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Some estimates for commutators of Hausdorff operators

Jie Sun[1,2*] and Xianliang Shi[2]

*Correspondence: sj800816@163.com
[1]College of Science, Mudanjiang Normal University, Mudanjiang, Heilongjiang 157011, P.R. China
[2]College of Mathematics and Computer Science, Key Laboratory of High Performance Computing and Stochastic Information Processing (HPCSIP) (Ministry of Education of China), Hunan Normal University, Changsha, Hunan 410081, P.R. China

Abstract

In this paper, we establish some new estimates for commutators of Hausdorff operators on homogeneous Herz and Morrey-Herz spaces, which extend some previous results.

MSC: 42B20; 42B25

Keywords: Hausdorff operator; central BMO function; Herz space; Morrey-Herz space; commutator

1 Introduction and results

In 2000, Liflyand and Móricz gave the definition of Hausdorff operator in [1]. Suppose f is a locally integrable function on $\mathbb{R}$, the Hausdorff operator is defined by

$$h_\Phi f(x) = \int_{\mathbb{R}} \frac{\Phi(\frac{x}{t})}{t} f(t)\,dt, \quad x \in \mathbb{R},$$

where Φ belongs to $L^1(\mathbb{R})$. The fractional Hausdorff operator in higher dimensional space $\mathbb{R}^n$ is defined in [2] as follows:

$$H_{\Phi,\beta} f(x) = \int_{\mathbb{R}^n} \frac{\Phi(\frac{x}{|y|})}{|y|^{n-\beta}} f(y)\,dy,$$

where Φ is a radial function defined on $\mathbb{R}^n$, $0 \le \beta < n$. When $\beta = 0$, $H_{\Phi,\beta} f = H_\Phi f$.

If we set $\Phi(t) = t^{\beta-n}\chi_{(1,\infty)}(t)$, we get

$$H_{\Phi,\beta} f(x) = H_\beta f(x),$$

where H_β is the fractional Hardy operator defined by

$$H_\beta f(x) = \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} f(y)\,dy, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

If we set $\Phi(t) = \chi_{(0,1)}(t)$, we obtain

$$H_{\Phi,\beta} f(x) = H^*_\beta f(x),$$

where H_β^* is defined by

$$H_\beta^* f(x) = \int_{|y|\geq|x|} \frac{1}{|y|^{n-\beta}} f(y)\,dy.$$

The properties of Hausdorff operator in L^p, H^p, h^p and other spaces can be found in [1, 3–7].

Let T be a classical Calderón-Zygmund operator with its kernel satisfying the standard estimates; the commutator generated by T and a function $b \in BMO(\mathbb{R}^n)$ is defined by

$$[b,T]f(x) = b(x)T(f)(x) - T(bf)(x).$$

In 1976, Coifman *et al.* [8] proved that $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$ for any $1 < p < \infty$. In 2012, Gao and Jia studied the boundedness of commutator $[b, H_\Phi]$ of Hausdorff operator with central BMO function in Lebesgue space, Morrey-Herz spaces, and Herz spaces in [9]. The theory of commutators of singular integrals has been studied extensively.

In 1981, Cohen [10] considered the following generalized commutators:

$$T_A f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}} f(y)\bigl(A(x) - A(y) - \nabla A(y)(x-y)\bigr)\,dy,$$

where $\Omega \in L^1(S^{n-1})$ satisfies $\Omega(\lambda x) = \Omega(x)$, $\lambda > 0$, $\forall x$, $\int_{S^{n-1}} |x|\Omega(x)\,dx = 0$. And he proved when $\Omega \in \mathrm{Lip}_1(S^{n-1})$, $\nabla A \in BMO(\mathbb{R}^n)$, T_A is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

In 2002, Pérez and Trujillo-González [11] established the boundedness for the multilinear commutators of the classical Calderón-Zygmund operators $T_{\vec{b}} f$, which are defined by $T_{\vec{b}} f(x) = \int_{\mathbb{R}^n} \prod_{j=1}^m (b_j(x) - b_j(y))K(x,y)\,dy$. Later the multilinear commutators were widely studied by many authors.

In this paper, we mainly discuss the properties of generalized commutators and multilinear commutators of Hausdorff operators with central BMO functions in some function spaces.

Let us first of all recall the definition of homogeneous central BMO space.

Definition 1 [12] Let $1 \leq q < \infty$, $CBMO_q(\mathbb{R}^n)$ is the space of all functions $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying

$$\|f\|_{CBMO_q(\mathbb{R}^n)} = \sup_{r>0} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} \bigl| f(x) - f_{B(0,r)} \bigr|^q \,dx \right)^{1/q} < \infty,$$

where $B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$ and $f_{B(0,r)}$ is the mean value of f on $B(0,r)$.

It is easy to see $BMO(\mathbb{R}^n) \subsetneqq CBMO_q(\mathbb{R}^n)$, $1 \leq q < \infty$, and $CBMO_q(\mathbb{R}^n) \subsetneqq CBMO_p(\mathbb{R}^n)$, $1 \leq p < q < \infty$.

Let us give the definitions of the generalized commutators of Hausdorff operator and the multilinear commutators of Hausdorff operators.

Let A be a function on $\mathbb{R}^n$ having derivatives of order one in $CBMO_q(\mathbb{R}^n)$. For $x, t \in \mathbb{R}^n$, set

$$R(A; x, t) = A(x) - A(t) - \nabla A(t)(x-t).$$

We define the fractional generalized commutators of Hausdorff operators as follows:

$$\begin{aligned} H_{\Phi,\beta,A}f(x) &= \int_{\mathbb{R}^n} \frac{\Phi(\frac{x}{|y|})}{|y|^{n-\beta}|x-y|} f(y)R(A;x,y)\,dy \\ &= \int_{|y|<|x|} \frac{\Phi(\frac{x}{|y|})}{|y|^{n-\beta}|x-y|} f(y)R(A;x,y)\,dy \\ &\quad + \int_{|y|\geq|x|} \frac{\Phi(\frac{x}{|y|})}{|y|^{n-\beta}|x-y|} f(y)R(A;x,y)\,dy \\ &= H^1_{\Phi,\beta,A}f(x) + H^2_{\Phi,\beta,A}f(x), \end{aligned}$$

where Φ is a radial function. When $\beta = 0$, write $H_{\Phi,\beta,A}f(x) = H_{\Phi,A}f(x)$, and $H^1_{\Phi,\beta,A}f(x) = H^1_{\Phi,A}f(x)$, $H^2_{\Phi,\beta,A}f(x) = H^2_{\Phi,A}f(x)$.

Set $\Phi(t) = t^{\beta-n}\chi_{(1,\infty)}(t)$, then

$$H_{\Phi,\beta,A}f(x) = H_{\beta,A}f(x),$$

and set $\Phi(t) = \chi_{(0,1)}(t)$, we have

$$H_{\Phi,\beta,A}f(x) = H^*_{\beta,A}f(x).$$

In 2010, Lu and Zhao [13] considered the properties of generalized commutators of Hardy operators. They got the following results.

Theorem A *Let* $1 < p < \infty$, $u \geq p'$, $s > n$, *and* $\lambda \geq 0$. *Suppose* $\nabla A \in CBMO_{\max\{s,u\}}$. *If* $\alpha < np/p'$, *then*

$$\|H_A f\|_{L^p(|x|^\alpha\,dx)} \leq C\|\nabla A\|_{CBMO_{\{s,u\}}}\|f\|_{L^p(|x|^\alpha\,dx)}.$$

Theorem B *Let* $1 < p < \infty$, $1 < q < \infty$. *Suppose* $u \geq q'$, $s > n$ *and* $\nabla A \in CBMO_{\max\{s,u\}}$. *If* $\alpha < n/q'$, *then*

$$\|H_A f\|_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)} \leq C\|\nabla A\|_{CBMO_{\{s,u\}}}\|f\|_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)}.$$

Theorem C *Let* $1 < p < \infty$, $1 < q < \infty$ *and* $\lambda \geq 0$. *Suppose* $u \geq q'$, $s > n$ *and* $\nabla A \in CBMO_{\max\{s,u\}}$. *If* $\alpha < n/q' + \lambda$, *then*

$$\|H_A f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \leq C\|\nabla A\|_{CBMO_{\{s,u\}}}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}.$$

Let $\vec{b} = (b_1,\ldots,b_m)$, $b_j \in CBMO_q(\mathbb{R}^n)$, $1 \leq j \leq m$, $1 \leq q < \infty$. Similarly to [14], we consider the following the multilinear commutator $H_{\Phi,\beta,\vec{b}}$ generalized by the n dimensional fractional Hausdorff operator $H_{\Phi,\beta}$ and $\vec{b}$:

$$H_{\Phi,\beta,\vec{b}}f(x) = \int_{\mathbb{R}^n}\left[\prod_{j=1}^m\bigl(b_j(x) - b_j(y)\bigr)\right]\frac{\Phi(\frac{x}{|y|})}{|y|^{n-\beta}}f(y)\,dy,$$

where m is a nonpositive integer, Φ is a radial function.

We decompose the integral

$$\begin{aligned} H_{\Phi,\beta,\vec{b}}f(x) &= \int_{\mathbb{R}^n} \frac{\Phi(\frac{x}{|y|})}{|y|^{n-\beta}} \Biggl[\prod_{j=1}^{m} \bigl(b_j(x) - b_j(y)\bigr) \Biggr] f(y)\,dy \\ &= \int_{|y|<|x|} \frac{\Phi(\frac{x}{|y|})}{|y|^{n-\beta}} \Biggl[\prod_{j=1}^{m} \bigl(b_j(x) - b_j(y)\bigr) \Biggr] f(y)\,dy \\ &\quad + \int_{|y|\geq|x|} \frac{\Phi(\frac{x}{|y|})}{|y|^{n-\beta}} \Biggl[\prod_{j=1}^{m} \bigl(b_j(x) - b_j(y)\bigr) \Biggr] f(y)\,dy \\ &= H^1_{\Phi,\beta,\vec{b}}f(x) + H^2_{\Phi,\beta,\vec{b}}f(x). \end{aligned}$$

When $m = 1$, $\beta = 0$, denote

$$H_{\Phi,\beta,\vec{b}}f(x) = H_{\Phi,b}f(x) = [b, H_\Phi]f(x)$$

and

$$H^1_{\Phi,\beta,\vec{b}}f(x) = H^1_{\Phi,b}f(x), \qquad H^2_{\Phi,\beta,\vec{b}}f(x) = H^2_{\Phi,b}f(x).$$

Select

$$\Phi_1(t) = t^{\beta-n}\chi_{(1,\infty)}(t)$$

and

$$\Phi_2(t) = \chi_{(0,1)}(t),$$

we have

$$H^1_{\Phi_1,\beta,\vec{b}}f(x) = H_{\beta,\vec{b}}f(x)$$

and

$$H^2_{\Phi_2,\beta,\vec{b}}f(x) = H^*_{\beta,\vec{b}}f(x).$$

Before we formulate the main theorems, we give some remarks

$$\begin{aligned} \mathscr{A}^n_{\Phi,\beta,r} &= \biggl(\int_0^\infty \bigl|\Phi(t)\bigr|^{r'} t^{-1-\beta r' + \frac{nr'}{r}}\,dt \biggr)^{\frac{1}{r'}}, \\ \mathscr{B}^n_{\Phi,r} &= \biggl(\int_0^\infty \bigl|\Phi(t)\bigr|^{r'} t^{-1+n}\,dt \biggr)^{\frac{1}{r'}}. \end{aligned}$$

When $\beta = 0$, denote by $\mathscr{A}^n_{\Phi,\beta,r} = \mathscr{A}^n_{\Phi,r}$.

