsetneq Q_0}} \right) 2^{-\nu(n-\alpha)} \chi_Q(x) \int_{3Q} \left|b(x) - b(y)\right|^m \left|f(y)\right| \, dy \\
&=: C(A + B).
\end{aligned}$$

We evaluate A and B in Sections 4.1 and 4.2, respectively.

4.1 The estimate of A

By $|b(x) - b(y)|^m \le 2^{m-1}(|b(x) - m_Q(b)|^m + |m_Q(b) - b(y)|^m)$, we obtain

$$\begin{aligned}
A &= \sum_{Q\in\mathcal{D}(Q_0)} l(Q)^{\alpha-n} \chi_Q(x) \int_{3Q} \left|b(x) - b(y)\right|^m \left|f(y)\right| \, dy \\
&\le C \sum_{Q\in\mathcal{D}(Q_0)} l(Q)^\alpha \chi_Q(x) \left|b(x) - m_Q(b)\right|^m ⨍_{3Q} \left|f(y)\right| \, dy \\
&\quad + C \sum_{Q\in\mathcal{D}(Q_0)} l(Q)^\alpha \chi_Q(x) ⨍_{3Q} \left|m_Q(b) - b(y)\right|^m \left|f(y)\right| \, dy.
\end{aligned}$$

We take $\theta_1 > 1$ as in Lemma 2. By Hölder's inequality for $\theta_1 > 1$, we have

$$\begin{aligned}
A &\le C \sum_{Q\in\mathcal{D}(Q_0)} l(Q)^\alpha \chi_Q(x) \left|b(x) - m_Q(b)\right|^m ⨍_{3Q} \left|f(y)\right| \, dy \\
&\quad + C \sum_{Q\in\mathcal{D}(Q_0)} l(Q)^\alpha \chi_Q(x) \left(⨍_{3Q} \left|m_Q(b) - b(y)\right|^{m\theta_1'} \, dy \right)^{\frac{1}{\theta_1'}} \left(⨍_{3Q} \left|f(y)\right|^{\theta_1} \, dy \right)^{\frac{1}{\theta_1}}.
\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned} A &\le C \sum_{Q\in\mathcal{D}(Q_0)} l(Q)^\alpha \chi_Q(x)\bigl|b(x)-m_Q(b)\bigr|^m \fint_{3Q}\bigl|f(y)\bigr|\,dy \\ &\quad + C\|b\|_{\mathrm{BMO}}^m \sum_{Q\in\mathcal{D}(Q_0)} l(Q)^\alpha \chi_Q(x)\biggl(\fint_{3Q}\bigl|f(y)\bigr|^{\theta_1}\,dy\biggr)^{\frac{1}{\theta_1}} \\ &= C\bigl(I+\|b\|_{\mathrm{BMO}}^m II\bigr). \end{aligned}$$

We evaluate I. Let

$$\mathcal{D}_0(Q_0) := \biggl\{Q\in\mathcal{D}(Q_0); \biggl(\fint_{3Q}\bigl|f(y)\bigr|\,dy\biggr)\le\gamma A\biggr\}$$

and

$$\mathcal{D}_{k,j}(Q_0) := \biggl\{Q\in\mathcal{D}(Q_0); Q\subset Q_{k,j}, \gamma A^k < \biggl(\fint_{3Q}\bigl|f(y)\bigr|\,dy\biggr)\le\gamma A^{k+1}\biggr\},$$

where $Q_{k,j}$ is in Lemma 2. Then we have

$$\mathcal{D}(Q_0)=\mathcal{D}_0(Q_0)\cup\biggl(\bigcup_{k,j}\mathcal{D}_{k,j}(Q_0)\biggr).$$

By the duality argument, we have

$$\biggl(\int_{Q_0} I^q\cdot v(x)^q\,dx\biggr)^{\frac{1}{q}} = \sup_{\|g\|_{L^{q'}(Q_0)}=1}\biggl(\int_{Q_0} I\cdot v(x)\bigl|g(x)\bigr|\,dx\biggr).$$

Let $g\ge 0$, $\mathrm{supp}(g)\subset Q_0$, $\|g\|_{L^{q'}(Q_0)}=1$. Then we have

$$\begin{aligned} \int_{Q_0} I\cdot v(x)\bigl|g(x)\bigr|\,dx &\le C\sum_{Q\in\mathcal{D}(Q_0)} l(Q)^\alpha\biggl(\fint_{3Q}\bigl|f(y)\bigr|\,dy\biggr) \\ &\quad\times\int_Q\bigl|b(x)-m_Q(b)\bigr|^m v(x)g(x)\,dx \\ &= C\biggl(\sum_{Q\in\mathcal{D}_0(Q_0)}+\sum_{k,j}\sum_{Q\in\mathcal{D}_{k,j}(Q_0)}\biggr) l(Q)^\alpha\biggl(\fint_{3Q}\bigl|f(y)\bigr|\,dy\biggr) \\ &\quad\times\int_Q\bigl|b(x)-m_Q(b)\bigr|^m v(x)g(x)\,dx \\ &= I_0+\sum_{k,j} I_{k,j}. \end{aligned}$$

We evaluate $I_{k,j}$. If $Q\in\mathcal{D}_{k,j}(Q_0)$, then we have

$$\fint_{3Q}\bigl|f(y)\bigr|\,dy\le\gamma A^{k+1}.$$

Hence we obtain

$$I_{k,j} \le \sum_{Q\in\mathcal{D}_{k,j}(Q_0)} l(Q)^\alpha \gamma A^{k+1} \int_Q \big|b(x)-m_Q(b)\big|^m \nu(x)g(x)\,dx$$
$$\le A \sum_{Q\in\mathcal{D}_{k,j}(Q_0)} l(Q_{k,j})^\alpha \gamma A^{k} \int_Q \big|b(x)-m_Q(b)\big|^m \nu(x)g(x)\,dx.$$

Since

$$\gamma A^k < \fint_{3Q_{k,j}} \big|f(y)\big|\,dy,$$

we obtain

$$I_{k,j} \le A \sum_{Q\in\mathcal{D}_{k,j}(Q_0)} l(Q_{k,j})^\alpha \fint_{3Q_{k,j}} \big|f(y)\big|\,dy \int_Q \big|b(x)-m_Q(b)\big|^m \nu(x)g(x)\,dx.$$

By Hölder's inequality for $\theta_2 > 1$ as in Lemma 3, we obtain

$$I_{k,j} \le Al(Q_{k,j})^\alpha m_{3Q_{k,j}}\big(|f|\big) \sum_{Q\in\mathcal{D}_{k,j}(Q_0)} \left(\int_Q \big|b(x)-m_Q(b)\big|^m \nu(x)g(x)\,dx\right)$$
$$\le Al(Q_{k,j})^\alpha m_{3Q_{k,j}}\big(|f|\big) \sum_{Q\in\mathcal{D}_{k,j}(Q_0)} |Q| \left(\fint_Q \big|b(x)-m_Q(b)\big|^{m\theta_2'}\,dx\right)^{\frac{1}{\theta_2'}}$$
$$\times \left(\fint_Q \nu(x)^{\theta_2} g(x)^{\theta_2}\,dx\right)^{\frac{1}{\theta_2}}.$$

