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Bounds for q-integrals of ${}_{r+1}\psi_{r+1}$ with applications

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Abstract

In this paper, we establish an inequality for the q-integral of the bilateral basic hypergeometric function ${}_{r+1}\psi_{r+1}$. As applications of the inequality, we give some sufficient conditions for the convergence of q-series.

MSC: Primary 26D15; secondary 33D15

Keywords: inequality; q-integral; the bilateral basic hypergeometric function ${}_{r+1}\psi_{r+1}$; convergence

1 Introduction and main result

q-series, which are also called basic hypergeometric series, play a very important role in many fields, such as affine root systems, Lie algebras and groups, number theory, orthogonal polynomials, and physics. The inequality technique is one of the useful tools in the study of special functions. There are many papers about the inequalities and the q-integral; see [1–9]. In this paper, we derive an inequality for the q-integral of the bilateral basic hypergeometric function ${}_{r+1}\psi_{r+1}$. Some applications of the inequality are also given. The main result of this paper is the following inequality for q-integrals.

Theorem 1.1 *Let a, b be any real numbers such that $0 < q < b < a < 1$, and let a_i, b_i be any real numbers such that $|a_i| > q$, $|b_i| < 1$ for $i = 1, 2, \ldots, r$ with $r \geq 1$ and $|b_1 b_2 \cdots b_r| \leq |a_1 a_2 \cdots a_r|$. Then for any $c > 0$, $t > 0$, such that $c > b/a$, $c + t < 1$, we have*

$$\left| \int_0^t {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ b, b_1, \ldots, b_r \end{matrix}; q, c + z \right) d_q z \right| \leq \frac{Mt(q, b/a; q)_\infty}{(c + t, b/ac; q)_\infty}, \tag{1.1}$$

where

$$M = \max\left\{ \prod_{i=1}^{r} \frac{(-|a_i|; q)_\infty}{(|b_i|; q)_\infty}, \prod_{i=1}^{r} \frac{(-q/|b_i|; q)_\infty}{(q/|a_i|; q)_\infty} \right\}.$$

Before we give the proof of the theorem, we recall some definitions, notation, and well-known results which will be used in this paper. Throughout the whole paper, it is supposed that $0 < q < 1$. The q-shifted factorials are defined as

$$(a; q)_0 = 1, \qquad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \qquad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k). \tag{1.2}$$

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We also adopt the following compact notation for the multiple q-shifted factorial:

$$(a_1,a_2,\ldots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n, \tag{1.3}$$

where n is an integer or ∞. We may extend the definition (1.2) of $(a;q)_n$ to

$$(a;q)_\alpha=\frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \tag{1.4}$$

for any complex number α. In particular,

$$(a;q)_{-n}=\frac{(a;q)_\infty}{(aq^{-n};q)_\infty}=\frac{1}{(aq^{-n};q)_n}=\frac{(-q/a)^n}{(q/a;q)_n}q^{\binom{n}{2}}. \tag{1.5}$$

The following is the well-known Ramanujan ${}_1\psi_1$ summation formula [10, 11],

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n=\frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty},\quad |b/a|<|z|<1. \tag{1.6}$$

The bilateral basic hypergeometric series ${}_r\psi_s$ is defined by

$${}_r\psi_s\left(\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_s\end{matrix};q,z\right)=\sum_{n=-\infty}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n}(-1)^{(s-r)n}q^{(s-r)\binom{n}{2}}z^n. \tag{1.7}$$

Jackson defined the q-integral by [12]

$$\int_0^d f(t)\,d_qt=d(1-q)\sum_{n=0}^{\infty}f\left(dq^n\right)q^n \tag{1.8}$$

and

$$\int_c^d f(t)\,d_qt=\int_0^d f(t)\,d_qt-\int_0^c f(t)\,d_qt. \tag{1.9}$$

In [13], the author uses Ramanujan's ${}_1\psi_1$ summation formula to give the following inequality: Let a, b be any real numbers such that $q<b<a<1$ or $a<b<0$, and let a_i, b_i be any real numbers such that $|a_i|>q$, $|b_i|<1$ for $i=1,2,\ldots,r$ with $r\geq 1$ and $|b_1b_2\cdots b_r|\leq|a_1a_2\cdots a_r|$. Then for any $b/a<|z|<1$, we have

$$\left|{}_{r+1}\psi_{r+1}\left(\begin{matrix}a,a_1,\ldots,a_r\\ b,b_1,\ldots,b_r\end{matrix};q,z\right)\right|\leq M\frac{(q,b/a,a|z|,q/a|z|;q)_\infty}{(b,q/a,|z|,b/a|z|;q)_\infty}, \tag{1.10}$$

where

$$M=\max\left\{\prod_{i=1}^{r}\frac{(-|a_i|;q)_\infty}{(|b_i|;q)_\infty},\prod_{i=1}^{r}\frac{(-q/|b_i|;q)_\infty}{(q/|a_i|;q)_\infty}\right\}.$$

2 The proof of theorem

In this section, we use (1.10) to prove Theorem 1.1.