Our main results are the following theorems.

Theorem 1 *Suppose* $1 < p, q < \infty$, $0 \le \beta < n$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{n}$, *and* $\nabla A \in CBMO_{\max\{s, \frac{pr}{p-r}\}}(\mathbb{R}^n)$ *with* $s > n$, $r < \min\{p, q'\}$.

(a) *If* $\mathscr{A}^n_{\Phi,\beta,r} < \infty$, *then* $H^1_{\Phi,\beta,A}$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.
(b) *If* $\mathscr{B}^n_{\Phi,r} < \infty$, *then* $H^2_{\Phi,\beta,A}$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.
(c) *If* $\mathscr{A}^n_{\Phi,\beta,r} < \infty$, $\mathscr{B}^n_{\Phi,r} < \infty$, *then* $H_{\Phi,\beta,A}$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.

Theorem 2 *Suppose* $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $0 \le \beta < n$, $\frac{1}{q_1} - \frac{1}{q_2} = \frac{\beta}{n}$, *and* $\nabla A \in CBMO_{\max\{s, \frac{q_1 r}{q_1 - r}\}}(\mathbb{R}^n)$ *with* $1 < r < q_1$, $s > n$.

(a) *If* $\mathscr{A}^n_{\Phi,\beta,r} < \infty$, $\alpha < \frac{n}{r} - \frac{n}{q_1}$, *then* $H^1_{\Phi,\beta,A}$ *is bounded from* $\dot{K}^{\alpha,p}_{q_1}(\mathbb{R}^n)$ *to* $\dot{K}^{\alpha,p}_{q_2}(\mathbb{R}^n)$.
(b) *If* $\mathscr{B}^n_{\Phi,r} < \infty$, $\alpha > -(\frac{n}{q_2} - \frac{n}{r})$, *then* $H^2_{\Phi,\beta,A}$ *is bounded from* $\dot{K}^{\alpha,p}_{q_1}(\mathbb{R}^n)$ *to* $\dot{K}^{\alpha,p}_{q_2}(\mathbb{R}^n)$.
(c) *If* $\mathscr{A}^n_{\Phi,\beta,r} < \infty$, $\mathscr{B}^n_{\Phi,r} < \infty$, $-(\frac{n}{q_2} - \frac{n}{r}) < \alpha < \frac{n}{r} - \frac{n}{q_1}$, *then* $H_{\Phi,\beta,A}$ *is bounded from* $\dot{K}^{\alpha,p}_{q_1}(\mathbb{R}^n)$ *to* $\dot{K}^{\alpha,p}_{q_2}(\mathbb{R}^n)$.

Theorem 3 *Suppose* $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $0 \le \beta < n$, $\lambda \ge 0$, $\frac{1}{q_1} - \frac{1}{q_2} = \frac{\beta}{n}$, *and* $\nabla A \in CBMO_{\max\{s, \frac{q_1 r}{q_1 - r}\}}(\mathbb{R}^n)$ *with* $1 < r < q_1$, $s > n$.

(a) *If* $\mathscr{A}^n_{\Phi,\beta,r} < \infty$, $\alpha < \frac{n}{r} - \frac{n}{q_1} + \lambda$, *then* $H^1_{\Phi,\beta,A}$ *is bounded from* $M\dot{K}^{\alpha,\lambda}_{p,q_1}(\mathbb{R}^n)$ *to* $M\dot{K}^{\alpha,\lambda}_{p,q_2}(\mathbb{R}^n)$.
(b) *If* $\mathscr{B}^n_{\Phi,r} < \infty$, $\alpha > -(\frac{n}{q_2} - \frac{n}{r}) + \lambda$, *then* $H^2_{\Phi,\beta,A}$ *is bounded from* $M\dot{K}^{\alpha,\lambda}_{p,q_1}(\mathbb{R}^n)$ *to* $M\dot{K}^{\alpha,\lambda}_{p,q_2}(\mathbb{R}^n)$.
(c) *If* $\mathscr{A}^n_{\Phi,\beta,r} < \infty$, $\mathscr{B}^n_{\Phi,r} < \infty$, $-(\frac{n}{q_2} - \frac{n}{r}) + \lambda < \alpha < \frac{n}{r} - \frac{n}{q_1} + \lambda$, *then* $H_{\Phi,\beta,A}$ *is bounded from* $M\dot{K}^{\alpha,\lambda}_{p,q_1}(\mathbb{R}^n)$ *to* $M\dot{K}^{\alpha,\lambda}_{p,q_2}(\mathbb{R}^n)$.

Remark 1 Select $\Phi(t) = t^{\beta-n}\chi_{(1,\infty)}(t)$, it is easy to get $\mathscr{A}^n_{\Phi,\beta,r} < \infty$, and $H_{\Phi,\beta,A}f = H^1_{\Phi,\beta,A}f = H_{\beta,A}f$. We can get the boundedness of generalized commutator of fractional Hardy operator on Herz and Morrey-Herz spaces.

Remark 2 If we select $\Phi(t) = t^{\beta-n}\chi_{(1,\infty)}(t)$, it is easy to get $\mathscr{B}^n_{\Phi,r} < \infty$, and $H_{\Phi,\beta,A}f = H^2_{\Phi,\beta,A}f = H^*_{\beta,A}f$, by Theorems 2 and 3, we can get the boundedness of $H^*_{\beta,A}$ on Herz and Morrey-Herz spaces.

Remark 3 When $\beta = 0$, by Remark 1 and the definitions of Herz and Morrey-Herz spaces, we can easily get Theorem A, B, and C.

Theorem 4 *Let* $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $0 \le \beta < n$, $\frac{1}{q_1} - \frac{1}{q_2} = \frac{\beta}{n}$, $1 < r < q_1$, $s > n$, *and* $b_j \in CBMO_{\max\{r_j q_2, \frac{q_1 r}{q_1 - r} r_j\}}(\mathbb{R}^n)$, $r_j > 1$ $(1 \le j \le m)$, $\frac{1}{r_1} + \cdots + \frac{1}{r_m} = 1$.

(a) *If* $A^n_{\Phi,\beta,r} < \infty$, $\alpha < \frac{n}{r} - \frac{n}{q_1}$, *then* $H^1_{\Phi,\beta,\vec{b}}$ *is bounded from* $\dot{K}^{\alpha,p}_{q_1}(\mathbb{R}^n)$ *to* $\dot{K}^{\alpha,p}_{q_2}(\mathbb{R}^n)$.
(b) *If* $\mathscr{B}^n_{\Phi,r} < \infty$, $\alpha > -(\frac{n}{q_2} - \frac{n}{r})$, *then* $H^2_{\Phi,\beta,\vec{b}}$ *is bounded from* $\dot{K}^{\alpha,p}_{q_1}(\mathbb{R}^n)$ *to* $\dot{K}^{\alpha,p}_{q_2}(\mathbb{R}^n)$.
(c) *If* $\mathscr{A}^n_{\Phi,\beta,r} < \infty$, $\mathscr{B}^n_{\Phi,r} < \infty$, $-(\frac{n}{q_2} - \frac{n}{r}) < \alpha < \frac{n}{r} - \frac{n}{q_1}$, *then* $H_{\Phi,\beta,\vec{b}}$ *is bounded from* $\dot{K}^{\alpha,p}_{q_1}(\mathbb{R}^n)$ *to* $\dot{K}^{\alpha,p}_{q_2}(\mathbb{R}^n)$.

Theorem 5 *Let* $\lambda \ge 0$, $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $0 \le \beta < n$, $\frac{1}{q_1} - \frac{1}{q_2} = \frac{\beta}{n}$, $1 < r < q_1$, $s > n$, *and* $b_j \in CBMO_{\max\{r_j q_2, \frac{q_1 r}{q_1 - r} r_j\}}(\mathbb{R}^n)$, $r_j > 1$ $(1 \le j \le m)$, $\frac{1}{r_1} + \cdots + \frac{1}{r_m} = 1$.

(a) *If* $\mathscr{A}^n_{\Phi,\beta,r} < \infty$, $\alpha < \frac{n}{r} - \frac{n}{q_1} + \lambda$, *then* $H^1_{\Phi,\beta,\vec{b}}$ *is bounded from* $M\dot{K}^{\alpha,\lambda}_{p,q_1}(\mathbb{R}^n)$ *to* $M\dot{K}^{\alpha,\lambda}_{p,q_2}(\mathbb{R}^n)$.
(b) *If* $\mathscr{B}^n_{\Phi,r} < \infty$, $\alpha > -(\frac{n}{q_2} - \frac{n}{r}) + \lambda$, *then* $H^2_{\Phi,\beta,\vec{b}}$ *is bounded from* $M\dot{K}^{\alpha,\lambda}_{p,q_1}(\mathbb{R}^n)$ *to* $M\dot{K}^{\alpha,\lambda}_{p,q_2}(\mathbb{R}^n)$.
(c) *If* $\mathscr{A}^n_{\Phi,\beta,r} < \infty$, $\mathscr{B}^n_{\Phi,r} < \infty$, $-(\frac{n}{q_2} - \frac{n}{r}) + \lambda < \alpha < \frac{n}{r} - \frac{n}{q_1} + \lambda$, *then* $H_{\Phi,\beta,\vec{b}}$ *is bounded from* $M\dot{K}^{\alpha,\lambda}_{p,q_1}(\mathbb{R}^n)$ *to* $M\dot{K}^{\alpha,\lambda}_{p,q_2}(\mathbb{R}^n)$.