By Lemma 1, we obtain

$$I_{k,j} \le A\|b\|_{\mathrm{BMO}}^m l(Q_{k,j})^\alpha m_{3Q_{k,j}}\big(|f|\big) \sum_{Q\in\mathcal{D}_{k,j}(Q_0)} \int_Q \left(\fint_Q \big(\nu(y)g(y)\big)^{\theta_2}\,dy\right)^{\frac{1}{\theta_2}}\,dx$$
$$\le A\|b\|_{\mathrm{BMO}}^m l(Q_{k,j})^\alpha m_{3Q_{k,j}}\big(|f|\big) \sum_{Q\in\mathcal{D}_{k,j}(Q_0)} \int_Q M\big[(\nu_{k,j}g)^{\theta_2}\big](x)^{\frac{1}{\theta_2}}\,dx,$$

where $\nu_{k,j} = \nu\chi_{Q_{k,j}}$ and the symbol M is the ordinary Hardy-Littlewood maximal operator. By Lemma 2, we have

$$I_{k,j} \le A\|b\|_{\mathrm{BMO}}^m |Q_{k,j}| l(Q_{k,j})^\alpha m_{3Q_{k,j}}\big(|f|\big) \left(\fint_{Q_{k,j}} M\big[(\nu_{k,j}g)^{\theta_2}\big](x)^{\frac{1}{\theta_2}}\,dx\right)$$
$$\le 2A\|b\|_{\mathrm{BMO}}^m |E_{k,j}| l(Q_{k,j})^\alpha m_{3Q_{k,j}}\big(|f|\big) \left(\fint_{Q_{k,j}} M\big[(\nu_{k,j}g)^{\theta_2}\big](x)^{\frac{1}{\theta_2}}\,dx\right)$$
$$= 2A\|b\|_{\mathrm{BMO}}^m \int_{E_{k,j}} l(Q_{k,j})^\alpha m_{3Q_{k,j}}\big(|f|\big) \left(\fint_{Q_{k,j}} M\big[(\nu_{k,j}g)^{\theta_2}\big](x)^{\frac{1}{\theta_2}}\,dx\right)dy.$$

We take $s \in (q,r)$ and $L > 1$ as in Lemma 3. By Hölder's inequality for $s > 1$, we have

$$M\big[(\nu_{k,j}g)^{\theta_2}\big](x)^{\frac{1}{\theta_2}} \le M\big[\nu_{k,j}^{s\theta_2}\big](x)^{\frac{1}{s\theta_2}} M\big[g^{s'\theta_2}\big](x)^{\frac{1}{s'\theta_2}}.$$

By Hölder's inequality for $Lq > 1$, we obtain the following inequality:

$$\begin{aligned}&\left(\fint_{Q_{k,j}} M\big[(v_{k,j}g)^{\theta_2}\big](x)^{\frac{1}{\theta_2}}\,dx\right)\\&\quad\le\left(\fint_{Q_{k,j}} M\big[v_{k,j}^{s\theta_2}\big](x)^{\frac{Lq}{s\theta_2}}\,dx\right)^{\frac{1}{Lq}}\left(\fint_{Q_{k,j}} M\big[g^{s'\theta_2}\big](x)^{\frac{(Lq)'}{s'\theta_2}}\,dx\right)^{\frac{1}{(Lq)'}}.\end{aligned}$$

Since $s\theta_2 < Lq$, the boundedness of $M : L^{\frac{Lq}{s\theta_2}}(\mathbb{R}^n) \to L^{\frac{Lq}{s\theta_2}}(\mathbb{R}^n)$ gives us the following inequality:

$$\begin{aligned}&\left(\fint_{Q_{k,j}} M\big[(v_{k,j}g)^{\theta_2}\big](x)^{\frac{1}{\theta_2}}\,dx\right)\\&\quad\le C\left(\frac{1}{|Q_{k,j}|}\int_{\mathbb{R}^n} v_{k,j}(x)^{Lq}\,dx\right)^{\frac{1}{Lq}}\left(\fint_{Q_{k,j}} M\big[g^{s'\theta_2}\big](x)^{\frac{(Lq)'}{s'\theta_2}}\,dx\right)^{\frac{1}{(Lq)'}}.\end{aligned}$$

Since $a \ge L > 1$, by Hölder's inequality for $\frac{a}{L} \ge 1$,

$$\begin{aligned}&\left(\fint_{Q_{k,j}} M\big[(v_{k,j}g)^{\theta_2}\big](x)^{\frac{1}{\theta_2}}\,dx\right)\\&\quad\le C\left(\fint_{Q_{k,j}} v(x)^{aq}\,dx\right)^{\frac{1}{aq}}\left(\fint_{Q_{k,j}} M\big[g^{s'\theta_2}\big](x)^{\frac{(Lq)'}{s'\theta_2}}\,dx\right)^{\frac{1}{(Lq)'}}.\end{aligned}$$

By Lemma 2, this implies that

$$I_{k,j} \le 2A\|b\|_{\mathrm{BMO}}^m \int_{E_{k,j}} M_{\alpha,aq}(f,v)(x)\cdot M\Big[M\big[g^{s'\theta_2}\big]^{\frac{(Lq)'}{s'\theta_2}}\Big](x)^{\frac{1}{(Lq)'}}\,dx,$$

where

$$M_{\alpha,aq}(f,v)(x) := \sup_{Q\ni x} l(Q)^{\alpha} m_{3Q}(f)\left(\fint_Q v(x)^{aq}\,dx\right)^{\frac{1}{aq}}.$$

A similar argument gives us the following estimate:

$$I_0 \le 2A\|b\|_{\mathrm{BMO}}^m \int_{E_0} M_{\alpha,aq}(f,v)(x)\cdot M\Big[M\big[g^{s'\theta_2}\big]^{\frac{(Lq)'}{s'\theta_2}}\Big](x)^{\frac{1}{(Lq)'}}\,dx.$$

By summing up I_0 and $I_{k,j}$, we obtain

$$I_0 + \sum_{k,j} I_{k,j} \le 2A\|b\|_{\mathrm{BMO}}^m \int_{Q_0} M_{\alpha,aq}(f,v)(x)\cdot M\Big[M\big[g^{s'\theta_2}\big]^{\frac{(Lq)'}{s'\theta_2}}\Big](x)^{\frac{1}{(Lq)'}}\,dx.$$

By Hölder's inequality for $q > 1$, we have

$$\begin{aligned}&\int_{Q_0} M_{\alpha,aq}(f,v)(x)\cdot M\Big[M\big[g^{s'\theta_2}\big]^{\frac{(Lq)'}{s'\theta_2}}\Big](x)^{\frac{1}{(Lq)'}}\,dx\\&\quad\le\left(\int_{Q_0} M_{\alpha,aq}(f,v)(x)^q\,dx\right)^{\frac{1}{q}}\left(\int_{Q_0} M\Big[M\big[g^{s'\theta_2}\big]^{\frac{(Lq)'}{s'\theta_2}}\Big](x)^{\frac{q'}{(Lq)'}}\,dx\right)^{\frac{1}{q'}}.\end{aligned}$$

