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A note on fractional integral operators on Herz spaces with variable exponent

Meng Qu* and Jie Wang

*Correspondence:
qumeng@mail.ahnu.edu.cn
School of Mathematical and Computer Sciences, Anhui Normal University, Wuhu, 241003, China

Abstract

In this note, we prove that the fractional integral operators from Herz spaces with variable exponent $\dot{K}^{\alpha}_{p(\cdot),q}$ to Lipschitz-type spaces are bounded provided $p(\cdot)$ is locally log-Hölder continuous and log-Hölder continuous at infinity.

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1 Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclid space. For $0 < \beta < n$, the fractional integral operator I_β is defined by

$$I_\beta f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}}\,dy.$$

The famous Hardy-Littlewood-Sobolev theorem tells us that I_β is a bounded operator from the usual Lebesgue spaces L^p to L^q with $1/q = 1/p - \beta/n$, where $1 < p < n/\beta$. Also I_β is bounded from $L^{\frac{n}{\beta}}$ into BMO. As for $p > n/\beta$, Gatto and Vagi [1] proved that $\tilde{I}_\beta$ is bounded from L^p into Lipschitz spaces whose smoothness is controlled by p and α, where $\tilde{I}_\beta$ is defined as

$$\tilde{I}_\beta f(x) = \int_{\mathbb{R}^n} \left(\frac{1}{|x-y|^{n-\beta}} - \frac{\chi_{\{|\cdot|\geq 1\}}(y)}{|y|^{n-\beta}} \right) f(y)\,dy.$$

Indeed Gatto and Vagi's result was proved in the setting of the spaces of homogeneous type. Also there are extensions like weighted function spaces theory, see [2]. Recently, Ramseyer *et al.* [3] extended Gatto and Vagi's result in the variable exponent function spaces case.

For the sake of convenience, we briefly recall some basic elements of the Lebesgue spaces with variable exponent, while more results can be found in [4, 5] and the references therein. Let Ω be a non-empty open set in $\mathbb{R}^n$ and $p(\cdot) : \Omega \to [1,\infty)$ be a measurable function. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) := \left\{ f \text{ is measurable} : \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx < \infty \text{ for some constant } \lambda > 0 \right\}.$$

It is easy to check that $L^{p(\cdot)}(\Omega)$ is a Banach space with the norm defined by

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

We let

$$p^-(\Omega) := \operatorname*{ess\,inf}_{x\in\Omega} p(x), \qquad p^+(\Omega) := \operatorname*{ess\,sup}_{x\in\Omega} p(x),$$

and we denote by $\mathcal{P}(\Omega)$ the set of measurable function $p(\cdot)$ on Ω with value in $[1,\infty)$ such that $1 < p^-(\Omega) \le p(\cdot) \le p^+(\Omega) < \infty$. For the sake of simplicity, we write $L^{p(\cdot)}(\mathbb{R}^n)$ as $L^{p(\cdot)}$ and $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ as $\|f\|_{p(\cdot)}$, respectively.

We say a function $p(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}$ is *locally log-Hölder continuous*, if there exists a constant C such that

$$\left|p(x) - p(y)\right| \le \frac{C}{\log(e + 1/|x-y|)}$$

for all $x, y \in \mathbb{R}^n$. If, for some $p_\infty \in \mathbb{R}$ and $C > 0$, we have

$$\left|p(x) - p_\infty\right| \le \frac{C}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$, then we say $p(\cdot)$ is *log-Hölder continuous at infinity*.

The notation $\mathcal{P}^{\log}(\mathbb{R}^n)$ is used for all those exponents $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ which are *locally log-Hölder continuous* and *log-Hölder continuous at infinity* with $p_\infty := \lim_{|x|\to\infty} p(x)$. Moreover, we can easily show that $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ implies $p'(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$.

Ramseyer, Salinas and Viviani introduced the following function space, which can be viewed as the variable exponent counterpart of Lipschitz space defined by Peetre in [6].

Definition 1 ([3]) Let $0 < \beta < n$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and denote the Lebesgue measure of B by $|B|$. We say that a locally integrable function f belongs to $\mathcal{L}_{\beta,p(\cdot)}(\mathbb{R}^n)$ if there exists a constant C such that

$$\frac{1}{|B|^{\frac{\beta}{n}} \|\chi_Q\|_{p'(\cdot)}} \int_B |f - m_B f|\, dx \le C, \tag{1.1}$$

for every ball $B = B(x,R) \subset \mathbb{R}^n$, with $m_B f = \frac{1}{|B|}\int_B f$. The least constant C in (1.1) will be denoted by $\|f\|_{Lip_{\beta,p(\cdot)}}$.

Ramseyer, Salinas and Viviani proved the following theorem.