Remark 4 If we select $\Phi(t) = t^{\beta-n}\chi_{(1,\infty)}(t)$, it is obvious that $\mathscr{A}^n_{\Phi,\beta,r} < \infty$, we may obtain the boundedness of $H_{\beta,\vec{b}}$ on Herz and Morrey-Herz spaces. If we select $\Phi(t) = \chi_{(0,1)}(t)$, $\mathscr{B}^n_{\Phi,r} < \infty$, we may obtain the boundedness of $H^*_{\beta,\vec{b}}$ on Herz and Morrey-Herz spaces.

Corollary 1 *Let* $1 < p,q < \infty$, $s > n$, $0 \le \beta < n$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{n}$, $r < \min\{p,q'\}$, *and* $b_j \in CBMO_{\max\{r_j q, \frac{pr}{p-r} r_j\}}(\mathbb{R}^n)$, $r_j > 1$ $(1 \le j \le m)$, $\frac{1}{r_1} + \cdots + \frac{1}{r_m} = 1$.

(a) *If* $\mathscr{A}^n_{\Phi,\beta,r} < \infty$, *then* $H^1_{\Phi,\beta,\vec{b}}$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.
(b) *If* $\mathscr{B}^n_{\Phi,r} < \infty$, *then* $H^2_{\Phi,\beta,\vec{b}}$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.
(c) *If* $\mathscr{A}^n_{\Phi,\beta,r} < \infty$, $\mathscr{B}^n_{\Phi,r} < \infty$, *then* $H_{\Phi,\beta,\vec{b}}$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.

Remark 5 When $\beta = 0$, $m = 1$, the boundedness of commutator $H_{\Phi,b}$ can be obtained on Lebesgue, Herz, and Morrey-Herz spaces.

The rest of this paper is organized as follows. After recalling some preliminary notations and lemmas in Section 2, we will prove our results in Section 3. We would like to remark that the main methods employed in this paper are a combination of ideas and arguments from [8, 9] and [13].

Throughout this paper, we let p' satisfy $1/p + 1/p' = 1$. The letter C, sometimes with additional parameters, will stand for positive constants, not necessarily the same at each occurrence, but C is independent of the essential variables.

2 Preliminaries and lemmas

In order to prove the theorems, we will formulate some lemmas and preliminaries. For the multi-indices $\gamma = (\gamma_1, \ldots, \gamma_n)$, we will always use notations $|\gamma| = \gamma_1 + \cdots + \gamma_n$, γ_j $(1 \le j \le n)$ being nonnegative integers, $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$. Let $\nabla A = (D_1 A, D_2 A, \ldots, D_n A)$ where $D_j A = \frac{\partial A}{\partial x_j}$, $j = 1, \ldots, n$.

For any positive integer m and j $(1 \le j \le m)$, we denote by $\mathscr{C}^m_j$ the family of all finite subsets $\sigma = \{\sigma(1), \sigma(2), \ldots, \sigma(j)\}$ of $\{1, 2, \ldots, m\}$ of j different elements. For any $\sigma \in \mathscr{C}^m_j$, we associate the complementary sequence $\sigma' \in \mathscr{C}^j_{m-j}$ given by $\sigma' = \{1, 2, \ldots, m\} \setminus \sigma$. We also denote by $|\sigma|$ the number of elements in σ, and

$$\mathscr{C}^m_j = \bigl\{\sigma : \sigma = \bigl\{\sigma(1), \sigma(2), \ldots, \sigma(j)\bigr\}\bigr\}, \quad 1 \le j \le m.$$

Let $b_j \in CBMO_{qr_j}(\mathbb{R}^n)$ $(1 \le j \le m)$, $\vec{b} = (b_1, b_2, \ldots, b_m)$, for $1 < r_i, q < \infty$,

$$\frac{1}{r_1} + \cdots + \frac{1}{r_m} = 1,$$

denote by $\|\vec{b}\|_{CBMO_{qr_i}(\mathbb{R}^n)} = \|b_1\|_{CBMO_{qr_1}(\mathbb{R}^n)} \|b_2\|_{CBMO_{qr_2}(\mathbb{R}^n)} \cdots \|b_m\|_{CBMO_{qr_m}(\mathbb{R}^n)}$.

For all $1 \le j \le m$ and $\sigma = \{\sigma(1), \sigma(2), \ldots, \sigma(j)\} \in \mathscr{C}^m_j$. Denote $\vec{b}_\sigma = (b_{\sigma(1)}, b_{\sigma(2)}, \ldots, b_{\sigma(j)})$, $\vec{b}_{\sigma'} = (b_{\sigma(j+1)}, \ldots, b_{\sigma(m)})$. Denote

$$\bigl(\vec{b}(x) - \vec{b}(y)\bigr)_\sigma = \bigl(b_{\sigma(1)}(x) - b_{\sigma(1)}(y)\bigr) \cdots \bigl(b_{\sigma(j)}(x) - b_{\sigma(j)}(y)\bigr)$$

and

$$\bigl(\vec{b}_B - \vec{b}(y)\bigr)_\sigma = \bigl((b_{\sigma(1)})_B - b_{\sigma(1)}(y)\bigr) \cdots \bigl((b_{\sigma(j)})_B - b_{\sigma(j)}(y)\bigr),$$

where B is any ball in $\mathbb{R}^n$, $(b_{\sigma(j)})_B$ is the average of $b_{\sigma(j)}$ over ball B. Denote by

$$\|\vec{b}_\sigma\|_{CBMO_{qr_\sigma}(\mathbb{R}^n)} = \|b_{\sigma(1)}\|_{CBMO_{qr_{\sigma_1}}(\mathbb{R}^n)}\|b_{\sigma(2)}\|_{CBMO_{qr_{\sigma_2}}(\mathbb{R}^n)}\cdots\|b_{\sigma(j)}\|_{CBMO_{qr_{\sigma_j}}(\mathbb{R}^n)},$$

where $1 < q < \infty$, $r_{\sigma(j)} \in \{r_1, \dots, r_m\}$, $1 \le j \le m$,

$$\frac{1}{r_{\sigma(1)}} + \cdots + \frac{1}{r_{\sigma(j)}} = \frac{1}{r_\sigma}$$

and

$$\frac{1}{r_\sigma} + \frac{1}{r_{\sigma'}} = 1.$$

For all $\sigma \in \mathscr{C}_j^m$, denote

$$H_{\Phi,\beta,\vec{b}_\sigma} f(x) = \int_{\mathbb{R}^n} \prod_{j=1}^{|\sigma|} \big(b_{\sigma(j)}(x) - b_{\sigma(j)}(y)\big) \frac{\Phi(\frac{x}{|y|})}{|y|^{n-\beta}} f(y)\,dy,$$

when $\sigma = \{1, 2, \dots, m\}$, $\sigma' = \emptyset$, write $\vec{b}_\sigma = \vec{b}$, $H_{\Phi,\beta,\vec{b}_\sigma} = H_{\Phi,\beta,\vec{b}}$, $H_{\Phi,\beta,\vec{b}_{\sigma'}} = H_{\Phi,\beta}$.

For $k \in \mathbb{Z}$, let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $C_k = B_k \setminus B_{k-1}$, and χ_k ($k \in \mathbb{Z}$) denote the characteristic function of the set C_k.

Definition 2 [15] Let $\alpha \in \mathbb{R}$, $0 < p \le \infty$, $0 < q < \infty$. The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \big\{f \in L_{\mathrm{loc}}^q(\mathbb{R}^n \setminus \{0\}), \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty\big\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|_{L^q(\mathbb{R}^n)}^p\right\}^{1/p},$$

with the usual modification made when $p = \infty$.

Obviously, $\dot{K}_q^{0,q}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$, $\dot{K}_q^{\frac{\alpha}{q},q}(\mathbb{R}^n) = L^q(\mathbb{R}^n, |x|^\alpha)$, so the Herz space is the natural generalization of the Lebesgue spaces with power weight $|x|^\alpha$.

Definition 3 [16] Let $\alpha \in \mathbb{R}$, $0 < p \le \infty$, $0 < q < \infty$, and $\lambda \ge 0$. The homogeneous Morrey-Herz space $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$ is defined by

$$M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n) = \big\{f \in L_{\mathrm{loc}}^q(\mathbb{R}^n \setminus \{0\}), \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} < \infty\big\},$$

where

$$\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|f\chi_k\|_{L^q(\mathbb{R}^n)}^p\right)^{1/p},$$

with the usual modification made when $p = \infty$.

Obviously, $M\dot{K}^{\alpha,0}_{p,q}(\mathbb{R}^n) = \dot{K}^{\alpha,p}_q(\mathbb{R}^n)$.

Similarly to the discussion of Lemma 3.1 in [9], it is easy to get the following results.

Lemma 1 *Let $\beta \geq 0$, $1 < r \leq p < \infty$, Φ is a radial function, then*

$$\text{(i)}\quad \int_{C_i} \frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}} |f(y)|\,dy \leq \mathscr{A}_{\Phi,\beta,r} 2^{in(\frac{1}{r}-\frac{1}{p})} |x|^{\beta-\frac{n}{r}} \|f\chi_i\|_{L^p(\mathbb{R}^n)};$$

$$\text{(ii)}\quad \int_{C_i} \frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}} |f(y)|\,dy \leq \mathscr{B}_{\Phi,r} 2^{i\beta+in(\frac{1}{r'}-\frac{1}{p})} |x|^{-\frac{1}{r'}} \|f\chi_i\|_{L^p(\mathbb{R}^n)}.$$

Lemma 2 [14] *Let A be a function on $\mathbb{R}^n$ with derivatives of order one in $L^q(\mathbb{R}^n)$ for some $q > n$. Then*

$$|A(x) - A(y)| \leq C|x-y| \left(\frac{1}{|I_x^y|} \int_{I_x^y} |\nabla A(z)|^q\,dz \right)^{\frac{1}{q}},$$

where I_x^y is the cube centered at x with sides parallel to the axes and whose side length is $2\sqrt{n}|x-y|$.

Lemma 3 [10] *Suppose that $f \in CBMO_q(\mathbb{R}^n)$, $1 \leq q < \infty$, and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{B(0,r_1)} \int_{B(0,r_1)} |f(x) - f_{B(0,r_2)}|^q\,dx \right)^{\frac{1}{q}} \leq C\left(1 + \left|\log\left(\frac{r_1}{r_2}\right)\right|\right) \|f\|_{CBMO_q}.$$

3 Proofs of main theorems

It is easily to see that Theorem 1 can be immediately deduced from Theorem 2 by letting $\alpha = 0$, $1 < p = q_1 < \infty$, $1 < q_2 = q < \infty$. Thus it is sufficient to prove Theorems 2 and 3.

Proof of Theorem 2 We only consider the case $1 < p < \infty$, while the case $p = \infty$ follows after slight modifications.