Since $(Lq)' < q'$, the boundedness of $M : L^{\frac{q'}{(Lq)'}}(\mathbb{R}^n) \to L^{\frac{q'}{(Lq)'}}(\mathbb{R}^n)$ gives us the following inequality:

$$\left(\int_{Q_0} M\bigl[M\bigl[g^{s'\theta_2}\bigr]^{\frac{(Lq)'}{s'\theta_2}}\bigr](x)^{\frac{q'}{(Lq)'}}\,dx\right)^{\frac{1}{q'}} \le C\left(\int_{\mathbb{R}^n} M\bigl[g^{s'\theta_2}\bigr](x)^{\frac{(Lq)'}{s'\theta_2}\cdot\frac{q'}{(Lq)'}}\,dx\right)^{\frac{1}{q'}}$$
$$= C\left(\int_{\mathbb{R}^n} M\bigl[g^{s'\theta_2}\bigr](x)^{\frac{q'}{s'\theta_2}}\,dx\right)^{\frac{1}{q'}}.$$

Since $s'\theta_2 < q'$, the boundedness of $M : L^{\frac{q'}{s'\theta_2}}(\mathbb{R}^n) \to L^{\frac{q'}{s'\theta_2}}(\mathbb{R}^n)$ gives us the following inequality:

$$\left(\int_{\mathbb{R}^n} M\bigl[g^{s'\theta_2}\bigr](x)^{\frac{q'}{s'\theta_2}}\,dx\right)^{\frac{1}{q'}} \le C\left(\int_{Q_0} \bigl|g(x)\bigr|^{s'\theta_2\cdot\frac{q'}{s'\theta_2}}\,dx\right)^{\frac{1}{q'}}$$
$$= C\left(\int_{Q_0} \bigl|g(x)\bigr|^{q'}\,dx\right)^{\frac{1}{q'}} = C.$$

By Hölder's inequality for $\frac{p}{a} > 1$, we obtain

$$M_{\alpha,aq}(f,v)(x) \le \sup_{Q\ni x} l(Q)^{\alpha} m_{3Q}\bigl(|fw|^{\frac{p}{a}}\bigr)^{\frac{a}{p}} \left(⨍_Q v(y)^{aq}\,dy\right)^{\frac{1}{aq}} \left(⨍_{3Q} w(y)^{-(p/a)'}\,dy\right)^{\frac{1}{(p/a)'}}.$$

By the condition (2), we obtain

$$M_{\alpha,aq}(f,v)(z) \le C[v,w]_{aq_0,r_0,aq,p/a} \sup_{Q\ni z} l(Q)^{\alpha-\frac{n}{r_0}} m_{3Q}\bigl(|fw|^{\frac{p}{a}}\bigr)^{\frac{a}{p}}$$
$$\le C[v,w]_{aq_0,r_0,aq,p/a} M_{(\alpha-\frac{n}{r_0})\frac{p}{a}}\bigl((fw)^{\frac{p}{a}}\bigr)(z)^{\frac{a}{p}}.$$

This implies that

$$|Q_0|^{\frac{1}{q_0}}\left(⨍_{Q_0} M_{\alpha,aq}(f,v)(z)^q\,dz\right)^{\frac{1}{q}} \le C[v,w]_{aq_0,r_0,aq,p/a} \bigl\| M_{(\alpha-\frac{n}{r_0})\frac{p}{a}}\bigl((fw)^{\frac{p}{a}}\bigr)\bigr\|^{\frac{a}{p}}_{\mathcal{M}^{\frac{aq_0}{p}}_{\frac{aq}{p}}}.$$

Since

$$\frac{1}{q_0}\cdot\frac{p}{a} = \frac{1}{p_0}\cdot\frac{p}{a} - \frac{(\alpha - \frac{n}{r_0})\cdot\frac{p}{a}}{n} \quad \text{and} \quad \frac{\frac{ap_0}{p}}{\frac{aq_0}{p}} = \frac{a}{\frac{aq}{p}},$$

by Theorem A, we have

$$|Q_0|^{\frac{1}{q_0}}\left(⨍_{Q_0} M_{\alpha,aq}(f,v)(z)^q\,dz\right)^{\frac{1}{q}}$$
$$\le C[v,w]_{aq_0,r_0,aq,p/a} \bigl\|(fw)^{\frac{p}{a}}\bigr\|^{\frac{a}{p}}_{\mathcal{M}^{\frac{ap_0}{p}}_a}$$
$$= C[v,w]_{aq_0,r_0,aq,p/a}\left(\sup_Q |Q|^{\frac{p}{ap_0}}\left(⨍_Q \bigl|f(x)w(x)\bigr|^{\frac{p}{a}\cdot a}\,dx\right)^{\frac{1}{a}}\right)^{\frac{a}{p}}$$

$$
\begin{aligned}
&= C[v,w]_{aq_0,r_0,aq,p/a} \sup_Q |Q|^{\frac{1}{p_0}} \left(\fint_Q \left|f(x)\right|^p w(x)^p \, dx \right)^{\frac{1}{p}} \\
&= C[v,w]_{aq_0,r_0,aq,p/a} \|fw\|_{\mathcal{M}^{p_0}_p}.
\end{aligned}
$$

We evaluate II. Let

$$
\mathcal{D}'_0(Q_0) := \left\{ Q \in \mathcal{D}(Q_0); \left(\fint_{3Q} \left|f(y)\right|^{\theta_1} dy \right)^{\frac{1}{\theta_1}} \le \gamma' A' \right\}
$$

and

$$
\mathcal{D}'_{k,j}(Q_0) := \left\{ Q \in \mathcal{D}(Q_0); Q \subset Q'_{k,j}, \gamma' A'^k < \left(\fint_{3Q} \left|f(y)\right|^{\theta_1} dy \right)^{\frac{1}{\theta_1}} \le \gamma' A'^{k+1} \right\},
$$

where $Q'_{k,j}$ is found in Lemma 2. Then we have

$$
\mathcal{D}(Q_0) = \mathcal{D}'_0(Q_0) \cup \left(\bigcup_{k,j} \mathcal{D}'_{k,j}(Q_0) \right).
$$

By the duality argument, we have

$$
\left(\int_{Q_0} II^q \cdot v(x)^q \, dx \right)^{\frac{1}{q}} = \sup_{\|g\|_{L^{q'}(Q_0)}=1} \left(\int_{Q_0} II \cdot v(x) \left|g(x)\right| dx \right).
$$

Let $g \ge 0$ be such that $\mathrm{supp}(g) \subset Q_0$ and $\|g\|_{L^{q'}(Q_0)} = 1$. We have

$$
\begin{aligned}
\int_{Q_0} II \cdot v(x) g(x) \, dx &\le \sum_{Q \in \mathcal{D}(Q_0)} l(Q)^\alpha \left(\fint_{3Q} \left|f(y)\right|^{\theta_1} dy \right)^{\frac{1}{\theta_1}} \|vg\|_{L^1(Q)} \\
&\le \left(\sum_{Q \in \mathcal{D}'_0(Q_0)} + \sum_{k,j} \sum_{Q \in \mathcal{D}'_{k,j}(Q_0)} \right) l(Q)^\alpha \left(\fint_{3Q} \left|f(y)\right|^{\theta_1} dy \right)^{\frac{1}{\theta_1}} \|vg\|_{L^1(Q)} \\
&\le \left(II_0 + \sum_{k,j} II_{k,j} \right).
\end{aligned}
$$

We evaluate $II_{k,j}$. If $Q \in \mathcal{D}'_{k,j}(Q_0)$, then we have

$$
\left(\fint_{3Q} \left|f(y)\right|^{\theta_1} dy \right)^{\frac{1}{\theta_1}} \le \gamma' A'^{k+1}.
$$