Theorem 1.1 ([3]) *Given $0 < \beta < n$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then the following two statements are equivalent.*

(1) *$\tilde{I}_\beta$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ into $\mathcal{L}_{\beta,p(\cdot)}(\mathbb{R}^n)$.*

(2) *$p(\cdot) \in P_\beta$, i.e., there exists a positive constant C such that for any ball B,*

$$\left\|\frac{\chi_{\mathbb{R}^n\setminus B}}{|x_B - \cdot|^{n-\beta+1}}\right\|_{p'(\cdot)} \le C|B|^{\frac{\beta}{n}-\frac{1}{n}-1}\|\chi_B\|_{p'(\cdot)} \tag{1.2}$$

hold for every ball B, where x_B denotes its center.

Corollary 2.16 in [3] says that if $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $0 < \beta - 1 < \frac{n}{p^+}$, then $p(\cdot)$ satisfies (1.2). With the help of Theorem 1.1, $\tilde{I}_\beta$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $\mathcal{L}_{\beta,p(\cdot)}(\mathbb{R}^n)$. It is natural to ask what the target space is when $L^{p(\cdot)}(\mathbb{R}^n)$ is replaced by other more general spaces. The main result of this note is that the target space of mapping $\tilde{I}_\beta$ is just the variant Lipschitz space when $L^{p(\cdot)}(\mathbb{R}^n)$ is replaced by the so-called variable exponent Herz space.

2 Herz spaces and main results

Variable exponent Herz spaces were considered by many authors in recent years. Especially Herz spaces with two variable exponents and even with three variable exponents were produced by Almeida and Drihem [7] and Samko [8], respectively. For brevity, we only consider the Herz space with one variable exponent case, which was introduced by Izuki in [9]. Let $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$, $B_k = \{x \in \mathbb{R}^n : |x| < 2^k\}$, $A_k = B_k \setminus B_{k-1}$, and $\chi_{A_k} = \chi_k$ be the characteristic function of the set A_k for $k \in \mathbb{Z}$.

Definition 2 ([9]) Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space $\dot{K}^{\alpha}_{p(\cdot),q}(\mathbb{R}^n)$ is defined as the set of all $f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}^{\alpha}_{p(\cdot),q}(\mathbb{R}^n)} := \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|f\chi_k\|^q_{p(\cdot)}\right)^{1/q} < \infty.$$

It is obvious that if $p(\cdot)$ is a constant, then $\dot{K}^{\alpha}_{p(\cdot),q}(\mathbb{R}^n) = \dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ are classical Herz spaces. We can refer to [10] for more properties of the classical one.

Our main result is to establish a result of mapping property of $\tilde{I}_\beta$ on $\dot{K}^{\alpha}_{p(\cdot),q}(\mathbb{R}^n)$. For this purpose we need to define a variant of the Lipschitz space.

Definition 3 Given $-\infty < \lambda < +\infty$, $0 < \beta < n$, and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. We say that a locally integrable function f belongs to $\mathcal{L}^{\lambda}_{\beta,p(\cdot)}$ if there exists a constant C such that

$$\frac{1}{(|x|+R)^\lambda}\frac{1}{|B|^{\frac{\beta}{n}}\|\chi_B\|_{p'(\cdot)}}\int_B |f - m_B f|\,dx \leq C, \tag{2.1}$$

for every ball $B = B(x,R) \subset \mathbb{R}^n$, with $m_B f = \frac{1}{|B|}\int_B f$. The least constant C in (2.1) will be denoted by $\|f\|_{\mathcal{L}^{\lambda}_{\beta,p(\cdot)}}$.

Remark 2.1 It is easy to see that in Definition 3 the average $m_B f$ can be replaced by a constant in the following sense:

$$\frac{1}{2}\|f\|_{Lip^{\lambda}_{\beta,p(\cdot)}} \leq \sup_{B\in\mathbb{R}^n, R>0}\inf_{c\in\mathbb{R}}\frac{1}{(|x|+R)^\lambda}\frac{1}{|B|^{\frac{\beta}{n}}\|\chi_B\|_{p'(\cdot)}}\int_B |f-c|\,dx \leq \|f\|_{Lip^{\lambda}_{\beta,p(\cdot)}}.$$

Also by Definition 3, we obtain $\mathcal{L}^{\lambda_1}_{\beta_1,p(\cdot)} \subset \mathcal{L}^{\lambda}_{\beta,p(\cdot)}$, where $\lambda - \lambda_1 = \beta_1 - \beta \geq 0$. Especially, $\mathcal{L}^{\lambda}_{\beta,p(\cdot)} \subset \mathcal{L}_{\beta+\lambda,p(\cdot)}$ for $\lambda < 0$ and $\mathcal{L}_{\beta+\lambda,p(\cdot)} \subset \mathcal{L}^{\lambda}_{\beta,p(\cdot)}$ for $\lambda > 0$.

Now we are in a position to state our results.

Theorem 2.1 *Suppose that* $0 < q < \infty$ *and* $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$. *If* $\beta - \frac{n}{p^+} - 1 < \alpha < n - \frac{n}{p^-}$, $1 < \beta < \frac{n}{p^+} + 1$, *then the operator* $\tilde{I}_\beta$ *is bounded from* $\dot{K}^{\alpha}_{p(\cdot),q}(\mathbb{R}^n)$ *to* $\mathcal{L}^{-\alpha}_{\beta,p(\cdot)}(\mathbb{R}^n)$.