For simplicity, we denote $q = \frac{q_1 r}{q_1 - r}$.

(a) When $\mathscr{A}^n_{\Phi,\beta,r} < \infty$, we get

$$\begin{aligned}
\|(H^1_{\Phi,\beta,A} f)\chi_k\|^{q_2}_{L^{q_2}(\mathbb{R}^n)} &\leq \int_{C_k} \left(\int_{B_i} \frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}|x-y|} |R(A;x,y)f(y)|\,dy \right)^{q_2} dx \\
&\leq \int_{C_k} \left(\sum_{i=-\infty}^{k-2} \int_{C_i} \frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}|x-y|} |R(A;x,y)f(y)|\,dy \right)^{q_2} dx \\
&\quad + \int_{C_k} \left(\sum_{i=k-1}^{k} \int_{C_i} \frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}|x-y|} |R(A;x,y)f(y)|\,dy \right)^{q_2} dx \\
&:= I + II.
\end{aligned}$$

For fixed k, set

$$A_k(y) = A(y) - m_{B_k}(\nabla A) y^\alpha, \quad \text{for all multi-indices } \alpha \text{ such that } |\alpha| = 1,$$

where $m_{B_k}(\nabla A)$ is the mean value of ∇A on B_k. By a simple calculation, we get $R(A;x,y) = R(A_k;x,y)$. (More details may be found in [17].)

Since $y \in C_i$ and $x \in C_k$, $i \le k-2$, then $|x-y| \sim |x| \sim 2^k$. It is easy to deduce that $\nabla A_k(y) = \nabla A(y) - m_{B_k}(\nabla A)$. Then by Lemma 2 for $s > n$, similarly as the discussion in [15] we have

$$\begin{aligned}
\left|R(A_k;x,y)\right| &\le \left|A_k(x) - A_k(y)\right| + \left|\nabla A(y) - m_{B_k}(\nabla A)\right||x-y| \\
&\le C|x-y|\left\{\left(\frac{1}{|I_x^y|}\int_{I_x^y}\left|\nabla A_k(z)\right|^s dz\right)^{1/s} + \left|\nabla A(y) - m_{B_k}(\nabla A)\right|\right\} \\
&\le C|x-y|\left\{\left(\frac{1}{2^{kn}}\int_{B(0,C2^k)}\left|\nabla A(z) - m_{B_k}(\nabla A)\right|^s dz\right)^{1/s}\right. \\
&\quad \left. + \left|\nabla A(y) - m_{B_k}(\nabla A)\right|\right\} \\
&\le C|x-y|\left(\|\nabla A\|_{CBMO_s} + \left|\nabla A(y) - m_{B_k}(\nabla A)\right|\right).
\end{aligned}$$

Then we can split I into two parts

$$\begin{aligned}
I &\le \int_{C_k}\left(\sum_{i=-\infty}^{k-2}\int_{C_i}\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}|x-y|}\left|R(A;x,y)f(y)\right| dy\right)^{q_2} dx \\
&\le C\int_{C_k}\left(\sum_{i=-\infty}^{k-2}\int_{C_i}\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}}\|\nabla A\|_{CBMO_s}\left|f(y)\right| dy\right)^{q_2} dx \\
&\quad + C\int_{C_k}\left(\sum_{i=-\infty}^{k-2}\int_{C_i}\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}}\left|\nabla A(y) - m_{B_k}(\nabla A)\right|\left|f(y)\right| dy\right)^{q_2} dx \\
&:= I_1 + I_2.
\end{aligned}$$

Now we estimate the I_1. Using Lemma 1(i) and noting that $\frac{1}{q_2} = \frac{1}{q_1} - \frac{\beta}{n}$, we get

$$\begin{aligned}
I_1 &\le C\mathscr{A}_{\Phi,\beta,r}^{n}{}^{q_2}\|\nabla A\|_{CBMO_s}^{q_2}\int_{B_k}\left(\sum_{i=-\infty}^{k-2}2^{in(\frac{1}{r}-\frac{1}{q_1})}2^{k\beta-\frac{kn}{r}}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}\right)^{q_2} dx \\
&\le C\mathscr{A}_{\Phi,\beta,r}^{n}{}^{q_2}\|\nabla A\|_{CBMO_s}^{q_2}|B_k|\left(\sum_{i=-\infty}^{k-2}2^{in(\frac{1}{r}-\frac{1}{q_1})}2^{k\beta-\frac{kn}{r}}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}\right)^{q_2} \\
&\le C\mathscr{A}_{\Phi,\beta,r}^{n}{}^{q_2}\|\nabla A\|_{CBMO_s}^{q_2}\left(\sum_{i=-\infty}^{k-2}2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}\right)^{q_2}.
\end{aligned}$$

For $y \in C_i$, we have

$$\left|\nabla A(y) - m_{B_k}(\nabla A)\right| \le C\left|\nabla A(y) - m_{B_{i+1}}(\nabla A)\right| + (k-i)\|\nabla A\|_{CBMO_1}.$$

Then we can decompose I_2 into two parts

$$\begin{aligned}
I_2 &\le C\int_{C_k}\left(\sum_{i=\infty}^{k-2}\int_{C_i}\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}}\left|\nabla A(y) - m_{B_{i+1}}(\nabla A)\right|\left|f(y)\right| dy\right)^{q_2} dx \\
&\quad + C\int_{C_k}\left(\sum_{i=\infty}^{k-2}\int_{C_i}\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}}(k-i)\|\nabla A\|_{CBMO_1}\left|f(y)\right| dy\right)^{q_2} dx \\
&:= I_2' + I_2''.
\end{aligned}$$

For I_2', by Lemma 1(i) for $r < q_1$ and Hölder's inequality, we have

$$\begin{aligned}
I_2' &\le C\mathscr{A}^n_{\Phi,\beta,r}{}^{q_2} \int_{C_k} \left(\sum_{i=-\infty}^{k-2} |x|^{\beta-\frac{n}{r}} \int_{C_i} \left|\nabla A(y) - m_{B_{i+1}}(\nabla A)\right|^r \left|f(y)\right|^r dy \right)^{q_2} dx \\
&\le C\mathscr{A}^n_{\Phi,\beta,r}{}^{q_2} \int_{C_k} \left\{ \sum_{i=-\infty}^{k-2} |x|^{\beta-\frac{n}{r}} \left(\int_{C_i} \left|\nabla A(y) - m_{B_{i+1}}(\nabla A)\right|^{\frac{rq_1}{q_1-r}} \right)^{\frac{q_1-r}{q_1 r}} \right. \\
&\quad \times \left. \left(\int_{C_i} \left|f(y)\right|^{q_1} dy \right)^{q_1} \right\}^{q_2} dx \\
&\le C\mathscr{A}^n_{\Phi,\beta,r}{}^{q_2} \int_{C_k} \left(\sum_{i=-\infty}^{k-2} 2^{k\beta-\frac{kn}{r}} \|\nabla A\|_{CBMO_q} |B_i|^{\frac{q_1-r}{q_1 r}} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^{q_2} \\
&\le C\mathscr{A}^n_{\Phi,\beta,r}{}^{q_2} \|\nabla A\|^{q_2}_{CBMO_q} \left(\sum_{i=-\infty}^{k-2} 2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^{q_2}.
\end{aligned}$$

For I_2'', using Lemma 1(i) and $\frac{1}{q_2} = \frac{1}{q_1} - \frac{\beta}{n}$, we get

$$\begin{aligned}
I_2'' &\le C\mathscr{A}^n_{\Phi,\beta,r}{}^{q_2} \|\nabla A\|^{q_2}_{CBMO_1} \int_{C_k} \left(\sum_{i=-\infty}^{k-2} (k-i) 2^{in(\frac{1}{r}-\frac{1}{q_1})} |x|^{\beta-\frac{n}{r}} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^{q_2} dx \\
&\le C\mathscr{A}^n_{\Phi,\beta,r}{}^{q_2} \|\nabla A\|^{q_2}_{CBMO_1} 2^{kn} \left(\sum_{i=-\infty}^{k-2} (k-i) 2^{in(\frac{1}{r}-\frac{1}{q_1})} 2^{k\beta-\frac{kn}{r}} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^{q_2} \\
&\le C\mathscr{A}^n_{\Phi,\beta,r}{}^{q_2} \|\nabla A\|^{q_2}_{CBMO_1} \left(\sum_{i=-\infty}^{k-2} (k-i) 2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^{q_2}.
\end{aligned}$$

To estimate II, take a $\phi \in C_0^\infty$, such that $\operatorname{supp}\phi \subset B(0,2)$ and $\phi = 1$ in $B(0,1)$. Set $M = \max\{\|\phi\|_\infty, \|\nabla\phi\|_\infty\}$. Take $y_0 \in C_{k+4}$, and let

$$A_k^\phi(x) = \left(A_k(x) - A_k(y_0)\right)\phi\left(2^{-k}x\right).$$

For $x \in C_k$, $y \in C_i$, $k-1 \le i \le k$, and $\phi = 1$ in $B(0,1)$, we may have $\phi(2^{-k}x) = 1$, $\phi(2^{-k}y) = 1$. Since

$$R\left(A_k^\phi; x, y\right) = A_k^\phi(x) - A_k^\phi(y) - \nabla A_k^\phi(y)(x-y),$$

we have

$$\begin{aligned}
R\left(A_k^\phi; x, y\right) &= \left(A_k(x) - A_k(y_0)\right)\phi\left(2^{-k}x\right) - \left(A_k(y) - A_k(y_0)\right)\phi\left(2^{-k}y\right) - \nabla A_k^\phi(y)(x-y) \\
&= A_k(x) - A_k(y) - \nabla\left(A_k(y) - A_k(y_0)\right)(x-y) \\
&= A_k(x) - A_k(y) - \nabla A_k(y)(x-y) \\
&= R(A_k, x, y) \\
&= R(A, x, y).
\end{aligned}$$

Then we have

$$\begin{aligned}
II &\le C\int_{C_k}\left(\sum_{i=k-1}^{k}\int_{C_i}\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}|x-y|}\big|f(y)\big|\big|R\big(A_k^{\phi};x,y\big)\big|\,dy\right)^{q_2}dx \\
&\le C\int_{C_k}\left(\sum_{i=k-1}^{k}\int_{C_i}\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}|x-y|}\big|f(y)\big|\big|A_k^{\phi}(x)-A_k^{\phi}(y)\big|\,dy\right)^{q_2}dx \\
&\quad + C\int_{C_k}\left(\sum_{i=k-1}^{k}\int_{C_i}\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}}\big|f(y)\big|\big|\nabla A_k^{\phi}(y)\big|\,dy\right)^{q_2}dx \\
&:= II_1 + II_2.
\end{aligned}$$