Therefore we obtain

$$
\begin{aligned}
II_{k,j} &\le \sum_{Q \in \mathcal{D}'_{k,j}(Q_0)} l(Q)^\alpha \left(\fint_{3Q} \left|f(y)\right|^{\theta_1} dy \right)^{\frac{1}{\theta_1}} \int_Q v(x) g(x) \, dx \\
&\le \gamma' A'^{k+1} \sum_{Q \in \mathcal{D}'_{k,j}(Q_0)} l(Q)^\alpha \int_Q v(x) g(x) \, dx.
\end{aligned}
$$

Since

$$\gamma' A'^k \le \left(\fint_{3Q'_{k,j}} |f(y)|^{\theta_1} \, dy \right)^{\frac{1}{\theta_1}},$$

we obtain

$$II_{k,j} \le A' \left(\fint_{3Q'_{k,j}} |f(y)|^{\theta_1} \, dy \right)^{\frac{1}{\theta_1}} l(Q'_{k,j})^\alpha \left(\fint_{Q'_{k,j}} v(x) g(x) \, dx \right) |Q'_{k,j}|.$$

By Hölder's inequality for $\theta_3 > 1$ as in Lemma 3, we have

$$\begin{aligned} II_{k,j} &\le A' \left(\fint_{3Q'_{k,j}} |f(y)|^{\theta_1} \, dy \right)^{\frac{1}{\theta_1}} l(Q'_{k,j})^\alpha \left(\fint_{Q'_{k,j}} v(x)^{\theta_3 q} \, dx \right)^{\frac{1}{\theta_3 q}} \\ &\quad \times \left(\fint_{Q'_{k,j}} g(x)^{(\theta_3 q)'} \, dx \right)^{\frac{1}{(\theta_3 q)'}} |Q'_{k,j}|. \end{aligned}$$

By Lemma 2, we obtain

$$\begin{aligned} II_{k,j} &\le 2A' \left(\fint_{3Q'_{k,j}} |f(y)|^{\theta_1} \, dy \right)^{\frac{1}{\theta_1}} l(Q'_{k,j})^\alpha \left(\fint_{Q'_{k,j}} v(x)^{\theta_3 q} \, dx \right)^{\frac{1}{\theta_3 q}} \\ &\quad \times \left(\fint_{Q'_{k,j}} g(x)^{(\theta_3 q)'} \, dx \right)^{\frac{1}{(\theta_3 q)'}} |E'_{k,j}| \\ &= 2A' \int_{E'_{k,j}} l(Q'_{k,j})^\alpha \left(\fint_{3Q'_{k,j}} |f(x)|^{\theta_1} \, dx \right)^{\frac{1}{\theta_1}} \left(\fint_{Q'_{k,j}} v(x)^{\theta_3 q} \, dx \right)^{\frac{1}{\theta_3 q}} \\ &\quad \times \left(\fint_{Q'_{k,j}} g(x)^{(\theta_3 q)'} \, dx \right)^{\frac{1}{(\theta_3 q)'}} \, dy \\ &\le 2A' \int_{E'_{k,j}} \tilde{M}_{\alpha,\theta_1,\theta_3 q}(f^{\theta_1}, v)(y) \cdot M[g^{(\theta_3 q)'}](y)^{\frac{1}{(\theta_3 q)'}} \, dy, \end{aligned}$$

where

$$\tilde{M}_{\alpha,\theta_1,\theta_3 q}(f^{\theta_1}, v)(y) := \sup_{Q \ni y} l(Q)^\alpha \left(\fint_{3Q} |f(x)|^{\theta_1} \, dx \right)^{\frac{1}{\theta_1}} \left(\fint_Q v(x)^{\theta_3 q} \, dx \right)^{\frac{1}{\theta_3 q}}.$$

A similar argument gives us the following estimate:

$$II_0 \le 2A' \int_{E'_0} \tilde{M}_{\alpha,\theta_1,\theta_3 q}(f^{\theta_1}, v)(y) \cdot M[g^{(\theta_3 q)'}](y)^{\frac{1}{(\theta_3 q)'}} \, dy.$$

By summing up II_0 and $II_{k,j}$, we obtain

$$II_0 + \sum_{k,j} II_{k,j} \le 2A' \int_{Q_0} \tilde{M}_{\alpha,\theta_1,\theta_3 q}(f^{\theta_1}, v)(y) \cdot M[g^{(\theta_3 q)'}](y)^{\frac{1}{(\theta_3 q)'}} \, dy.$$

By Hölder's inequality for $q > 1$, we have

$$\int_{Q_0} \tilde{M}_{\alpha,\theta_1,\theta_3 q}\big(f^{\theta_1}, \nu\big)(y) \cdot M\big[g^{(\theta_3 q)'}\big](y)^{\frac{1}{(\theta_3 q)'}}\, dy \leq \left(\int_{Q_0} \tilde{M}_{\alpha,\theta_1,\theta_3 q}\big(f^{\theta_1}, \nu\big)(y)^q\, dy\right)^{\frac{1}{q}} \cdot \left(\int_{Q_0} M\big[g^{(\theta_3 q)'}\big](y)^{\frac{q'}{(\theta_3 q)'}}\, dy\right)^{\frac{1}{q'}}.$$

Since $(\theta_3 q)' < q'$ and $\operatorname{supp}(g) \subset Q_0$, by the boundedness of $M : L^{\frac{q'}{(\theta_3 q)'}}(\mathbb{R}^n) \to L^{\frac{q'}{(\theta_3 q)'}}(\mathbb{R}^n)$, we have

$$\left(\int_{Q_0} M\big[g^{(\theta_3 q)'}\big](y)^{\frac{q'}{(\theta_3 q)'}}\, dy\right)^{\frac{1}{q'}} \leq C\left(\int_{Q_0} g(x)^{(\theta_3 q)' \cdot \frac{q'}{(\theta_3 q)'}}\, dx\right)^{\frac{1}{q'}} = C.$$

Therefore we have

$$\int_{Q_0} \tilde{M}_{\alpha,\theta_1,\theta_3 q}\big(f^{\theta_1}, \nu\big)(y) \cdot M\big[g^{(\theta_3 q)'}\big](y)^{\frac{1}{(\theta_3 q)'}}\, dy \leq C\left(\int_{Q_0} \tilde{M}_{\alpha,\theta_1,\theta_3 q}\big(f^{\theta_1}, \nu\big)(y)^q\, dy\right)^{\frac{1}{q}}.$$

By Hölder's inequality for $\frac{p}{a_* \theta_1} > 1$ as in Lemma 3, we have

$$\begin{aligned} \tilde{M}_{\alpha,\theta_1,\theta_3 q}\big(f^{\theta_1}, \nu\big)(x) &\leq C \sup_{Q \ni x} l(Q)^{\alpha} m_{3Q}\big(|fw|^{\frac{p}{a_*}}\big)^{\frac{a_*}{p}} \\ &\quad \times \left(\fint_{3Q} w(y)^{-\theta_1 (\frac{p}{\theta_1 a_*})'}\, dy\right)^{\frac{1}{\theta_1} \frac{1}{(\frac{p}{\theta_1 a_*})'}} \left(\fint_{Q} \nu(y)^{\theta_3 q}\, dy\right)^{\frac{1}{\theta_3 q}}. \end{aligned}$$