Theorem 2.2 *Suppose that* $0 < q < \infty$ *and* $p(x) \in \mathcal{P}^{\log}(\mathbb{R}^n)$. *If* $\varepsilon > 0$, *then* $\tilde{I}_\beta$ *is not bounded from* $\dot{K}^{\alpha}_{p(\cdot),q}(\mathbb{R}^1)$ *to* $\mathcal{L}^{-\alpha-\varepsilon}_{\beta+\varepsilon,p(\cdot)}(\mathbb{R}^1)$.

Remark 2.2 According to Remark 2.1, $\mathcal{L}^{-\alpha-\varepsilon}_{\beta+\varepsilon,p(\cdot)}(\mathbb{R}^1) \subset \mathcal{L}^{-\alpha}_{\beta,p(\cdot)}(\mathbb{R}^1)$ when $\varepsilon > 0$. This shows that Theorem 2.1 is optimal.

We give some lemmas in Section 3 and then prove the above theorems in Section 4. C always means a positive constant independent of the main parameters and it may change from one occurrence to another. $f \sim g$ means $C^{-1}g \leq f \leq Cg$.

3 Technique lemmas

Lemma 3.1 ([11]) *Let* $\Omega \subset \mathbb{R}^n$. *If* $p(\cdot) \in \mathcal{P}(\Omega)$, *then for all* $f \in L^{p(\cdot)}(\Omega)$ *and all* $g \in L^{p'(\cdot)}(\Omega)$ *we have*

$$\int_\Omega \left|f(x)g(x)\right| dx \leq r_p \|f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{p'(\cdot)}(\Omega)},$$

where $r_p := 1 + \frac{1}{p^-(\Omega)} - \frac{1}{p^+(\Omega)}$.

Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) := \sup_{r>0} r^{-n} \int_{B(x,r)} \left|f(y)\right| dy, \quad x \in \mathbb{R}^n,$$

and we say $\mathcal{B}(\mathbb{R}^n)$ is the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the condition that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 3.2 ([12]) $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ *implies* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.

Lemma 3.3 ([13]) *Let* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, *then there exists a positive constant C such that*

$$C^{-1}|B| \leq \|\chi_B\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)} \leq C|B|$$

hold for every ball B.

Remark 3.1 According to Lemma 3.2, the conclusion of Lemma 3.3 is correct when the condition $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ is replaced by $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$.

Lemma 3.4 (Corollary 4.5.9 in [5]) *Let* $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, *then for every ball* $B \subset \mathbb{R}^n$,

$$\|\chi_B\|_{p(\cdot)} \sim |B|^{\frac{1}{p(x)}}, \quad \text{if } |B| \leq 2^n, x \in B,$$

and

$$\|\chi_B\|_{p(\cdot)} \sim |B|^{\frac{1}{p_\infty}}, \quad \text{if } |B| \geq 1.$$

Lemma 3.5 ([3]) *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *and* $0 < \beta - 1 < \frac{n}{p^+}$.

(1) *If* $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ *and* $0 < \beta - 1 < \frac{n}{p^+}$, *then* $p(\cdot) \in P_\beta$.

(2) *If* $p(\cdot) \in P_\beta$, *then there exists a positive constant* C *such that*

$$\|\chi_{2B}\|_{p(\cdot)} \le C\|\chi_B\|_{p(\cdot)}, \tag{3.1}$$

for every ball B, *where* $2B$ *is the ball having the same center as* B *but whose diameter is two times as large.*

We point out that the two results collected in Lemma 3.5 are from [3]. The result (1) is Corollary 2.16 and (2) is Lemma 2.9 therein, respectively.

Lemma 3.6 *Let* $p(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$, *then there exists a constant* $C > 0$ *such that for all balls* B *and all measurable subsets* $S = B(x_0, r_0) \subset B = B(x_1, r_1)$,

$$\frac{\|\chi_S\|_{p(\cdot)}}{\|\chi_B\|_{p(\cdot)}} \le C\left(\frac{|S|}{|B|}\right)^{\frac{1}{p^+}}.$$

Proof We proved the lemma in the following three cases: (1) $|S| < |B| < 1$; (2) $|S| < 1 < |B|$; (3) $1 \le |S| < |B|$. Cases (2) and (3) are easy, we omit the details. Now for case (1). By Lemma 3.4,

$$\frac{\|\chi_S\|_{p(\cdot)}}{\|\chi_B\|_{p(\cdot)}} \sim \frac{|S|^{\frac{1}{p(x_S)}}}{|B|^{\frac{1}{p(x_S)}}}|B|^{\frac{1}{p(x_S)}-\frac{1}{p(x_B)}} \le C\left(\frac{|S|}{|B|}\right)^{\frac{1}{p^+}}.$$