Now we consider II_1. Since $k-1\le i\le k$, $y\in C_i$, $y_0\in C_{k+4}$, $|y-y_0|\sim 2^k$, by Lemma 2, we get

$$\begin{aligned}
\big|\nabla A_k^{\phi}(y)\big| &= \big|\nabla\big(\big(A_k(y)-A_k(y_0)\big)\phi\big(2^{-k}y\big)\big)\big| \\
&\le \big|\nabla A_k(y)\phi\big(2^{-k}y\big)\big| \\
&\quad + 2^{-k}\big|A_k(y)-A_k(y_0)\big|\big|\nabla\phi\big(2^{-k}y\big)\big| \\
&\le M\big(\big|\nabla A_k(y)\big| + 2^{-k}\big|A_k(y)-A_k(y_0)\big|\big) \\
&\le CM\left(\big|\nabla A_k(y)\big| + 2^{-k}|y-y_0|\left(\frac{1}{|I_y^{y_0}|}\int_{I_y^{y_0}}\big|\nabla A_k(z)\big|^s\,dz\right)^{1/s}\right) \\
&\le CM\big(\big|\nabla A_k(y)\big| + \|\nabla A\|_{CBMO_s}\big).
\end{aligned}$$

Then we see

$$\begin{aligned}
II_2 &\le C\int_{C_k}\left(\sum_{i=k-1}^{k}\int_{C_i}\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}}\big|f(y)\big|\big|\nabla A_k(y)\big|\,dy\right)^{q_2}dx \\
&\quad + C\int_{C_k}\left(\sum_{i=k-1}^{k}\int_{C_i}\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}}\big|f(y)\big|\|\nabla A\|_{CBMO_s}\,dy\right)^{q_2}dx \\
&:= II_2' + II_2''.
\end{aligned}$$

Since $\nabla A_k(y) = \nabla A(y) - m_{B_k}(\nabla A)$, we get

$$II_2' \le C\int_{C_k}\left(\sum_{i=k-1}^{k}\int_{C_i}\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}}\big|f(y)\big|\big|\nabla A(y)-m_{B_k}(\nabla A)\big|\,dy\right)^{q_2}dx.$$

Employing the same idea for estimating I_2, we have

$$II_2' \le C{\mathscr{A}^{n}_{\Phi,\beta,r}}^{q_2}\|\nabla A\|_{CBMO_q}^{q_2}\left(\sum_{i=k-1}^{k}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}\right)^{q_2}.$$

We now estimate II_2''. Applying Lemma 1(i), we get

$$\begin{aligned}
II_2'' &\le C\|\nabla A\|_{CBMO_s}^{q_2} \mathscr{A}_{\Phi,\beta,r}^{n}{}^{q_2} \int_{C_k} \left(\sum_{i=k-1}^{k} 2^{in(\frac{1}{r}-\frac{1}{q_1})} |x|^{\beta-\frac{n}{r}} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^{q_2} dx \\
&\le C\|\nabla A\|_{CBMO_s}^{q_2} \mathscr{A}_{\Phi,\beta,r}^{n}{}^{q_2} 2^{kn} \left(\sum_{i=k-1}^{k} 2^{in(\frac{1}{r}-\frac{1}{q_1})} |x|^{\beta-\frac{n}{r}} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^{q_2} \\
&\le C\|\nabla A\|_{CBMO_s}^{q_2} \mathscr{A}_{\Phi,\beta,r}^{n}{}^{q_2} \left(\sum_{i=k-1}^{k} 2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^{q_2}.
\end{aligned}$$

Next we turn to estimate II_1. By Lemma 1(i), we get the following estimate:

$$\begin{aligned}
II_1 &\le C\|\nabla A\|_{CBMO_s}^{q_2} \int_{C_k} \left(\sum_{i=k-1}^{k} \int_{2^{i-1}<|y|<2^i} \frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}|x-y|} \right. \\
&\quad \left. \times \left|A_k^{\phi}(x) - A_k^{\phi}(y)\right| \left|f(y)\right| dy \right)^{q_2} dx \\
&= C\|\nabla A\|_{CBMO_s}^{q_2} \int_{C_k} \left(\sum_{i=k-1}^{k} \sum_{j=-\infty}^{i} \int_{2^{i-1}<|y|<2^i, 2^j<|x-y|<2^{j+1}} \frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}|x-y|} \right. \\
&\quad \left. \times \left|A_k^{\phi}(x) - A_k^{\phi}(y)\right| \left|f(y)\right| dy \right)^{q_2} dx.
\end{aligned}$$

Now we estimate $|A_k^{\phi}(x) - A_k^{\phi}(y)|$. By Lemma 2 and Lemma 3, we obtain

$$\begin{aligned}
\left|A_k^{\phi}(x) - A_k^{\phi}(y)\right| &\le C|x-y| \left(\frac{1}{|I_x^y|} \int_{I_x^y} \left|\nabla A_k^{\phi}(z)\right|^s dz \right)^{1/s} \\
&\le C|x-y| \left(\frac{1}{2^{jn}} \int_{B(0,C2^j)} \left|\nabla A_k(z) + \|\nabla A\|_{CBMO_s}\right|^s dz \right)^{1/s} \\
&\le CM|x-y| \left(\left(\frac{1}{2^{jn}} \int_{B(0,C2^j)} \left|\nabla A_k(z)\right|^s dz \right)^{1/s} + \|\nabla A\|_{CBMO_s} \right) \\
&\le M|x-y| \big(|k-j| + C\big) \|\nabla A\|_{CBMO_s}.
\end{aligned}$$

Then by $\frac{1}{q_2} = \frac{1}{q_1} - \frac{\beta}{n}$, we get

$$\begin{aligned}
II_1 &\le C\|\nabla A\|_{CBMO_s}^{q_2} \int_{C_k} \left(\sum_{i=k-1}^{k} \sum_{j=-\infty}^{i} \int_{2^{i-1}<|y|<2^i, 2^j<|x-y|<2^{j+1}} \frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}} \right. \\
&\quad \left. \times M\big[(k-j) + C\big] \left|f(y)\right| dy \right)^{q_2} dx \\
&\le C\mathscr{A}_{\Phi,\beta,r}^{n}{}^{q_2} \|\nabla A\|_{CBMO_s}^{q_2} \int_{C_k} \left(\sum_{i=k-1}^{k} \sum_{j=-\infty}^{i} (k-j) 2^{jn(\frac{1}{r}-\frac{1}{q_1})} |x|^{\beta-\frac{n}{r}} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^{q_2} dx \\
&\le C\mathscr{A}_{\Phi,\beta,r}^{n}{}^{q_2} \|\nabla A\|_{CBMO_s}^{q_2} 2^{kn} \left(\sum_{i=k-1}^{k} \sum_{j=-\infty}^{i} (k-j) 2^{jn(\frac{1}{r}-\frac{1}{q_1})} 2^{k\beta-\frac{kn}{r}} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^{q_2}
\end{aligned}$$

$$\leq C{\mathscr{A}^n_{\Phi,\beta,r}}^{q_2}\|\nabla A\|^{q_2}_{CBMO_s}\left(\sum_{i=k-1}^{k}\sum_{j=-\infty}^{i}(k-j)2^{(j-k)n(\frac{1}{r}-\frac{1}{q_1})}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}\right)^{q_2}$$

$$\leq C{\mathscr{A}^n_{\Phi,\beta,r}}^{q_2}\|\nabla A\|^{q_2}_{CBMO_s}\left(\sum_{i=k-1}^{k}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}\right)^{q_2}.$$

Then we have

$$\begin{aligned}
&\|H^1_{\Phi,\beta,A}f\|_{\dot{K}^{\alpha,p}_{q_2}}\\
&\quad\leq C\|\nabla A\|_{CBMO_{\max\{s,q\}}}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{i=-\infty}^{k-2}(k-i)2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}\right)^p\right\}^{\frac{1}{p}}\\
&\qquad+C\|\nabla A\|_{CBMO_{\max\{s,q\}}}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{i=k-1}^{k}(k-i)2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}\right)^p\right\}^{\frac{1}{p}}\\
&\qquad+C\|\nabla A\|_{CBMO_s}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{i=k-1}^{k}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}\right)^p\right\}^{\frac{1}{p}}:=L+J+K.
\end{aligned}$$

Case 1. $0<p\leq 1$. By the well-known inequality

$$\left(\sum_{i=1}^{\infty}|a_i|\right)^p\leq\sum_{i=1}^{\infty}|a_i|^p,$$

we have the following:

$$\begin{aligned}
L&\leq C\|\nabla A\|_{CBMO_{\max\{s,q\}}}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\sum_{i=-\infty}^{k-2}(k-i)^p2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})p}\|f\chi_i\|^p_{L^{q_1}(\mathbb{R}^n)}\right\}^{\frac{1}{p}}\\
&\leq C\|\nabla A\|_{CBMO_{\max\{s,q\}}}\left\{\sum_{i=-\infty}^{\infty}2^{i\alpha p}\sum_{k=i+2}^{\infty}(k-i)^p2^{(i-k)(\frac{n}{r}-\frac{n}{q_1}-\alpha)p}\|f\chi_i\|^p_{L^{q_1}(\mathbb{R}^n)}\right\}^{\frac{1}{p}}\\
&\leq C\|\nabla A\|_{CBMO_{\max\{s,q\}}}\left\{\sum_{i=-\infty}^{\infty}2^{i\alpha p}\|f\chi_i\|^p_{L^{q_1}(\mathbb{R}^n)}\sum_{k=i+2}^{\infty}(k-i)^p2^{(k-i)(\frac{n}{q_1}+\alpha-\frac{n}{r})p}\right\}^{\frac{1}{p}}\\
&\leq C\|\nabla A\|_{CBMO_{\max\{s,q\}}}\|f\|_{\dot{K}^{\alpha,p}_{q_1}(\mathbb{R}^n)}.
\end{aligned}$$

Similarly to the estimate of L, we obtain

$$J\leq C\|\nabla A\|_{CBMO_{\max\{s,q\}}}\|f\|_{\dot{K}^{\alpha,p}_{q_1}(\mathbb{R}^n)}.$$

For K, we have

$$\begin{aligned}
K&\leq C\|\nabla A\|_{CBMO_s}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\sum_{i=k-1}^{k}\|f\chi_i\|^p_{L^{q_1}(\mathbb{R}^n)}\right\}^{\frac{1}{p}}\\
&\leq C\|\nabla A\|_{CBMO_s}\left\{\sum_{i=-\infty}^{\infty}2^{i\alpha p}\|f\chi_i\|^p_{L^{q_1}(\mathbb{R}^n)}\right\}^{\frac{1}{p}}\leq C\|\nabla A\|_{CBMO_s}\|f\|_{\dot{K}^{\alpha,p}_{q_1}(\mathbb{R}^n)}.
\end{aligned}$$