By Lemma 3, we have $\theta_1(\frac{p}{\theta_1 a_*})' \leq (\frac{p}{a})'$. By Hölder's inequality, we have

$$\begin{aligned} \tilde{M}_{\alpha,\theta_1,\theta_3 q}\big(f^{\theta_1}, \nu\big)(x) &\leq C \sup_{Q \ni x} \big(l(Q)^{\alpha \cdot \frac{p}{a_*}} m_{3Q}\big(|fw|^{\frac{p}{a_*}}\big)\big)^{\frac{a_*}{p}} \left(\frac{|3Q|}{|Q|}\right)^{\frac{1}{aq_0}} |3Q|^{-\frac{1}{r_0}} \\ &\quad \times \left(\frac{|Q|}{|3Q|}\right)^{\frac{1}{aq_0}} |3Q|^{\frac{1}{r_0}} \left(\fint_{Q} \nu(y)^{\theta_3 q}\, dy\right)^{\frac{1}{\theta_3 q}} \left(\fint_{3Q} w(y)^{-(p/a)'}\, dy\right)^{\frac{1}{(p/a)'}}. \end{aligned}$$

By the condition (2), we obtain

$$\begin{aligned} \tilde{M}_{\alpha,\theta_1,\theta_3 q}\big(f^{\theta_1}, \nu\big)(x) &\leq C[\nu, w]_{aq_0, r_0, aq, p/a} \sup_{Q \ni x} \big(l(Q)^{\alpha \cdot \frac{p}{a_*} - \frac{n}{r_0} \cdot \frac{p}{a_*}} m_{3Q}\big(|fw|^{\frac{p}{a_*}}\big)\big)^{\frac{a_*}{p}} \\ &= C[\nu, w]_{aq_0, r_0, aq, p/a} \cdot M_{(\alpha - \frac{n}{r_0})\frac{p}{a_*}}\big(|fw|^{\frac{p}{a_*}}\big)(x)^{\frac{a_*}{p}}. \end{aligned}$$

This implies that

$$|Q_0|^{\frac{1}{q_0}} \left(\fint_{Q_0} \tilde{M}_{\alpha,\theta_1,\theta_3 q}\big(f^{\theta_1}, \nu\big)(x)^q\, dx\right)^{\frac{1}{q}} \leq C[\nu, w]_{aq_0, r_0, aq, p/a} \big\| M_{(\alpha - \frac{n}{r_0})\frac{p}{a_*}}\big(|fw|^{\frac{p}{a_*}}\big)\big\|^{\frac{a_*}{p}}_{\mathcal{M}^{\frac{a_* q_0}{p}}_{\frac{a_* q}{p}}}.$$

Since

$$\frac{1}{q_0} \cdot \frac{p}{a_*} = \frac{1}{p_0} \cdot \frac{p}{a_*} - \frac{(\alpha - \frac{n}{r_0}) \cdot \frac{p}{a_*}}{n} \quad \text{and} \quad \frac{\frac{a_* p_0}{p}}{\frac{a_* q_0}{p}} = \frac{a_*}{\frac{a_* q}{p}},$$

by Theorem A, we have

$$\left\| M_{(\alpha-\frac{n}{r_0})\frac{p}{a_*}}\left(|fw|^{\frac{p}{a_*}}\right)\right\|^{\frac{a_*}{p}}_{\mathcal{M}^{\frac{a_*q_0}{p}}_{\frac{a_*q}{p}}} \le \left\| |fw|^{\frac{p}{a_*}}\right\|^{\frac{a_*}{p}}_{\mathcal{M}^{\frac{a_*p_0}{p}}_{a_*}} = \|fw\|_{\mathcal{M}^{p_0}_p}.$$

Therefore we have

$$\|II\cdot v\|_{\mathcal{M}^{q_0}_q} \le C[v,w]_{aq_0,r_0,aq,p/a}\|fw\|_{\mathcal{M}^{p_0}_p}.$$

4.2 The estimate of B

Since $|b(x)-b(y)|^m \le 2^{m-1}(|b(x)-m_Q(b)|^m + |m_Q(b)-b(y)|^m)$, we have

$$\begin{aligned}&\int_{3Q}\left|b(x)-b(y)\right|^m\left|f(y)\right|dy\\&\quad\le C\int_{3Q}\left|b(x)-m_Q(b)\right|^m\left|f(y)\right|dy + C\int_{3Q}\left|m_Q(b)-b(y)\right|^m\left|f(y)\right|dy.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}v(x)B &\le Cv(x)\sum_{\substack{Q\supsetneq Q_0,\\ Q\in\mathcal{D}(\mathbb{R}^n)}}|Q|^{(\frac{\alpha}{n}-1)}\chi_Q(x)\int_{3Q}\left|b(x)-m_Q(b)\right|^m\left|f(y)\right|dy\\&\quad + Cv(x)\sum_{\substack{Q\supsetneq Q_0,\\ Q\in\mathcal{D}(\mathbb{R}^n)}}|Q|^{(\frac{\alpha}{n}-1)}\chi_Q(x)\int_{3Q}\left|m_Q(b)-b(y)\right|^m\left|f(y)\right|dy\\&=: CC_1[f,v](x)+CC_2[f,v](x).\end{aligned}$$

By Hölder's inequality and the definition of the Morrey norm we obtain

$$\begin{aligned}C_1[f,v](x) &= v(x)\sum_{\substack{Q\supsetneq Q_0,\\ Q\in\mathcal{D}(\mathbb{R}^n)}}|Q|^{(\frac{\alpha}{n}-1)}\chi_Q(x)\left|b(x)-m_Q(b)\right|^m\int_{3Q}\left|f(y)\right|dy\\&\le v(x)\sum_{\substack{Q\supsetneq Q_0,\\ Q\in\mathcal{D}(\mathbb{R}^n)}}|Q|^{(\frac{\alpha}{n}-1)}\chi_Q(x)\left|b(x)-m_Q(b)\right|^m|3Q|^{\frac{1}{p_0}}\left(\fint_{3Q}\left|f(y)\right|^p w(y)^p\,dy\right)^{\frac{1}{p}}\\&\quad\times|3Q|^{1-\frac{1}{p_0}}\left(\fint_{3Q}w(y)^{-p'}\,dy\right)^{\frac{1}{p'}}\\&\le C\|fw\|_{\mathcal{M}^{p_0}_p}v(x)\sum_{\substack{Q\supsetneq Q_0,\\ Q\in\mathcal{D}(\mathbb{R}^n)}}|3Q|^{\frac{\alpha}{n}-\frac{1}{p_0}}\left(\fint_{3Q}w(y)^{-p'}\,dy\right)^{\frac{1}{p'}}\left|b(x)-m_Q(b)\right|^m.\end{aligned}$$