Indeed in the last inequality in the above equation, since $|x_B - x_S| \le 2r_1$, we make use of the local-Hölder continuity of $p(x)$, so

$$\begin{aligned}\left|\frac{1}{p(x_S)} - \frac{1}{p(x_B)}\right| \log\frac{1}{r_1} &\le \frac{\log\frac{1}{r_1}}{\log(e + \frac{1}{|x_S - x_B|})} \\ &\le \frac{\log\frac{1}{r_1}}{\log(e + \frac{1}{2r_1})} \le C.\end{aligned}$$

The lemma is proved. □

4 Proofs of theorems

Proof of Theorem 2.1 Fix a ball $Q = B(x_0, R)$. To prove Theorem 2.1, we need to estimate

$$\frac{(|x_0| + R)^\alpha}{|Q|^{\frac{\beta}{n}}\|\chi_Q\|_{p'(\cdot)}}\int_Q \left|\tilde{I}_\beta f(x) - c\right| dx.$$

Let k be the least integer such that $Q \subset B(0, 2^k)$, hence $|x_0| + R \sim 2^k$. We consider three cases:

(1) $Q \cap B(0, 2^{k-2}) \ne \emptyset$,

(2) $Q \cap B(0, 2^{k-2}) = \emptyset$ and $R \ge 2^{k-4}$,

(3) $Q \cap B(0, 2^{k-2}) = \emptyset$ and $R < 2^{k-4}$.

Case (1) or (2). Note that $|Q| \geq C2^{kn}$ in both cases. We write

$$f(x) = f\chi_{B(0,2^{k+1})}(x) + f\chi_{\mathbb{R}^n\setminus B(0,2^{k+1})}(x) =: f_1(x) + f_2(x). \tag{4.1}$$

First we estimate $\tilde{I}_\beta f_1$.

Let $c_1 = -\int_{|y|\geq 1} \frac{f_1(y)}{|y|^{n-\beta}}\,dy$, then $\tilde{I}_\beta f_1 - c_1 = I_\beta f_1$. For any $x \in Q$, $|I_\beta f_1(x)| \leq \int_{B(0,2^{k+1})} \frac{|f(y)|}{|x-y|^{n-\beta}}\,dy$. Then by Fubini's theorem, we have

$$\begin{aligned}\int_Q \left|I_\beta f_1(x)\right| dx &\leq \int_{B(0,2^{k+1})} \left|f(y)\right| \int_Q \frac{1}{|x-y|^{n-\beta}}\,dx\,dy \\ &\leq C|Q|^{\frac{\beta}{n}} \int_{B(0,2^{k+1})} \left|f(y)\right| dy \\ &= C|Q|^{\frac{\beta}{n}} \sum_{j=-\infty}^{k+1} \int_{A_j} \left|f(y)\right| dy.\end{aligned}$$

Using Lemma 3.1 and Lemma 3.3, we derive the estimate

$$\begin{aligned}\frac{(|x_0|+R)^\alpha}{|Q|^{\frac{\beta}{n}}\|\chi_Q\|_{p'(\cdot)}} \int_Q \left|I_\beta f_1(x)\right| dx &\leq C \sum_{j=-\infty}^{k+1} \frac{(|x_0|+R)^\alpha}{\|\chi_Q\|_{p'(\cdot)}} \int_{A_j} \left|f(y)\right| dy \\ &\leq C \sum_{j=-\infty}^{k+1} 2^{k\alpha} \frac{\|\chi_Q\|_{p(\cdot)}}{|Q|} \|f\chi_j\|_{p(\cdot)} \|\chi_j\|_{p'(\cdot)} \\ &\leq C \sum_{j=-\infty}^{k+1} 2^{k\alpha} 2^{-kn} \|f\chi_j\|_{p(\cdot)} \|\chi_{B_j}\|_{p'(\cdot)} \|\chi_{B_k}\|_{p(\cdot)}.\end{aligned} \tag{4.2}$$

Now we can distinguish three cases as follows, by Lemma 3.4:

(1) If $0 \leq j-1 \leq k$, we have

$$\begin{aligned}\|\chi_{B_j}\|_{p'(\cdot)}\|\chi_{B_k}\|_{p(\cdot)} &\sim |B_j|^{\frac{1}{p'_\infty}} |B_k|^{\frac{1}{p_\infty}} \sim \left(2^{jn}\right)^{\frac{1}{p'_\infty}} \left(2^{kn}\right)^{\frac{1}{p_\infty}} \\ &\sim 2^{jn} 2^{(k-j)\frac{n}{p_\infty}} \leq C 2^{jn} 2^{(k-j)\frac{n}{p^-}}.\end{aligned}$$

(2) If $j-1 < 0 \leq k$, we obtain

$$\begin{aligned}\|\chi_{B_j}\|_{p'(\cdot)}\|\chi_{B_k}\|_{p(\cdot)} &\sim |B_j|^{\frac{1}{p'(x_j)}} |B_k|^{\frac{1}{p_\infty}} \sim \left(2^{jn}\right)^{\frac{1}{p'(x_j)}} \left(2^{kn}\right)^{\frac{1}{p_\infty}} \\ &\sim 2^{jn} \left(2^{-jn}\right)^{\frac{1}{p(x_j)}} \left(2^{kn}\right)^{\frac{1}{p_\infty}} \leq C 2^{jn} 2^{(k-j)\frac{n}{p^-}}.\end{aligned}$$