Case 2. $1 < p < \infty$. Using Hölder's inequality, we get

$$
\begin{aligned}
L &\le C\|\nabla A\|_{CBMO_{\max\{s,q\}}}\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{i=-\infty}^{k-2}(k-i)2^{(k-i)(\frac{n}{q_1}+\alpha-\frac{n}{r})}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}\right)^p\right\}^{\frac{1}{p}}\\
&\le C\|\nabla A\|_{CBMO_{\max\{s,q\}}}\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{i=-\infty}^{k-2}2^{i\alpha p}(k-i)^p2^{(k-i)(\frac{n}{q_1}+\alpha-\frac{n}{r})\frac{p}{2}}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}^p\right)\right.\\
&\quad\left.\times\left(\sum_{i=-\infty}^{k-2}2^{(k-i)(\frac{n}{q_1}+\alpha-\frac{n}{r})\frac{p'}{2}}\right)^{\frac{p}{p'}}\right\}^{\frac{1}{p}}\\
&\le C\|\nabla A\|_{CBMO_{\max\{s,q\}}}\left\{\sum_{i=-\infty}^{\infty}2^{i\alpha p}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}^p\left(\sum_{k=i+2}^{\infty}2^{(k-i)(\alpha+\frac{n}{q_1}-\frac{n}{r})\frac{p'}{2}}\right)\right\}^{1/p}\\
&\le C\|\nabla A\|_{CBMO_{\max\{s,q\}}}\|f\|_{\dot{K}_{q_1}^{\alpha,p}(\mathbb{R}^n)}.
\end{aligned}
$$

Similarly to the estimate of L, we obtain

$$
J \le C\|\nabla A\|_{CBMO_{\max\{s,q\}}}\|f\|_{\dot{K}_{q_1}^{\alpha,p}(\mathbb{R}^n)}.
$$

For K, we have

$$
\begin{aligned}
K &\le C\|\nabla A\|_{CBMO_s}\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{i=k-1}^{k}2^{i\alpha}2^{(k-i)\alpha}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}\right)^p\right\}^{\frac{1}{p}}\\
&\le C\|\nabla A\|_{CBMO_s}\left\{\sum_{k=-\infty}^{\infty}2^{i\alpha p}\left(\sum_{i=k-1}^{k}2^{(k-i)\alpha\frac{p}{2}}\|f\chi_i\|_{L^{q_1}}^p(\mathbb{R}^n)\right)\right.\\
&\quad\left.\times\left(\sum_{i=k-1}^{k}2^{(k-i)\alpha\frac{p'}{2}}\right)^{\frac{p}{p'}}\right\}^{\frac{1}{p}}\\
&\le C\|\nabla A\|_{CBMO_s}\left(\sum_{i=-\infty}^{\infty}2^{i\alpha p}\|f\chi_i\|_{L^{q_1}}^p(\mathbb{R}^n)\right)^{\frac{1}{p}}\\
&= C\|\nabla A\|_{CBMO_s}\|f\|_{\dot{K}_{q_1}^{\alpha,p}(\mathbb{R}^n)}.
\end{aligned}
$$

This completes the proofs of (a). The proofs of (b) are similar to that for (a), thus we omit the details. By combining the estimate (a) and (b), we get (c). □

Proof of Theorem 3 (a) We only give the proof when $\lambda > 0$. By a similar method to the proof of I, II in Theorem 2, we have

$$
\begin{aligned}
&\left\|H_{\Phi,\beta,A}^1 f\right\|_{M\dot{K}_{p,q_2}^{\alpha,\lambda}}\\
&= \sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha p}\left\|(H_{\Phi,\beta,A}f)\chi_k\right\|_{L^{q_2}(\mathbb{R}^n)}^p\right)^{1/p}\\
&\le C\|\nabla A\|_{CBMO_{\max\{s,\frac{q_1 r}{q_1-r}\}}}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}2^{k\alpha p}\left(\sum_{i=-\infty}^{k-2}2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}\right)^p\right\}^{1/p}
\end{aligned}
$$

$$
\begin{aligned}
&+ C\|\nabla A\|_{CBMO_{\max\{s,\frac{q_1 r}{q_1-r}\}}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \\
&\times \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{i=k-1}^{k} (k-i) 2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
&+ C\|\nabla A\|_{CBMO_s} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{i=k-1}^{k} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
= {}& C\|\nabla A\|_{CBMO_{\max\{s,\frac{q_1 r}{q_1-r}\}}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \\
&\times \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} 2^{-kn(\frac{1}{r}-\frac{1}{q_1})} \left(\sum_{i=-\infty}^{k-2} 2^{in(\frac{1}{r}-\frac{1}{q_1})} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
&+ C\|\nabla A\|_{CBMO_{\max\{s,\frac{q_1 r}{q_1-r}\}}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \\
&\times \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} 2^{-kn(\frac{1}{r}-\frac{1}{q_1})} \left(\sum_{i=k-1}^{k} (k-i) 2^{in(\frac{1}{r}-\frac{1}{q_1})} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
&+ C\|\nabla A\|_{CBMO_s} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{i=k-1}^{k} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
:= {}& E_1 + E_2 + E_3.
\end{aligned}
$$

For E_1, noting that $\alpha < \frac{n}{r} - \frac{n}{q_1} + \lambda$, we have

$$
\begin{aligned}
E_1 \le {}& C\|\nabla A\|_{CBMO_{\max\{s,\frac{q_1 r}{q_1-r}\}}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \\
&\times \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{i=-\infty}^{k-2} 2^{k\alpha} 2^{-kn(\frac{1}{r}-\frac{1}{q_1})} 2^{in(\frac{1}{r}-\frac{1}{q_1})} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
\le {}& C\|\nabla A\|_{CBMO_{\max\{s,\frac{q_1 r}{q_1-r}\}}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \\
&\times \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{i=-\infty}^{k-2} 2^{k\alpha} 2^{-kn(\frac{1}{r}-\frac{1}{q_1})} 2^{in(\frac{1}{r}-\frac{1}{q_1})} 2^{-i\alpha} \right.\right. \\
&\left.\left. \times 2^{i\lambda} 2^{-i\lambda} \left(\sum_{l=-\infty}^{i} 2^{l\alpha p} \|f\chi_l\|_{L^{q_1}(\mathbb{R}^n)}^p \right)^{1/p} \right)^p \right\}^{1/p} \\
\le {}& C\|\nabla A\|_{CBMO_{\max\{s,\frac{q_1 r}{q_1-r}\}}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \\
&\times \left\{ \sum_{k=-\infty}^{k_0} 2^{pk\lambda} \left(\sum_{i=-\infty}^{k-2} 2^{(i-k)(-\alpha+\frac{n}{r}-\frac{n}{q_1}+\lambda)} \|f\|_{M\dot{K}_{p,q_1}^{\alpha,\lambda}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
\le {}& C\|\nabla A\|_{CBMO_{\max\{s,\frac{q_1 r}{q_1-r}\}}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{pk\lambda} \right)^{1/p} \|f\|_{M\dot{K}_{p,q_1}^{\alpha,\lambda}(\mathbb{R}^n)} \\
\le {}& C\|\nabla A\|_{CBMO_{\max\{s,\frac{q_1 r}{q_1-r}\}}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} 2^{pk_0\lambda} \|f\|_{M\dot{K}_{p,q_1}^{\alpha,\lambda}(\mathbb{R}^n)} \\
= {}& C\|\nabla A\|_{CBMO_{\max\{s,\frac{q_1 r}{q_1-r}\}}} \|f\|_{M\dot{K}_{p,q_1}^{\alpha,\lambda}(\mathbb{R}^n)}.
\end{aligned}
$$

Similarly to the proof of E_1, we have

$$E_2 \le C\|\nabla A\|_{CBMO_{\max\{s,\frac{q_1 r}{q_1-r}\}}}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q_1}(\mathbb{R}^n)}.$$

For E_3, we have the following estimate:

$$\begin{aligned}
E_3 &= C\|\nabla A\|_{CBMO_s}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda} \\
&\quad\times\left\{\sum_{k=-\infty}^{k_0}2^{k\alpha p}\left(\sum_{i=k-1}^{k}2^{-k\lambda}2^{k\alpha}2^{-i\alpha}2^{i\lambda}2^{-i\lambda}\left(\sum_{l=-\infty}^{i}2^{l\alpha p}\|f\chi_l\|^p_{L^{q_1}(\mathbb{R}^n)}\right)^{1/p}\right)^p\right\}^{1/p} \\
&\le C\|\nabla A\|_{CBMO_s}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha p}\right)^{1/p} \\
&\quad\times\sum_{i=k-1}^{k}2^{(i-k)(-\alpha+\lambda)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q_1}(\mathbb{R}^n)} \\
&\le C\|\nabla A\|_{CBMO_s}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q_1}(\mathbb{R}^n)}.
\end{aligned}$$

This finishes the proof of (a). The proof of Theorem 3(b) is similar to that for (a). We omit the details. By combining the estimates of (a) and (b), we can easily get (c). □

Proof of Theorem 4 We prove (a) first. When $\mathscr{A}^n_{\Phi,\beta,r}<\infty$, we get

$$\begin{aligned}
&\left\|\left(H^1_{\Phi,\beta,\vec{b}}f\right)\chi_k\right\|^{q_2}_{L^{q_2}(\mathbb{R}^n)} \\
&\le\int_{C_k}\left(\int_{B_i}\left|\prod_{j=1}^{m}\left(b_{\sigma(j)}(x)-b_{\sigma(j)}(y)\right)\right|\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}}\left|f(y)\right|dy\right)^{q_2}dx \\
&\le\int_{C_k}\left(\sum_{i=-\infty}^{k}\int_{C_i}\left|\prod_{j=0}^{m}\left(b_{\sigma(j)}(x)-b_{\sigma(j)}(y)\right)\right|\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}}\left|f(y)\right|dy\right)^{q_2}dx \\
&=\int_{C_k}\left(\sum_{i=-\infty}^{k}\int_{C_i}\left|\sum_{j=0}^{m}\sum_{\sigma\in\mathscr{C}^m_j}\left(\vec{b}(x)-\vec{b}_{B_k}\right)_\sigma\left(\vec{b}_{B_k}-\vec{b}(y)\right)_{\sigma'}\right|\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}}\left|f(y)\right|dy\right)^{q_2}dx \\
&\le\int_{C_k}\left|\prod_{j=1}^{m}\left(b_j(x)-(b_j)_{B_k}\right)\right|^{q_2}\left(\sum_{i=-\infty}^{k}\int_{C_i}\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}}\left|f(y)\right|dy\right)^{q_2}dx \\
&\quad+\int_{C_k}\left(\sum_{i=-\infty}^{k}\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}^m_j}\int_{C_i}\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}}\right. \\
&\quad\left.\times\left|\left(\vec{b}(x)-\vec{b}_{B_k}\right)_\sigma\left(\vec{b}_{B_k}-\vec{b}(y)\right)_{\sigma'}\right|\left|f(y)\right|dy\right)^{q_2}dx \\
&\quad+\int_{C_k}\left(\sum_{i=-\infty}^{k}\int_{C_i}\left|\prod_{j=1}^{m}\left(b_j(y)-(b_j)_{B_k}\right)\right|\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}}f(y)\,dy\right)^{q_2}dx \\
&:=I+II+III.
\end{aligned}$$