Proof Under the conditions of the theorem 1.1, it is easy to see that

$$b/a < c + tq^n < 1. \tag{2.1}$$

Letting $z = c + tq^n$ in (1.10) gives

$$\begin{aligned}&\left|{}_{r+1}\psi_{r+1}\left(\begin{matrix}a, a_1, \ldots, a_r\\ b, b_1, \ldots, b_r\end{matrix}; q, c + tq^n\right)\right|\\ &\quad\le M\frac{(q, b/a, a(c + tq^n), q/a(c + tq^n); q)_\infty}{(b, q/a, c + tq^n, b/a(c + tq^n); q)_\infty}, \quad n = 0, 1, 2, \ldots.\end{aligned} \tag{2.2}$$

Since $0 < b < a(c + tq^n) < a < 1$, we have

$$\bigl(a\bigl(c + tq^n\bigr), q/a\bigl(c + tq^n\bigr); q\bigr)_\infty < (b, q/a; q)_\infty \tag{2.3}$$

and

$$\bigl(c + tq^n, b/a\bigl(c + tq^n\bigr); q\bigr)_\infty \ge (c + t, b/ac; q)_\infty. \tag{2.4}$$

Combining (2.2), (2.3), and (2.4), we get

$$\left|{}_{r+1}\psi_{r+1}\left(\begin{matrix}a, a_1, \ldots, a_r\\ b, b_1, \ldots, b_r\end{matrix}; q, c + tq^n\right)\right| \le M\frac{(q, b/a, ; q)_\infty}{(c + t, b/ac; q)_\infty}, \quad n = 0, 1, 2, \ldots. \tag{2.5}$$

By the definition of the q-integral (1.8), we get

$$\begin{aligned}&\int_0^t {}_{r+1}\psi_{r+1}\left(\begin{matrix}a, a_1, \ldots, a_r\\ b, b_1, \ldots, b_r\end{matrix}; q, c + z\right) d_qz\\ &\quad= t(1 - q)\sum_{n=0}^{\infty} q^n {}_{r+1}\psi_{r+1}\left(\begin{matrix}a_1, a_2, \ldots, a_{r+1}\\ b_1, b_2, \ldots, b_r\end{matrix}; q, tq^n\right).\end{aligned} \tag{2.6}$$

Consequently,

$$\begin{aligned}&\left|\int_0^t {}_{r+1}\psi_{r+1}\left(\begin{matrix}a, a_1, \ldots, a_r\\ b, b_1, \ldots, b_r\end{matrix}; q, c + z\right) d_qz\right|\\ &\quad= \left|t(1 - q)\sum_{n=0}^{\infty} q^n {}_{r+1}\psi_{r+1}\left(\begin{matrix}a_1, a_2, \ldots, a_{r+1}\\ b_1, b_2, \ldots, b_r\end{matrix}; q, tq^n\right)\right|\\ &\quad\le t(1 - q)\sum_{n=0}^{\infty} q^n \left|{}_{r+1}\psi_{r+1}\left(\begin{matrix}a_1, a_2, \ldots, a_{r+1}\\ b_1, b_2, \ldots, b_r\end{matrix}; q, tq^n\right)\right|.\end{aligned} \tag{2.7}$$

Using (2.5) one gets

$$\begin{aligned}&\left|\int_0^t {}_{r+1}\psi_{r+1}\left(\begin{matrix}a, a_1, \ldots, a_r\\ b, b_1, \ldots, b_r\end{matrix}; q, c + z\right) d_qz\right|\\ &\quad\le t(1 - q)M\frac{(q, b/a, ; q)_\infty}{(c + t, b/ac; q)_\infty}\sum_{n=0}^{\infty} q^n = \frac{Mt(q, b/a, ; q)_\infty}{(c + t, b/ac; q)_\infty}.\end{aligned} \tag{2.8}$$

Thus, we complete the proof. □

From (1.1) and the definition of the q-integral (1.9), we can easily get the following result.