(3) If $j-1 \leq k < 0$, we get

$$\begin{aligned}\|\chi_{B_j}\|_{p'(\cdot)}\|\chi_{B_k}\|_{p(\cdot)} &\sim |B_j|^{\frac{1}{p'(x_j)}} |B_k|^{\frac{1}{p(x_k)}} \sim \left(2^{jn}\right)^{\frac{1}{p'(x_j)}} \left(2^{kn}\right)^{\frac{1}{p(x_k)}} \\ &\sim 2^{jn} 2^{(k-j)\frac{n}{p(x_j)}} \left(2^{kn}\right)^{\frac{1}{p(x_k)}-\frac{1}{p(x_j)}} \leq C 2^{jn} 2^{(k-j)\frac{n}{p^-}}.\end{aligned}$$

Here in the last inequality we using the following facts: If $k \geq 0$, $|x_k| < 2^k$, and $|x_j| < 2^j \leq 2^k$, then the local-Hölder continuity of $p(x)$ at the origin yields

$$\begin{aligned}\left|\frac{1}{p(x_k)} - \frac{1}{p(x_j)}\right| \log\frac{1}{2^k} &\leq \left|\frac{1}{p(x_k)} - \frac{1}{p(0)}\right| \log\frac{1}{2^k} + \left|\frac{1}{p(x_j)} - \frac{1}{p(0)}\right| \log\frac{1}{2^k} \\ &\leq C\frac{\log\frac{1}{2^k}}{\log(e+\frac{1}{2^k})} \leq C\end{aligned}$$

with $C > 0$ independent of k, j, x_k, x_j.

Therefore,

$$\frac{(|x_0|+R)^\alpha}{|Q|^{\frac{\beta}{n}}\|\chi_Q\|_{p'(\cdot)}} \int_Q \left|I_\beta f_1(x)\right| dx \leq C \sum_{j=-\infty}^{k+1} 2^{(k-j)(\alpha-n+\frac{n}{p^-})} 2^{j\alpha} \|f\chi_j\|_{p(\cdot)}.$$

Since

$$2^{j\alpha}\|f\chi_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \left(2^{jq\alpha}\|f\chi_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q} \leq \left(\sum_{i=-\infty}^{\infty} 2^{iq\alpha}\|f\chi_i\|^q_{L^{p(\cdot)}}(\mathbb{R}^n)\right)^{1/q} \leq \|f\|_{\dot{K}^{\alpha}_{p(\cdot),q}(\mathbb{R}^n)}.$$

Thus by the condition $\alpha - n + \frac{n}{p^-} < 0$, it follows that

$$\frac{(|x_0|+R)^\alpha}{|Q|^{\frac{\beta}{n}}\|\chi_Q\|_{p'(\cdot)}} \int_Q \left|I_\beta f_1(x)\right| dx \leq C\|f\|_{\dot{K}^{\alpha}_{p(\cdot),q}(\mathbb{R}^n)}. \tag{4.3}$$

Next we estimate $\tilde{I}_\beta f_2$. Let $c_2 = \tilde{I}_\beta f_2(x_0)$. For any $x \in Q$, $y \in A_j$ and $j \geq k+2$, we have $|x_0 - y| \geq |y| - |x_0| > 2^{j-1} - 2^k \geq 2^{j-2}$. Then

$$\begin{aligned}\left|\tilde{I}_\beta f_2(x) - c_2\right| &\leq \int_{\mathbb{R}^n} \left|\frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x_0-y|^{n-\beta}}\right| \left|f_2(y)\right| dy \\ &\leq CR\int_{\mathbb{R}^n\setminus B(0,2^{k+1})} \frac{|f(y)|}{|x_0-y|^{n-\beta+1}}\, dy \\ &\leq CR\sum_{j=k+2}^{\infty}\int_{A_j} \frac{|f(y)|}{|x_0-y|^{n-\beta+1}}\, dy \\ &\leq CR\sum_{j=k+2}^{\infty} 2^{-j(n-\beta+1)}\int_{A_j} \left|f(y)\right| dy.\end{aligned}$$