To estimate I, by Lemma 1(i) and $\frac{1}{q_2} = \frac{1}{q_1} - \frac{\beta}{n}$, $\frac{1}{r_1} + \cdots + \frac{1}{r_m} = 1$, $1 < r_j < \infty$, for $1 < r < q_1$, we get

$$\begin{aligned}
I &= C \int_{C_k} \left| \prod_{j=1}^{m} \bigl(b_j(x) - (b_j)_{B_k}\bigr) \right|^{q_2} \left(\sum_{i=\infty}^{k} \int_{C_i} \frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}} \bigl|f(y)\bigr|\, dy \right)^{q_2} dx \\
&\le C \prod_{j=1}^{m} \|b_j\|_{CBMO_{q_2 r_j}}^{q_2} |B_k| \left(\sum_{i=-\infty}^{k} 2^{in(\frac{1}{r} - \frac{1}{q_1})} 2^{k\beta - \frac{kn}{r}} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^{q_2} \\
&\le C \prod_{j=1}^{m} \|b_j\|_{CBMO_{q_2 r_j}}^{q_2} \left(\sum_{i=-\infty}^{k} 2^{(i-k)n(\frac{1}{r} - \frac{1}{q_1})} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^{q_2}.
\end{aligned}$$

For II, we have

$$\begin{aligned}
II &= C \int_{C_k} \left(\sum_{i=-\infty}^{k} \sum_{j=1}^{m-1} \sum_{\sigma \in \mathscr{C}_j^m} \int_{C_i} \frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}} \bigl| \bigl(\vec{b}(x) - \vec{b}_{B_k}\bigr)_\sigma \bigl(\vec{b}_{B_k} - \vec{b}(y)\bigr)_{\sigma'} \bigr| \bigl|f(y)\bigr|\, dy \right)^{q_2} dx \\
&\le C \sum_{j=1}^{m-1} \sum_{\sigma \in \mathscr{C}_j^m} \int_{C_k} \bigl| \bigl(\vec{b}(x) - \vec{b}_{B_k}\bigr)_\sigma \bigr|^{q_2} \left(\sum_{i=-\infty}^{k} \int_{C_i} \frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}} \bigl| \bigl(\vec{b}_{B_k} - \vec{b}(y)\bigr)_{\sigma'} \bigr| \bigl|f(y)\bigr|\, dy \right)^{q_2} dx.
\end{aligned}$$

By Lemma 1(i), we have

$$\begin{aligned}
II &\le C \sum_{j=1}^{m-1} \sum_{\sigma \in \mathscr{C}_j^m} \int_{C_k} \bigl| \bigl(\vec{b}(x) - \vec{b}_{B_k}\bigr)_\sigma \bigr|^{q_2} \\
&\quad \times \left(\sum_{i=-\infty}^{k} |x|^{\beta - \frac{n}{r}} \left(\int_{C_i} \bigl| \bigl(\vec{b}_{B_i} - \vec{b}(y)\bigr)_{\sigma'} \bigr|^r \bigl|f(y)\bigr|^r\, dy \right)^{1/r} \right)^{q_2} dx.
\end{aligned}$$

And by Hölder's inequality for $\frac{1}{r_{\sigma(1)}} + \frac{1}{r_{\sigma(2)}} + \cdots + \frac{1}{r_{\sigma(j)}} + \frac{1}{r_{\sigma'}} = 1$, and noting that $\frac{1}{q_2} = \frac{1}{q_1} - \frac{\beta}{n}$, we get

$$\begin{aligned}
II &\le \sum_{j=1}^{m-1} \sum_{\sigma \in \mathscr{C}_j^m} \left(\int_{C_k} \bigl| b_{\sigma(1)}(x) - (b_{\sigma(1)})_{B_k} \bigr|^{q_2 r_{\sigma(1)}}\, dx \right)^{\frac{1}{r_{\sigma(1)}}} \times \cdots \\
&\quad \times \left(\int_{C_k} \bigl| b_{\sigma(j)}(x) - (b_{\sigma(j)})_{B_k} \bigr|^{q_2 r_{\sigma(j)}}\, dx \right)^{\frac{1}{r_{\sigma(j)}}} \\
&\quad \times \left(\int_{C_k} \left(\sum_{i=-\infty}^{k} 2^{k\beta - \frac{kn}{r}} \left(\int_{C_i} \bigl| \bigl(\vec{b}_{B_k} - \vec{b}(y)\bigr)_{\sigma'} \bigr|^r \bigl|f(y)\bigr|^r\, dy \right)^{1/r} \right)^{q_2 r_{\sigma'}} dx \right)^{1/r_{\sigma'}} \\
&\le C \sum_{j=1}^{m-1} \sum_{\sigma \in \mathscr{C}_j^m} \|\vec{b}_\sigma\|_{CBMO_{q_2 r_\sigma}(\mathbb{R}^n)}^{q_2} |B_k| \\
&\quad \times \left(\left(\sum_{i=-\infty}^{k} 2^{k\beta - \frac{kn}{r}} \int_{C_i} \bigl| \bigl(\vec{b}_{B_k} - \vec{b}(y)\bigr)_{\sigma'} \bigr|^r \bigl|f(y)\bigr|^r\, dy \right)^{1/r} \right)^{q_2}
\end{aligned}$$

$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\|\vec{b}_\sigma\|^{q_2}_{CBMO_{q_2r_\sigma}(\mathbb{R}^n)}|B_k|$$
$$\times\left(\sum_{i=-\infty}^{k}2^{k\beta-\frac{kn}{r}}\left(\int_{C_i}\left|\left(\vec{b}_{B_k}-\vec{b}(y)\right)_{\sigma'}\right|^{\frac{rq_1}{q_1-r}}dy\right)^{\frac{q_1-r}{q_1r}}\left(\int_{C_i}\left|f(y)\right|^{q_1}dy\right)^{1/q_1}\right)^{q_2}.$$

By Lemma 3, we have

$$II\le C\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\|\vec{b}_\sigma\|^{q_2}_{CBMO_{q_2r_\sigma}(\mathbb{R}^n)}(k-i)^{|\sigma'|}\|\vec{b}_{\sigma'}\|^{q_2}_{CBMO_{\frac{rq_1}{q_1-r}r_{\sigma'}}(\mathbb{R}^n)}$$
$$\times\left(\sum_{i=-\infty}^{k}2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}\right)^{q_2}$$
$$\le C\prod_{j=1}^{m}\|b_j\|^{q_2}_{CBMO_{\max\{q_2r_j,\frac{rq_1}{q_1-r}r_j\}}(\mathbb{R}^n)}\left(\sum_{i=-\infty}^{k}(k-i)^{|\sigma'|}2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}\right)^{q_2}.$$

Now we estimate III. Since

$$\prod_{j=1}^{m}\left(b_j(y)-(b_j)_{B_k}\right)=\sum_{j=0}^{m}\sum_{\sigma\in\mathscr{C}_j^m}\left[\left(\vec{b}(y)-\vec{b}_{B_i}\right)_\sigma(\vec{b}_{B_i}-\vec{b}_{B_k})_{\sigma'}\right],$$

and by Lemma 3 and Minkowski's inequality, we get

$$III\le\sum_{i=-\infty}^{k}\int_{C_k}\left(\int_{C_i}\left|\prod_{j=1}^{m}\left(b_j(y)-(b_j)_{B_i}\right)+\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\left(\vec{b}(y)-\vec{b}_{B_i}\right)_\sigma(\vec{b}_{B_i}-\vec{b}_{B_k})_{\sigma'}\right.\right.$$
$$\left.\left.+\prod_{j=1}^{m}\left((b_j)_{B_i}-(b_j)_{B_k}\right)\right|\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}}\left|f(y)\right|dy\right)^{q_2}dx$$
$$\le C\sum_{i=-\infty}^{k}\int_{C_k}\left(\int_{C_i}\left|\prod_{j=1}^{m}\left(b_j(y)-(b_j)_{B_i}\right)\right|\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}}\left|f(y)\right|dy\right)^{q_2}dx$$
$$+C\sum_{i=-\infty}^{k}\int_{C_k}\left(\int_{C_i}\left|\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\left(\vec{b}(y)-\vec{b}_{B_i}\right)_\sigma(\vec{b}_{B_i}-\vec{b}_{B_k})_{\sigma'}\right|\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}}\left|f(y)\right|dy\right)^{q_2}dx$$
$$+C\sum_{i=-\infty}^{k}\int_{C_k}\left(\int_{C_i}\left|\prod_{j=1}^{m}\left((b_j)_{B_i}-(b_j)_{B_k}\right)\right|\frac{|\Phi(\frac{x}{|y|})|}{|y|^{n-\beta}}\left|f(y)\right|dy\right)^{q_2}dx$$
$$=III_1+III_2+III_3.$$

Then by Hölder's inequality, Lemma 1, and Lemma 3, we have

$$III_1\le C\sum_{i=-\infty}^{k}\int_{C_k}\left(|x|^{\beta-\frac{n}{r}}\int_{C_i}\left|\prod_{j=1}^{m}\left(b_j(y)-(b_j)_{B_i}\right)\right|^r\left|f(y)\right|^r dy\right)^{q_2}dx$$
$$\le C\sum_{i=-\infty}^{k}\int_{C_k}\left(2^{k\beta-\frac{kn}{r}}\left(\int_{C_i}\left|\prod_{j=1}^{m}\left(b_j(y)-(b_j)_{B_i}\right)\right|^{\frac{rq_1}{q_1-r}}dy\right)^{\frac{q_1-r}{rq_1}}\right.$$

$$\begin{aligned}
&\times \left(\left(\int_{C_i} |f(y)|^{q_1}\,dy \right)^{1/q_1} \right)^{q_2} dx \\
&\le C \prod_{j=1}^{m} \|b_j\|^{q_2}_{CBMO_{\frac{rq_1}{q_1-r}r_j}(\mathbb{R}^n)} \left(\sum_{i=-\infty}^{k} 2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^{q_2},
\end{aligned}$$

$$III_2 \le C \prod_{j=1}^{m} \|b_j\|^{q_2}_{CBMO_{\frac{rq_1}{q_1-r}r_j}(\mathbb{R}^n)} \left(\sum_{i=-\infty}^{k} \sum_{j=1}^{m-1} \sum_{\sigma\in\mathscr{C}_j^m} (k-i)^{|\sigma'|} 2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^{q_2}.$$