Since $\frac{1}{q_0}=\frac{1}{p_0}+\frac{1}{r_0}-\frac{\alpha}{n}$, the integral of $C_1[f,v](x)^q$ on Q_0 is evaluated as follows:

$$\begin{aligned}&|Q_0|^{\frac{1}{q_0}}\left(\fint_{Q_0}C_1[f,v](x)^q\,dx\right)^{\frac{1}{q}}\\&\quad\le C\|fw\|_{\mathcal{M}^{p_0}_p}\sum_{\substack{Q\supsetneq Q_0,\\ Q\in\mathcal{D}(\mathbb{R}^n)}}|Q_0|^{\frac{1}{q_0}}|3Q|^{\frac{1}{r_0}-\frac{1}{q_0}}\left(\fint_{3Q}w(y)^{-p'}\,dy\right)^{\frac{1}{p'}}\end{aligned}$$

$$\times \left(\fint_{Q_0} v(x)^q \left| b(x) - m_Q(b) \right|^{mq} dx \right)^{\frac{1}{q}}$$

$$= C\|fw\|_{\mathcal{M}_p^{p_0}} \sum_{\substack{Q \supsetneqq Q_0, \\ Q \in \mathcal{D}(\mathbb{R}^n)}} \left(\frac{|Q_0|}{|3Q|} \right)^{\frac{1}{q_0}} |3Q|^{\frac{1}{r_0}} \left(\fint_{3Q} w(y)^{-p'} dy \right)^{\frac{1}{p'}}$$

$$\times \left(\fint_{Q_0} v(x)^q \left| b(x) - m_Q(b) \right|^{mq} dx \right)^{\frac{1}{q}}.$$

By Hölder's inequality for $\theta_4 > 1$ as in Lemma 3, we have

$$\begin{aligned} &\left(\fint_{Q_0} v(x)^q \left| b(x) - m_Q(b) \right|^{mq} dx \right)^{\frac{1}{q}} \\ &\quad \le \left(\fint_{Q_0} v(x)^{q\theta_4} dx \right)^{\frac{1}{q\theta_4}} \left(\fint_{Q_0} \left| b(x) - m_Q(b) \right|^{mq\theta_4'} dx \right)^{\frac{1}{q\theta_4'}}. \end{aligned} \tag{6}$$

We evaluate $|b(x) - m_Q(b)|$. If $Q \supsetneqq Q_0$ and $Q \in \mathcal{D}(\mathbb{R}^n)$, then there exists $k = 1, 2, \ldots,$ such that $Q_k := Q$, $Q_j \in \mathcal{D}(\mathbb{R}^n)$, $Q_j \supsetneqq Q_{j-1}$ and $|Q_j| = 2^n |Q_{j-1}|$ $(j = 1, 2, \ldots, k)$. By the triangle inequality, we obtain

$$\begin{aligned} \left| b(x) - m_Q(b) \right| &\le \left| b(x) - m_{Q_0}(b) \right| + \left| m_{Q_0}(b) - m_Q(b) \right| \\ &= \left| b(x) - m_{Q_0}(b) \right| + \left| \sum_{j=1}^k \left(m_{Q_{j-1}}(b) - m_{Q_j}(b) \right) \right| \\ &\le \left| b(x) - m_{Q_0}(b) \right| + \sum_{j=1}^k \left| m_{Q_{j-1}}(b) - m_{Q_j}(b) \right|. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \left| m_{Q_{j-1}}(b) - m_{Q_j}(b) \right| &= \left| \fint_{Q_{j-1}} b(y)\, dy - m_{Q_j}(b) \right| \\ &= \left| \fint_{Q_{j-1}} \left(b(y) - m_{Q_j}(b) \right) dy \right| \\ &\le \fint_{Q_{j-1}} \left| b(y) - m_{Q_j}(b) \right| dy \\ &\le \frac{2^n}{|Q_j|} \int_{Q_j} \left| b(y) - m_{Q_j}(b) \right| dy \\ &\le 2^n \|b\|_{\mathrm{BMO}} \quad (j = 1, 2, \ldots), \end{aligned}$$

where we invoke Definition 2 for the last line. By the inequality $(a + b)^m \le 2^{m-1}(a^m + b^m)$:

$$\begin{aligned} \left| b(x) - m_Q(b) \right|^m &\le \left(\left| b(x) - m_{Q_0}(b) \right| + 2^n k \|b\|_{\mathrm{BMO}} \right)^m \\ &\le C\left(\left| b(x) - m_{Q_0}(b) \right|^m + 2^{mn} k^m \|b\|_{\mathrm{BMO}}^m \right). \end{aligned} \tag{7}$$

By the estimates (6), (7), and Hölder's inequality for $(p/a)' > p'$, we obtain

$$\begin{aligned}
&\sum_{\substack{Q\supsetneq Q_0,\\ Q\in\mathcal{D}(\mathbb{R}^n)}} \left(\frac{|Q_0|}{|3Q|}\right)^{\frac{1}{q_0}} |3Q|^{\frac{1}{r_0}} \left(\fint_{3Q} w(y)^{-p'}\,dy\right)^{\frac{1}{p'}} \left(\fint_{Q_0} v(x)^q \left|b(x)-m_Q(b)\right|^{mq}\,dx\right)^{\frac{1}{q}} \\
&\quad\le C\sum_{k=1}^{\infty} \sum_{\substack{Q_k\in\mathcal{D}(\mathbb{R}^n),\\ Q_k\supset Q_0, |Q_k|=2^{kn}|Q_0|}} \left(\frac{|Q_0|}{|3Q_k|}\right)^{\frac{1}{q_0}} |3Q_k|^{\frac{1}{r_0}} \left(\fint_{3Q_k} w(y)^{-(p/a)'}\,dy\right)^{\frac{1}{(p/a)'}} \\
&\qquad\times \left(\fint_{Q_0} v(x)^{q\theta_4}\,dx\right)^{\frac{1}{q\theta_4}} \left(\fint_{Q_0} \left(\left|b(x)-m_{Q_0}(b)\right|^m + 2^{mn}k^m\|b\|_{\mathrm{BMO}}^m\right)^{q\theta_4'}\,dx\right)^{\frac{1}{q\theta_4'}}.
\end{aligned}$$

By the triangle inequality on $L^{q\theta_4'}(\mathbb{R}^n)$, we obtain

$$\begin{aligned}
&\left(\fint_{Q_0} \left(\left|b(x)-m_{Q_0}(b)\right|^m + 2^{mn}k^m\|b\|_{\mathrm{BMO}}^m\right)^{q\theta_4'}\,dx\right)^{\frac{1}{q\theta_4'}} \\
&\quad\le \left(\fint_{Q_0} \left|b(x)-m_{Q_0}(b)\right|^{mq\theta_4'}\,dx\right)^{\frac{1}{q\theta_4'}} + \left(\fint_{Q_0} \left(2^{mn}k^m\|b\|_{\mathrm{BMO}}^m\right)^{q\theta_4'}\,dx\right)^{\frac{1}{q\theta_4'}}. \qquad (8)
\end{aligned}$$