By Lemma 3.1 and Lemma 3.3, we obtain

$$\begin{aligned}\frac{(|x_0|+R)^\alpha}{|Q|^{\frac{\beta}{n}}\|\chi_Q\|_{p'(\cdot)}} \int_Q \left|\tilde{I}_\beta f_2(x) - c_2\right| dx &\leq CR\sum_{j=k+2}^{\infty} \frac{2^{k\alpha}2^{-j(n-\beta+1)}}{|Q|^{\frac{\beta}{n}}\|\chi_Q\|_{p'(\cdot)}} |Q| \int_{A_j}\left|f(y)\right| dy \\ &\leq CR\sum_{j=k+2}^{\infty} \frac{2^{k\alpha}2^{-j(n-\beta+1)}\|\chi_Q\|_{p(\cdot)}}{|Q|^{\frac{\beta}{n}}} \|f\chi_j\|_{p(\cdot)}\|\chi_j\|_{p'(\cdot)} \\ &\leq CR\sum_{j=k+2}^{\infty} \frac{2^{k\alpha}2^{-j(n-\beta+1)}}{2^{k\beta}} \|f\chi_j\|_{p(\cdot)}\|\chi_{B_j}\|_{p'(\cdot)}\|\chi_{B_k}\|_{p(\cdot)}.\end{aligned}$$

Applying the arguments used in the corresponding step of the estimate of $\tilde{I}_\beta f_1$, we arrive at the inequality

$$\|\chi_{B_j}\|_{p'(\cdot)}\|\chi_{B_k}\|_{p(\cdot)} \leq C2^{jn}2^{(k-j)\frac{n}{p^+}}. \tag{4.4}$$

Since $\alpha - \beta + \frac{n}{p^+} + 1 > 0$,

$$\begin{aligned}\frac{(|x_0| + R)^\alpha}{|Q|^{\frac{\beta}{n}}\|\chi_Q\|_{p'(\cdot)}}\int_Q \big|\tilde{I}_\beta f_2(x) - c_2\big|\,dx &\leq C\sum_{j=k+2}^{\infty} 2^{(k-j)(\alpha-\beta+\frac{n}{p^+}+1)}2^{j\alpha}\|f\chi_j\|_{p(\cdot)} \\ &\leq C\|f\|_{\dot{K}^\alpha_{p(\cdot),q}(\mathbb{R}^n)}.\end{aligned} \tag{4.5}$$

Combining (4.1)-(4.5), cases (1) and (2) are proved.

Case (3). We write

$$\begin{aligned}f(x) &= f\chi_{B(x_0,2R)}(x) + f\chi_{B_{k+1}\setminus(B_{k-3}\cup B(x_0,2R))}(x) + f\chi_{B_{k-3}}(x) + f\chi_{\mathbb{R}^n\setminus B_{k+1}}(x) \\ &=: f_1(x) + f_2(x) + f_3(x) + f_4(x).\end{aligned} \tag{4.6}$$

First we estimate $\tilde{I}_\beta f_1$. Let $c_1 = -\int_{|y|\geq 1}\frac{f_1(y)}{|y|^{n-\beta}}\,dy$, then $\tilde{I}_\beta f_1 - c_1 = I_\beta f_1$. For any $x \in Q$, $|I_\beta f_1(x)| \leq \int_{B(x_0,2R)}\frac{|f(y)|}{|x-y|^{n-\beta}}\,dy$. Then by Fubini's theorem and Lemma 3.1, we obtain

$$\begin{aligned}\int_Q \big|I_\beta f_1(x)\big|\,dx &\leq \int_{B(x_0,2R)}\big|f(y)\big|\int_Q \frac{1}{|x-y|^{n-\beta}}\,dx\,dy \\ &\leq C|Q|^{\frac{\beta}{n}}\int_{B(x_0,2R)}\big|f(y)\big|\,dy \\ &\leq C|Q|^{\frac{\beta}{n}}\|f\chi_{B(x_0,2R)}\|_{p(\cdot)}\|\chi_{B(x_0,2R)}\|_{p'(\cdot)}.\end{aligned}$$

Note that $B(x_0, 2R) \subset \bigcup_{j=k-2}^{k+1} A_j$, so

$$\begin{aligned}\frac{(|x_0| + R)^\alpha}{|Q|^{\frac{\beta}{n}}\|\chi_Q\|_{p'(\cdot)}}\int_Q \big|\tilde{I}_\beta f_1(x) - c_1\big|\,dx &\leq C2^{k\alpha}\|f\chi_{B(x_0,2R)}\|_{p(\cdot)} \\ &\leq C\sum_{j=k-2}^{k+1} 2^{j\alpha}\|f\chi_j\|_{p(\cdot)} \\ &\leq C\|f\|_{\dot{K}^\alpha_{p(\cdot),q}(\mathbb{R}^n)}.\end{aligned} \tag{4.7}$$

Next we estimate $\tilde{I}_\beta f_2$. Let $c_2 = \tilde{I}_\beta f_2(x_0)$. By Lemma 3.1, and then by the condition $1 < \beta < \frac{n}{p^+} + 1$ with Lemma 3.5,

$$\begin{aligned}\big|\tilde{I}_\beta f_2(x) - c_2\big| &\leq CR\int_{|x_0-y|>2R}\frac{|f_2(y)|}{|x_0-y|^{n-\beta+1}}\,dy \\ &\leq CR\|f\chi_{B_{k+1}\setminus B_{k-3}}\|_{p(\cdot)}\left\|\frac{\chi_{\mathbb{R}^n\setminus B(x_0,2R)}}{|x_0-y|^{n-\beta+1}}\right\|_{p'(\cdot)} \\ &\leq C\big|B(x_0,2R)\big|^{\frac{\beta}{n}-1}\|f\chi_{B_{k+1}\setminus B_{k-3}}\|_{p(\cdot)}\|\chi_{B(x_0,2R)}\|_{p'(\cdot)} \\ &\leq C|Q|^{\frac{\beta}{n}-1}\|f\chi_{B_{k+1}\setminus B_{k-3}}\|_{p(\cdot)}\|\chi_Q\|_{p'(\cdot)}.\end{aligned}$$