Also

$$III_3 \le C \prod_{j=1}^{m} \|b_j\|^{q_2}_{CBMO_{\frac{rq_1}{q_1-r}r_j}(\mathbb{R}^n)} \left(\sum_{i=-\infty}^{k} (k-i)^m 2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^{q_2}.$$

Then

$$\begin{aligned}
&\left\| H^1_{\Phi,\beta,\vec{b}} f \right\|_{\dot{K}^{\alpha,p}_{q_2}} \\
&\le C \prod_{j=1}^{m} \|b_j\|_{CBMO_{\max\{q_2 r_j, \frac{rq_1}{q_1-r}r_j\}}(\mathbb{R}^n)} \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{i=-\infty}^{k} 2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^p \right\}^{\frac{1}{p}} \\
&\quad + \prod_{j=1}^{m} \|b_j\|_{CBMO_{q_2 r_j}(\mathbb{R}^n)} \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{i=-\infty}^{k} (k-i)^m 2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^p \right\}^{\frac{1}{p}} \\
&\quad + \prod_{j=1}^{m} \|b_j\|_{CBMO_{\frac{rq_1}{q_1-r}r_j}(\mathbb{R}^n)} \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{i=-\infty}^{k} (k-i)^m 2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^p \right\}^{\frac{1}{p}} \\
&\le C \prod_{j=1}^{m} \|b_j\|_{CBMO_{\max\{q_2 r_j, \frac{rq_1}{q_1-r}r_j\}}(\mathbb{R}^n)} \\
&\quad \times \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{i=-\infty}^{k} (k-i)^m 2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})} \|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)} \right)^p \right\}^{\frac{1}{p}} \\
&:= L.
\end{aligned}$$

For $0 < p \le 1$, by the well-known inequality

$$\left(\sum_{i=1}^{\infty} |a_i| \right)^p \le \sum_{i=1}^{\infty} |a_i|^p.$$

We have the following:

$$\begin{aligned}
L &\le C \prod_{j=1}^{m} \|b_j\|_{CBMO_{\max\{q_2 r_j, \frac{rq_1}{q_1-r}r_j\}}(\mathbb{R}^n)} \\
&\quad \times \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \sum_{i=-\infty}^{k} (k-i)^{mp} 2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})p} \|f\chi_i\|^p_{L^{q_1}(\mathbb{R}^n)} \right\}^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\prod_{j=1}^{m}\|b_j\|_{CBMO_{\max\{q_2r_j,\frac{rq_1}{q_1-r}r_j\}}(\mathbb{R}^n)} \\
&\quad\times\left\{\sum_{i=-\infty}^{\infty}2^{i\alpha p}\sum_{k=i}^{\infty}(k-i)^{mp}2^{(i-k)(\frac{n}{r}-\frac{n}{q_1}-\alpha)p}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}^p\right\}^{\frac{1}{p}} \\
&\leq C\prod_{j=1}^{m}\|b_j\|_{CBMO_{\max\{q_2r_j,\frac{rq_1}{q_1-r}r_j\}}(\mathbb{R}^n)} \\
&\quad\times\left\{\sum_{i=-\infty}^{\infty}2^{i\alpha p}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}^p\sum_{k=i+2}^{\infty}(k-i)^{mp}2^{(k-i)(\frac{n}{q_1}+\alpha-\frac{n}{r})p}\right\}^{\frac{1}{p}} \\
&\leq C\prod_{j=1}^{m}\|b_j\|_{CBMO_{\max\{q_2r_j,\frac{rq_1}{q_1-r}r_j\}}(\mathbb{R}^n)}\|f\chi_i\|_{\dot{K}_{q_1}^{\alpha,p}(\mathbb{R}^n)}.
\end{aligned}
$$

For $1<p<\infty$, using Hölder's inequality, we get

$$
\begin{aligned}
L&\leq C\prod_{j=1}^{m}\|b_j\|_{CBMO_{\max\{q_2r_j,\frac{rq_1}{q_1-r}r_j\}}(\mathbb{R}^n)} \\
&\quad\times\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{i=-\infty}^{k-2}(k-i)^{m}2^{(k-i)(\frac{n}{q_1}+\alpha-\frac{n}{r})}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}\right)^p\right\}^{\frac{1}{p}} \\
&\leq C\prod_{j=1}^{m}\|b_j\|_{CBMO_{\max\{q_2r_j,\frac{rq_1}{q_1-r}r_j\}}(\mathbb{R}^n)} \\
&\quad\times\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{i=-\infty}^{k-2}2^{i\alpha p}(k-i)^{mp}2^{(k-i)(\frac{n}{q_1}+\alpha-\frac{n}{r})\frac{p}{2}}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}^p\right)\right. \\
&\quad\times\left.\left(\sum_{i=-\infty}^{k}2^{(k-i)(\frac{n}{q_1}+\alpha-\frac{n}{r})\frac{p'}{2}}\right)^{\frac{p}{p'}}\right\}^{\frac{1}{p}} \\
&\leq C\prod_{j=1}^{m}\|b_j\|_{CBMO_{\max\{q_2r_j,\frac{rq_1}{q_1-r}r_j\}}(\mathbb{R}^n)}\left\{\sum_{i=-\infty}^{\infty}2^{i\alpha p}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}^p\left(\sum_{k=i}^{\infty}2^{(k-i)(\alpha+\frac{n}{q_1}-\frac{n}{r})\frac{p'}{2}}\right)\right\}^{1/p} \\
&\leq C\prod_{j=1}^{m}\|b_j\|_{CBMO_{\max\{q_2r_j,\frac{rq_1}{q_1-r}r_j\}}(\mathbb{R}^n)}\|f\chi_i\|_{\dot{K}_{q_1}^{\alpha,p}(\mathbb{R}^n)}.
\end{aligned}
$$

This finishes the proof of (a). The proof of Theorem 4(b) is similar to that for (a). We omit the details. By combining the estimates of (a) and (b), we can easily get (c). □

Proof of Theorem 5 We only give the proof when $\lambda>0$. By the definition of Morrey-Herz spaces and the estimates for I, II, III in the proof of Theorem 4, we have

$$
\begin{aligned}
&\left\|H^1_{\Phi,\beta,\vec{b}}f\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q_2}} \\
&\quad=\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha p}\left\|\left(H^1_{\Phi,\beta,\vec{b}}f\right)\chi_k\right\|_{L^{q_2}(\mathbb{R}^n)}^p\right)^{1/p}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\prod_{j=1}^{m}\|b_j\|_{CBMO_{\max\{q_2r_j,\frac{rq_1}{q_1-r}r_j\}}(\mathbb{R}^n)}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\\
&\quad\times\left\{\sum_{k=-\infty}^{k_0}2^{k\alpha p}\left(\sum_{i=-\infty}^{k}(k-i)^m2^{(i-k)n(\frac{1}{r}-\frac{1}{q_1})}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\
&= C\prod_{j=1}^{m}\|b_j\|_{CBMO_{\max\{q_2r_j,\frac{rq_1}{q_1-r}r_j\}}(\mathbb{R}^n)}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\\
&\quad\times\left\{\sum_{k=-\infty}^{k_0}2^{k\alpha p}2^{-kn(\frac{1}{r}-\frac{1}{q_1})}\left(\sum_{i=-\infty}^{k}(k-i)^m2^{in(\frac{1}{r}-\frac{1}{q_1})}\|f\chi_i\|_{L^{q_1}(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\
&= K.
\end{aligned}
$$

For K, noting that $\alpha < \frac{n}{r} - \frac{n}{q_1} + \lambda$, we have

$$
\begin{aligned}
K &\leq C\prod_{j=1}^{m}\|b_j\|_{CBMO_{\max\{q_2r_j,\frac{rq_1}{q_1-r}r_j\}}(\mathbb{R}^n)}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\\
&\quad\times\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{i=-\infty}^{k}(k-i)^m2^{k\alpha}2^{-kn(\frac{1}{r}-\frac{1}{q_1})}2^{in(\frac{1}{r}-\frac{1}{q_1})}2^{-i\alpha}\right.\right.\\
&\quad\left.\left.\times 2^{i\lambda}2^{-i\lambda}\left(\sum_{l=-\infty}^{i}2^{l\alpha p}\|f\chi_l\|_{L^{q_1}(\mathbb{R}^n)}^p\right)^{1/p}\right)^p\right\}^{1/p}\\
&\leq C\prod_{j=1}^{m}\|b_j\|_{CBMO_{\max\{q_2r_j,\frac{rq_1}{q_1-r}r_j\}}(\mathbb{R}^n)}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\\
&\quad\times\left\{\sum_{k=-\infty}^{k_0}2^{pk\lambda}\left(\sum_{i=-\infty}^{k-2}(k-i)^m2^{(i-k)(-\alpha+\frac{n}{r}-\frac{n}{q_1}+\lambda)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q_1}(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\
&\leq C\prod_{j=1}^{m}\|b_j\|_{CBMO_{\max\{q_2r_j,\frac{rq_1}{q_1-r}r_j\}}(\mathbb{R}^n)}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{pk\lambda}\right)^{1/p}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q_1}(\mathbb{R}^n)}\\
&= C\prod_{j=1}^{m}\|b_j\|_{CBMO_{\max\{q_2r_j,\frac{rq_1}{q_1-r}r_j\}}(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q_1}(\mathbb{R}^n)}.
\end{aligned}
$$

This finishes the proof of (a). The proof of Theorem 5(b) is similar to that for (a). We omit the details. By combining the estimates of (a) and (b), we can easily get (c). □

4 Conclusions

This paper proves the boundedness of the generalized commutators of Hausdorff operators $H_{\Phi,\beta,A}$ and the multilinear commutators of Hausdorff operators $H_{\Phi,\beta,\vec{b}}$ with central BMO function, not only in Herz spaces, but also in Morrey-Herz spaces, which promotes some results of Hardy operators or the multilinear commutators of Hausdorff operators $H^m_{\Phi,\beta,b}$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
The authors worked jointly in drafting and approving the final manuscript.

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