By the estimate (8), we obtain

$$\begin{aligned}
&\sum_{k=1}^{\infty} \sum_{\substack{Q_k\in\mathcal{D}(\mathbb{R}^n),\\ Q_k\supset Q_0, |Q_k|=2^{kn}|Q_0|}} \left(\frac{|Q_0|}{|3Q_k|}\right)^{\frac{1}{q_0}} |3Q_k|^{\frac{1}{r_0}} \left(\fint_{3Q_k} w(y)^{-(p/a)'}\,dy\right)^{\frac{1}{(p/a)'}} \left(\fint_{Q_0} v(x)^{q\theta_4}\,dx\right)^{\frac{1}{q\theta_4}} \\
&\qquad\times \left(\fint_{Q_0} \left(\left|b(x)-m_{Q_0}(b)\right|^m + 2^{mn}k^m\|b\|_{\mathrm{BMO}}^m\right)^{q\theta_4'}\,dx\right)^{\frac{1}{q\theta_4'}} \\
&\quad\le \sum_{k=1}^{\infty} \sum_{\substack{Q_k\in\mathcal{D}(\mathbb{R}^n),\\ Q_k\supset Q_0, |Q_k|=2^{kn}|Q_0|}} \left(\frac{|Q_0|}{|3Q_k|}\right)^{\frac{1}{q_0}} |3Q_k|^{\frac{1}{r_0}} \\
&\qquad\times \left(\fint_{3Q_k} w(y)^{-(p/a)'}\,dy\right)^{\frac{1}{(p/a)'}} \left(\fint_{Q_0} v(x)^{q\theta_4}\,dx\right)^{\frac{1}{q\theta_4}} \\
&\qquad\times \left\{\left(\fint_{Q_0} \left|b(x)-m_{Q_0}(b)\right|^{mq\theta_4'}\,dx\right)^{\frac{1}{q\theta_4'}} + \left(\fint_{Q_0} \left(2^{mn}k^m\|b\|_{\mathrm{BMO}}^m\right)^{q\theta_4'}\,dx\right)^{\frac{1}{q\theta_4'}}\right\}.
\end{aligned}$$

By Lemma 1, we have

$$\left(\fint_{Q_0} \left|b(x)-m_{Q_0}(b)\right|^{mq\theta_4'}\,dx\right)^{\frac{1}{q\theta_4'}} \le C\|b\|_{\mathrm{BMO}}^m. \qquad (9)$$

The estimate (9) gives us the following:

$$\begin{aligned}
&\left(\fint_{Q_0} \left|b(x)-m_{Q_0}(b)\right|^{mq\theta_4'}\,dx\right)^{\frac{1}{q\theta_4'}} + \left(\fint_{Q_0} \left(2^{mn}k^m\|b\|_{\mathrm{BMO}}^m\right)^{q\theta_4'}\,dx\right)^{\frac{1}{q\theta_4'}} \\
&\quad\le C\|b\|_{\mathrm{BMO}}^m\left(1+2^{mn}k^m\right). \qquad (10)
\end{aligned}$$

As a consequence of (10), we obtain the following inequality:

$$\begin{aligned}
&\sum_{\substack{Q\supsetneq Q_0,\\ Q\in\mathcal{D}(\mathbb{R}^n)}} \left(\frac{|Q_0|}{|3Q|}\right)^{\frac{1}{q_0}} |3Q|^{\frac{1}{r_0}} \left(⨍_{3Q} w(y)^{-p'}\,dy\right)^{\frac{1}{p'}} \left(⨍_{Q_0} v(x)^q \bigl|b(x)-m_Q(b)\bigr|^{mq}\,dx\right)^{\frac{1}{q}} \\
&\quad\le C\|b\|_{\mathrm{BMO}}^m \sum_{k=1}^{\infty} \sum_{\substack{Q_k\in\mathcal{D}(\mathbb{R}^n),\\ Q_k\supset Q_0, |Q_k|=2^{kn}|Q_0|}} \left(\frac{|Q_0|}{|3Q_k|}\right)^{\frac{1}{aq_0}} |3Q_k|^{\frac{1}{r_0}} \left(⨍_{3Q_k} w(y)^{-(p/a)'}\,dy\right)^{\frac{1}{(p/a)'}} \\
&\qquad\times \left(⨍_{Q_0} v(x)^{q\theta_4}\,dx\right)^{\frac{1}{q\theta_4}} \left(1+2^{mn}k^m\right) \left(\frac{|Q_0|}{|3Q_k|}\right)^{\frac{1}{q_0}(1-\frac{1}{a})}.
\end{aligned}$$

By the condition (2), we have

$$\begin{aligned}
&\sum_{\substack{Q\supsetneq Q_0,\\ Q\in\mathcal{D}(\mathbb{R}^n)}} \left(\frac{|Q_0|}{|3Q|}\right)^{\frac{1}{q_0}} |3Q|^{\frac{1}{r_0}} \left(⨍_{3Q} w(y)^{-p'}\,dy\right)^{\frac{1}{p'}} \left(⨍_{Q_0} v(x)^q \bigl|b(x)-m_Q(b)\bigr|^{mq}\,dx\right)^{\frac{1}{q}} \\
&\quad\le C\|b\|_{\mathrm{BMO}}^m [v,w]_{aq_0,r_0,aq,p/a} \sum_{k=1}^{\infty} \sum_{\substack{Q_k\in\mathcal{D}(\mathbb{R}^n),\\ Q_k\supset Q_0, |Q_k|=2^{kn}|Q_0|}} \left(1+2^{mn}k^m\right) 2^{-\frac{kn}{q_0}(1-\frac{1}{a})} \\
&\quad= C\|b\|_{\mathrm{BMO}}^m [v,w]_{aq_0,r_0,aq,p/a} \sum_{k=1}^{\infty} \left(1+2^{mn}k^m\right) 2^{-\frac{kn}{q_0}(1-\frac{1}{a})} \\
&\quad\le C\|b\|_{\mathrm{BMO}}^m [v,w]_{aq_0,r_0,aq,p/a}.
\end{aligned}$$

Therefore we obtain

$$\bigl\|C_1[f,v]\bigr\|_{\mathcal{M}_q^{q_0}} \le C[v,w]_{aq_0,r_0,aq,p/a} \|b\|_{\mathrm{BMO}}^m \|fw\|_{\mathcal{M}_p^{p_0}}. \tag{11}$$

Next, we evaluate $C_2[f,v](x)$. By Hölder's inequality for $\theta_5\in(1,p)$ in Lemma 3, we have

$$\begin{aligned}
C_2[f,v](x) &= v(x) \sum_{\substack{Q\supsetneq Q_0,\\ Q\in\mathcal{D}(\mathbb{R}^n)}} l(Q)^{\alpha-n} \chi_Q(x) \left(\int_{3Q} \bigl|m_Q(b)-b(y)\bigr|^m f(y)\,dy\right) \\
&\le v(x) \sum_{\substack{Q\supsetneq Q_0,\\ Q\in\mathcal{D}(\mathbb{R}^n)}} l(Q)^{\alpha-n} \chi_Q(x) \left(\int_{3Q} \bigl|m_Q(b)-b(y)\bigr|^{m\theta_5'}\,dy\right)^{\frac{1}{\theta_5'}} \\
&\quad\times \left(\int_{3Q} \bigl|f(y)\bigr|^{\theta_5}\,dy\right)^{\frac{1}{\theta_5}}.
\end{aligned}$$

By Hölder's inequality for $\frac{p}{\theta_5}>1$, we obtain

$$\begin{aligned}
C_2[f,v](x) &\le v(x) \sum_{\substack{Q\supsetneq Q_0,\\ Q\in\mathcal{D}(\mathbb{R}^n)}} l(Q)^{\alpha-\frac{n}{p_0}} \chi_Q(x) \left(⨍_{3Q} \bigl|m_Q(b)-b(y)\bigr|^{m\theta_5'}\,dy\right)^{\frac{1}{\theta_5'}} |3Q|^{\frac{1}{p_0}} \\
&\quad\times \left(⨍_{3Q} w(y)^{-\theta_5(\frac{p}{\theta_5})'}\,dy\right)^{\frac{1}{\theta_5(\frac{p}{\theta_5})'}} \left(⨍_{3Q} \bigl|f(y)\bigr|^p w(y)^p\,dy\right)^{\frac{1}{p}}.
\end{aligned}$$