Corollary 2.1 *Under the conditions of the theorem, we have*

$$\begin{aligned}
&\left|\int_s^t {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ b, b_1, \ldots, b_r \end{matrix}; q, c+z\right) d_q z\right| \\
&\quad \leq \frac{M(q, b/a, ; q)_\infty}{(c+t, c+s, b/ac; q)_\infty}\bigl[t(c+s; q)_\infty + s(c+t; q)_\infty\bigr], \qquad (2.9)
\end{aligned}$$

where $s > 0$ *and* $c + s < 1$.

Proof By the definition of the q-integral (1.9), we get

$$\begin{aligned}
&\left|\int_s^t {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ b, b_1, \ldots, b_r \end{matrix}; q, c+z\right) d_q z\right| \\
&\quad = \left|\int_0^t {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ b, b_1, \ldots, b_r \end{matrix}; q, c+z\right) d_q z \right. \\
&\qquad \left. - \int_0^s {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ b, b_1, \ldots, b_r \end{matrix}; q, c+z\right) d_q z\right| \\
&\quad \leq \left|\int_0^t {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ b, b_1, \ldots, b_r \end{matrix}; q, c+z\right) d_q z\right| \\
&\qquad + \left|\int_0^s {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ b, b_1, \ldots, b_r \end{matrix}; q, c+z\right) d_q z\right| \\
&\quad \leq \frac{Mt(q, b/a, ; q)_\infty}{(c+t, b/ac; q)_\infty} + \frac{Ms(q, b/a, ; q)_\infty}{(c+s, b/ac; q)_\infty} \\
&\quad = \frac{M(q, b/a, ; q)_\infty}{(c+t, c+s, b/ac; q)_\infty}\bigl[t(c+s; q)_\infty + s(c+t; q)_\infty\bigr]. \qquad (2.10)
\end{aligned}$$

Thus, the inequality (2.9) holds. □

3 Some applications of the inequality

In this section, we use the inequality obtained in this paper to give some sufficient conditions for the convergence of the q-series. Convergence is an important problem in the study of q-series. There are some results about it. For example, Ito used an inequality technique to give a sufficient condition for the convergence of a special q-series called the Jackson integral [14].

Theorem 3.1 *Suppose that*

(1) a, b, c *be any positive real numbers such that* $0 < q < b < a < 1$, $c > b/a$;
(2) a_i, b_i *be any real numbers such that* $|a_i| > q$, $|b_i| < 1$ *for* $i = 1, 2, \ldots, r$ *with* $r \geq 1$ *and* $|b_1 b_2 \cdots b_r| \leq |a_1 a_2 \cdots a_r|$;
(3) $\{t_n\}$ *be any positive number series, such that* $c + t_n < 1$ *and* $\sum_{n=1}^\infty t_n$ *converges.*

Then the q-*series*

$$\sum_{n=0}^\infty \int_0^{t_n} {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ b, b_1, \ldots, b_r \end{matrix}; q, c+z\right) d_q z \qquad (3.1)$$

converges absolutely.

Proof Since $\sum_{n=1}^{\infty} t_n$ converges, we get

$$\lim_{n\to\infty} t_n = 0. \tag{3.2}$$

So, there exists an integer N_0 such that, when $n > N_0$,

$$c + t_n \leq d < 1. \tag{3.3}$$

When $n > N_0$, letting $t = t_n$ in (1.1) gives

$$\begin{aligned} &\left| \int_0^{t_n} {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ b, b_1, \ldots, b_r \end{matrix}; q, c+z \right) d_q z \right| \\ &\quad \leq \frac{M t_n (q, b/a; q)_\infty}{(c + t_n, b/ac; q)_\infty} \leq \frac{M (q, b/a; q)_\infty}{(d, b/ac; q)_\infty} t_n. \end{aligned} \tag{3.4}$$

From (3.4) and the convergence of $\sum_{n=1}^{\infty} t_n$, it is sufficient to establish that (3.1) is absolutely convergent. □

Corollary 3.2 *Let $\{s_n\}$ be any positive number series such that $c + s_n < 1$ and $\sum_{n=1}^{\infty} s_n$ converges. Under the conditions of Theorem* 3.1, *then the q-series*

$$\sum_{n=0}^{\infty} \int_{s_n}^{t_n} {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ b, b_1, \ldots, b_r \end{matrix}; q, c+z \right) d_q z \tag{3.5}$$

converges absolutely.