Hence

$$\frac{(|x_0|+R)^\alpha}{|Q|^{\frac{\beta}{n}}\|\chi_Q\|_{p'(\cdot)}}\int_Q\left|\tilde{I}_\beta f_2(x)-c_2\right|dx \le C2^{k\alpha}\|f\chi_{B_{k+1}\setminus B_{k-3}}\|_{p(\cdot)}$$
$$\le C\|f\|_{\dot{K}^\alpha_{p(\cdot),q}(\mathbb{R}^n)}. \tag{4.8}$$

Now we estimate $\tilde{I}_\beta f_4$. Let $c_4=\tilde{I}_\beta f_4(x_0)$. For any $x\in Q$, $y\in B(0,2^{k-3})$, we have $|x_0-y|\ge |x_0|-|y|>2^{k-2}-2^{k-3}=2^{k-3}$. Then

$$\begin{aligned}\left|\tilde{I}_\beta f_4(x)-c_4\right| &\le \int_{\mathbb{R}^n}\left|\frac{1}{|x-y|^{n-\beta}}-\frac{1}{|x_0-y|^{n-\beta}}\right|\left|f_4(y)\right|dy\\ &\le CR\int_{B(0,2^{k-3})}\frac{|f(y)|}{|x_0-y|^{n-\beta+1}}\,dy\\ &\le CR2^{-k(n-\beta+1)}\int_{B(0,2^{k-3})}\left|f(y)\right|dy\\ &= CR2^{-k(n-\beta+1)}\sum_{j=-\infty}^{k-3}\int_{A_j}\left|f(y)\right|dy.\end{aligned}$$

Now Lemma 3.1 yields

$$\left|\tilde{I}_\beta f_4(x)-c_4\right|\le CR2^{-k(n-\beta+1)}\sum_{j=-\infty}^{k-3}\|f\chi_j\|_{p(\cdot)}\|\chi_j\|_{p'(\cdot)}.$$

Lemma 3.6 gives

$$\frac{\|\chi_Q\|_{p(\cdot)}}{\|\chi_{B_k}\|_{p(\cdot)}}\le C\left(\frac{|Q|}{|B_k|}\right)^{\frac{1}{p^+}}. \tag{4.9}$$

Since $\alpha-n+\frac{n}{p^-}<0$, we have

$$\begin{aligned}&\frac{(|x_0|+R)^\alpha}{|Q|^{\frac{\beta}{n}}\|\chi_Q\|_{p'(\cdot)}}\int_Q\left|\tilde{I}_\beta f_4(x)-c_4\right|dx\\ &\quad\le C\frac{2^{k\alpha}}{|Q|^{\frac{\beta-1}{n}}}2^{-k(n-\beta+1)}\sum_{j=-\infty}^{k-3}\|f\chi_j\|_{p(\cdot)}\|\chi_j\|_{p'(\cdot)}\|\chi_Q\|_{p(\cdot)}\\ &\quad\le C\frac{2^{k\alpha}}{|B_k|^{\frac{\beta-1}{n}}}2^{-k(n-\beta+1)}\left(\frac{|Q|}{|B_k|}\right)^{\frac{1}{p^+}-\frac{\beta-1}{n}}\sum_{j=-\infty}^{k-3}\|f\chi_j\|_{p(\cdot)}\|\chi_{B_j}\|_{p'(\cdot)}\|\chi_{B_k}\|_{p(\cdot)}\\ &\quad\le C\sum_{j=-\infty}^{k-3}2^{(k-j)(\alpha-n+\frac{n}{p^-})}2^{j\alpha}\|f\chi_j\|_{p(\cdot)}\\ &\quad\le C\|f\|_{\dot{K}^\alpha_{p(\cdot),q}(\mathbb{R}^n)}.\end{aligned} \tag{4.10}$$

Finally we estimate $\tilde{I}_\beta f_3$. Let $c_3 = \tilde{I}_\beta f_3(x_0)$. For any $x \in Q$, $y \in \mathbb{R}^n \setminus B(0, 2^{k+1})$, and $j \geq k+2$, we have $|x_0 - y| \geq |y| - |x_0| > 2^{k+1} - 2^k = 2^k$. Then we write

$$\begin{aligned} \left|\tilde{I}_\beta f_3(x) - c_3\right| &\leq CR \int_{\mathbb{R}^n \setminus B(0,2^{k+1})} \frac{|f(y)|}{|x_0 - y|^{n-\beta+1}}\, dy \\ &\leq CR \sum_{j=k+2}^{\infty} \int_{A_j} \frac{|f(y)|}{|x_0 - y|^{n-\beta+1}}\, dy \\ &\leq CR \sum_{j=k+2}^{\infty} 2^{-j(n-\beta+1)} \int_{A_j} \left|f(y)\right| dy. \end{aligned}$$