Taking the Morrey norm, we obtain

$$C_2[f,v](x) \le \|fw\|_{\mathcal{M}^{p_0}_p} v(x) \sum_{\substack{Q \supsetneq Q_0,\\ Q \in \mathcal{D}(\mathbb{R}^n)}} l(Q)^{\alpha - \frac{n}{p_0}} \chi_Q(x) \left(\fint_{3Q} \left| m_Q(b) - b(y) \right|^{m\theta_5'} dy \right)^{\frac{1}{\theta_5'}} \times \left(\fint_{3Q} w(y)^{-\theta_5 (\frac{p}{\theta_5})'} dy \right)^{\frac{1}{\theta_5 (\frac{p}{\theta_5})'}}.$$

Using Lemma 1, we have

$$C_2[f,v](x) \le \|b\|^m_{\mathrm{BMO}} \|fw\|_{\mathcal{M}^{p_0}_p} v(x) \sum_{\substack{Q \supsetneq Q_0,\\ Q \in \mathcal{D}(\mathbb{R}^n)}} l(Q)^{\alpha - \frac{n}{p_0}} \chi_Q(x) \left(\fint_{3Q} w(y)^{-\theta_5 (\frac{p}{\theta_5})'} dy \right)^{\frac{1}{\theta_5 (\frac{p}{\theta_5})'}}.$$

Since we have the assumption that $a \geqq \frac{p}{(\theta_5 (\frac{p}{\theta_5})')'} > 1$, using Hölder's inequality, we obtain

$$C_2[f,v](x) \le C\|b\|^m_{\mathrm{BMO}} \|fw\|_{\mathcal{M}^{p_0}_p} v(x) \sum_{\substack{Q \supsetneq Q_0,\\ Q \in \mathcal{D}(\mathbb{R}^n)}} l(Q)^{\alpha - \frac{n}{p_0}} \chi_Q(x) \left(\fint_{3Q} w(y)^{-(p/a)'} dy \right)^{\frac{1}{(p/a)'}}.$$

The integral of $C_2[f,v](x)^q$ on Q_0 is evaluated as follows:

$$\begin{aligned} &|Q_0|^{\frac{1}{q_0}} \left(\fint_{Q_0} C_2[f,v](x)^q \, dx \right)^{\frac{1}{q}} \\ &\quad \le C\|b\|^m_{\mathrm{BMO}} \|fw\|_{\mathcal{M}^{p_0}_p} \sum_{\substack{Q \supsetneq Q_0,\\ Q \in \mathcal{D}(\mathbb{R}^n)}} l(Q)^{\alpha - \frac{n}{p_0}} |Q_0|^{\frac{1}{q_0}} \\ &\qquad \times \left(\fint_{Q_0} v(x)^q \, dx \right)^{\frac{1}{q}} \left(\fint_{3Q} w(y)^{-(p/a)'} dy \right)^{\frac{1}{(p/a)'}} \\ &\quad \le C\|b\|^m_{\mathrm{BMO}} \|fw\|_{\mathcal{M}^{p_0}_p} \sum_{\substack{Q \supsetneq Q_0,\\ Q \in \mathcal{D}(\mathbb{R}^n)}} \left(\frac{|Q_0|}{|3Q|} \right)^{\frac{1}{aq_0}} |3Q|^{\frac{1}{r_0}} \\ &\qquad \times \left(\fint_{Q_0} v(x)^{aq} \, dx \right)^{\frac{1}{aq}} \left(\fint_{3Q} w(y)^{-(p/a)'} dy \right)^{\frac{1}{(p/a)'}} \left(\frac{|Q_0|}{|3Q|} \right)^{\frac{1}{q_0}(1-\frac{1}{a})}. \end{aligned}$$

By the condition (2), we have

$$\begin{aligned} &|Q_0|^{\frac{1}{q_0}} \left(\fint_{Q_0} C_2[f,v](x)^q \, dx \right)^{\frac{1}{q}} \\ &\quad \le C[v,w]_{aq_0,r_0,aq,p/a} \|b\|^m_{\mathrm{BMO}} \|fw\|_{\mathcal{M}^{p_0}_p} \sum_{\substack{Q \supsetneq Q_0,\\ Q \in \mathcal{D}(\mathbb{R}^n)}} \left(\frac{|Q_0|}{|3Q|} \right)^{\frac{1}{q_0}(1-\frac{1}{a})} \\ &\quad \le C[v,w]_{aq_0,r_0,aq,p/a} \|b\|^m_{\mathrm{BMO}} \|fw\|_{\mathcal{M}^{p_0}_p}. \end{aligned}$$

We obtain the desired result. □

Competing interests
The author declares that they have no competing interests.

Acknowledgements
The author wish to thank Professor Y Sawano for a rich lecture of higher order commutators generated by BMO-functions and the fractional integral operator.

Received: 16 September 2015 Accepted: 21 December 2015 Published online: 04 January 2016

References

1. Iida, T, Sato, E, Sawano, Y, Tanaka, H: Weighted norm inequalities for multilinear fractional operators on Morrey spaces. Stud. Math. **205**, 139-170 (2011)
2. Sawano, Y, Sugano, S, Tanaka, H: A bilinear estimate for commutators of fractional integral operators. In: Potential Theory and Its Related Fields. RIMS Kôkyûroku Bessatsu, vol. B43, pp. 155-170. Res. Inst. Math. Sci., Kyoto (2013)
3. Grafakos, L: Modern Fourier Analysis, 2nd edn. Graduate Texts in Math., vol. 250. Springer, New York (2008)
4. Lu, S, Ding, Y, Yan, D: Singular Integrals and Related Topics. World Scientific, Singapore (2007)
5. Adams, D: A note on Riesz potentials. Duke Math. J. **42**, 765-778 (1975)
6. Di-Fazio, G, Ragusa, MA: Commutators and Morrey spaces. Boll. UMI **7**, 323-332 (1991)
7. Komori, Y, Mizuhara, T: Notes on commutators and Morrey spaces. Hokkaido Math. J. **32**, 345-353 (2003)
8. Iida, T, Sawano, Y, Tanaka, H: Atomic decomposition for Morrey spaces. Z. Anal. Anwend. **33**, 149-170 (2014)
9. Sawano, Y, Sugano, S, Tanaka, H: Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces. Trans. Am. Math. Soc. **363**, 6481-6503 (2011)
10. Ding, Y, Lu, S: Weighted norm inequalities for fractional integral operators with rough kernel. Can. J. Math. **50**, 29-39 (1998)
11. Ding, Y, Lu, S: Higher order commutators for a class of rough operators. Ark. Mat. **37**, 33-44 (1999)
12. Iida, T: Weighted inequalities on Morrey spaces for linear and multilinear fractional integrals with homogeneous kernels. Taiwan. J. Math. **18**, 147-185 (2014)
13. Duoandikoetxea, J: Fourier Analysis. Grad. Studies in Math., vol. 29. Am. Math. Soc., Providence (2001)
14. Pérez, C: Two weighted inequalities for potential and fractional type maximal operators. Indiana Univ. Math. J. **43**, 663-683 (1994)
15. Pérez, C: On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between L^p spaces with different weights. Proc. Lond. Math. Soc. (3) **71**, 135-157 (1995)
16. Pérez, C: Sharp L^p-weighted Sobolev inequalities. Ann. Inst. Fourier (Grenoble) **45**, 809-824 (1995)