Proof By the definition of the q-integral (1.9), we get

$$\begin{aligned} &\int_{s_n}^{t_n} {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ b, b_1, \ldots, b_r \end{matrix}; q, c+z \right) d_q z \\ &\quad = \int_0^{t_n} {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ b, b_1, \ldots, b_r \end{matrix}; q, c+z \right) d_q z \\ &\qquad - \int_0^{s_n} {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ b, b_1, \ldots, b_r \end{matrix}; q, c+z \right) d_q z. \end{aligned} \tag{3.6}$$

Since both

$$\int_0^{t_n} {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ b, b_1, \ldots, b_r \end{matrix}; q, c+z \right) d_q z \tag{3.7}$$

and

$$\int_0^{s_n} {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ b, b_1, \ldots, b_r \end{matrix}; q, c+z \right) d_q z \tag{3.8}$$

are absolutely convergent, so (3.5) is absolutely convergent. □

Theorem 3.3 *Suppose that*

(1) *a, b, c, d be any positive real numbers such that* $0 < q < b < a < 1$, $c > b/a$, $c + d < 1$;

(2) a_i, b_i *be any real numbers such that* $|a_i| > q$, $|b_i| < 1$ *for* $i = 1, 2, \dots, r$ *with* $r \geq 1$ *and* $|b_1 b_2 \cdots b_r| \leq |a_1 a_2 \cdots a_r|$;
(3) $\{t_n\}$ *be any positive number series, such that* $t_n \leq d$ *and* $c + d < 1$;
(4) $\sum_{n=1}^{\infty} e_n$ *converges absolutely.*

Then the q*-series*

$$\sum_{n=0}^{\infty} e_n \int_0^{t_n} {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \dots, a_r \\ b, b_1, \dots, b_r \end{matrix}; q, c+z\right) d_q z \tag{3.9}$$

converges absolutely.

Proof Using (1.1) gives

$$\begin{aligned} &\left| e_n \int_0^{t_n} {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \dots, a_r \\ b, b_1, \dots, b_r \end{matrix}; q, c+z\right) d_q z \right| \\ &\quad \leq \frac{M t_n (q, b/a; q)_\infty}{(c + t_n, b/ac; q)_\infty} |e_n| \leq \frac{M d (q, b/a; q)_\infty}{(c + d, b/ac; q)_\infty} |e_n|. \end{aligned} \tag{3.10}$$

Because $\sum_{n=1}^{\infty} e_n$ converges absolutely, (3.10) is sufficient to establish that (3.9) is absolutely convergent. □

Corollary 3.4 *Suppose that*

(1) a, b, c, d *be any positive real numbers such that* $0 < q < b < a < 1$, $c > b/a$, $c + d < 1$;
(2) a_i, b_i *be any real numbers such that* $|a_i| > q$, $|b_i| < 1$ *for* $i = 1, 2, \dots, r$ *with* $r \geq 1$ *and* $|b_1 b_2 \cdots b_r| \leq |a_1 a_2 \cdots a_r|$;
(3) $\{t_n\}$, $\{s_n\}$ *be any positive number series, such that* $t_n \leq d$, $s_n \leq d$, *and* $c + d < 1$;
(4) $\sum_{n=1}^{\infty} e_n$ *converges absolutely.*

Then the q*-series*

$$\sum_{n=0}^{\infty} e_n \int_{s_n}^{t_n} {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \dots, a_r \\ b, b_1, \dots, b_r \end{matrix}; q, c+z\right) d_q z \tag{3.11}$$

converges absolutely.

Proof By the definition of the q-integral (1.9), we get

$$\begin{aligned} &\left| e_n \int_{s_n}^{t_n} {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \dots, a_r \\ b, b_1, \dots, b_r \end{matrix}; q, c+z\right) d_q z \right| \\ &\quad = \left| e_n \int_0^{t_n} {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \dots, a_r \\ b, b_1, \dots, b_r \end{matrix}; q, c+z\right) d_q z \right. \\ &\qquad \left. - e_n \int_0^{s_n} {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \dots, a_r \\ b, b_1, \dots, b_r \end{matrix}; q, c+z\right) d_q z \right| \\ &\quad \leq \left| e_n \int_0^{t_n} {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \dots, a_r \\ b, b_1, \dots, b_r \end{matrix}; q, c+z\right) d_q z \right| \\ &\qquad + \left| e_n \int_0^{s_n} {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \dots, a_r \\ b, b_1, \dots, b_r \end{matrix}; q, c+z\right) d_q z \right| \end{aligned}$$

$$\leq \frac{Mt_n(q, b/a; q)_\infty}{(c + t_n, b/ac; q)_\infty}|e_n| + \frac{Ms_n(q, b/a; q)_\infty}{(c + s_n, b/ac; q)_\infty}|e_n|$$

$$\leq \frac{2Md(q, b/a; q)_\infty}{(c + d, b/ac; q)_\infty}|e_n|. \tag{3.12}$$

Since $\sum_{n=1}^{\infty} e_n$ converges absolutely, (3.11) converges absolutely. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors equally have made contributions. All authors read and approved the final manuscript.

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