Lemma 3.1 implies

$$\int_{A_j} \left|f(y)\right| dy \leq C \|f\chi_j\|_{p(\cdot)} \|\chi_{B_j}\|_{p'(\cdot)}.$$

Applying Lemma 3.3 we obtain

$$\frac{(|x_0| + R)^\alpha}{|Q|^{\frac{\beta}{n}} \|\chi_Q\|_{p'(\cdot)}} \int_Q \left|\tilde{I}_\beta f_3(x) - c_3\right| dx \leq C \frac{2^{k\alpha}}{|Q|^{\frac{\beta-1}{n}}} 2^{-k(n-\beta+1)} \sum_{j=-\infty}^{k-3} \|f\chi_j\|_{p(\cdot)} \|\chi_{B_j}\|_{p'(\cdot)} \|\chi_Q\|_{p(\cdot)}.$$

Since $\alpha - \beta + \frac{n}{p^+} + 1 > 0$, by (4.4) and (4.9),

$$\begin{aligned} \frac{(|x_0| + R)^\alpha}{|Q|^{\frac{\beta}{n}} \|\chi_Q\|_{p'(\cdot)}} \int_Q \left|\tilde{I}_\beta f_3(x) - c_3\right| dx &\leq C \sum_{j=k+2}^{\infty} 2^{(k-j)(\alpha-\beta+\frac{n}{p^+}+1)} 2^{j\alpha} \|f\chi_j\|_{p(\cdot)} \\ &\leq C\|f\|_{\dot{K}^{\alpha}_{p(\cdot),q}(\mathbb{R}^n)}. \end{aligned} \tag{4.11}$$

Combining (4.6)-(4.8), (4.10), and (4.11), case (3) is proved and then the proof of the theorem is completed. □

Proof of Theorem 2.2 Let $f_i(x) = 2^{-i\alpha}\chi_{[2^i,2^i+1]}(x)$ for $i \geq 1$, then $\|f_i\|_{\dot{K}^{\alpha}_{p(\cdot),q}(\mathbb{R}^1)} \sim 1$ and

$$\begin{aligned} \tilde{I}_\beta f_i(x) - \tilde{I}_\beta f_i\left(2^i\right) &= \int_{\mathbb{R}^1} \left\{ \frac{1}{|x-y|^{1-\beta}} - \frac{1}{|2^i - y|^{1-\beta}} \right\} f_i(y)\, dy \\ &= \frac{2^{-i\alpha}}{\beta} \left\{ \left(2^i + 1 - x\right)^\beta - \left(2^i - x\right)^\beta - 1 \right\}. \end{aligned}$$

Let $B_i = (2^i - 1, 2^i)$ with $|B_i| = 1$, then

$$\frac{1}{|B_i|} \int_{B_i} \tilde{I}_\beta f_i(x)\, dx = \frac{2^{-i\alpha}(2^{\beta+1} - \beta - 3)}{\beta(\beta+1)} + \tilde{I}_\beta f_i\left(2^i\right).$$

Hence

$$\begin{aligned} &\frac{1}{|B_i|} \int_{B_i} \left|\tilde{I}_\beta f_i(x) - (\tilde{I}_\beta f_i)_{B_i}\right| dx \\ &\quad = \frac{2^{-i\alpha}}{\beta} \int_{2^i-1}^{2^i} \left| \left(2^i + 1 - x\right)^\beta - \left(2^i - x\right)^\beta - 1 - \frac{2^{\beta+1} - \beta - 3}{\beta+1} \right| dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{2^{-i\alpha}}{\beta} \int_0^1 \left| (2-x)^\beta - (1-x)^\beta - 1 - \frac{2^{\beta+1} - \beta - 3}{\beta + 1} \right| dx \\
&= C2^{-i\alpha}
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{i\to\infty} \frac{1}{(|2^i - \frac{1}{2}| + \frac{1}{2})^{-\alpha-\varepsilon} |B_i|^{\beta+\varepsilon} \|\chi_{B_i}\|_{p'(\cdot)}} \int_{B_i} \left| \tilde{I}_\beta f_i(x) - (\tilde{I}_\beta f_i)_{B_i} \right| dx \\
&\quad = \lim_{i\to\infty} \frac{C2^{-i\alpha}}{(|2^i - \frac{1}{2}| + \frac{1}{2})^{-\alpha-\varepsilon} |B_i|^{\beta+\varepsilon-1} \|\chi_{B_i}\|_{p'(\cdot)}} \\
&\quad = \lim_{i\to\infty} C2^{i\varepsilon} = \infty.
\end{aligned}
$$

This finishes the proof of Theorem 2.2. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors worked jointly in drafting and approved the final manuscript.

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