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Convexity and concavity of the complete elliptic integrals with respect to Lehmer mean

Yu-Ming Chu[1*] and Tie-Hong Zhao[2]

*Correspondence: chuyuming2005@126.com
[1] School of Mathematics and Computation Sciences, Hunan City University, Yiyang, 413000, China
Full list of author information is available at the end of the article

Abstract

In this paper, the authors present necessary and sufficient conditions for the complete elliptic integrals of the first and second kind to be convex or concave with respect to the Lehmer mean.

MSC: 33C05; 26E60

Keywords: complete elliptic integral; generalized convexity; Lehmer mean

1 Introduction and main results

1.1 Legendre's complete elliptic integrals

For $r \in [0,1]$, Legendre's complete elliptic integrals of the first and second kind [1, 2] are defined by

$$\mathcal{K}(r) = \int_0^{\pi/2} \left(1 - r^2 \sin^2\theta\right)^{-1/2} d\theta \tag{1.1}$$

and

$$\mathcal{E}(r) = \int_0^{\pi/2} \left(1 - r^2 \sin^2\theta\right)^{1/2} d\theta, \tag{1.2}$$

respectively. Note that $\mathcal{K}(0) = \mathcal{E}(0) = \pi/2$ and $\mathcal{K}(1) = \infty$, $\mathcal{E}(1) = 1$.

It is well known that the complete elliptic integrals have many important applications in geometric function theory, theory of mean values, number theory, and many other areas of mathematics, as well as physics and engineering [3–9].

Because of the importance of the complete elliptic integrals, they have been studied extensively by many researchers from different points. The asymptotic behavior of $\mathcal{K}(r)$ near the singularity $r = 1$ has been explored by Kühnau [10], and Alzer and Qiu [11]. Many remarkable inequalities and monotonicity properties for $\mathcal{K}(r)$ and $\mathcal{E}(r)$ can be found in the literature [4, 12–23]. The generalization of the complete elliptic integrals was first introduced by Vuorinen *et al.* in [24], and subsequently they were studied intensively in [25–29].

1.2 Generalized convexity

In order to introduce the generalized convexity, we first recall the definition of mean function and several classical means.

Definition 1.1 ([14], Definition 2.1) A function $M:(0,\infty)\times(0,\infty)\to(0,\infty)$ is said to be a *mean function* if

(1) $\min(x,y)\le M(x,y)\le \max(x,y)$,
(2) $M(x,x)=x$,
(3) $M(x,y)=M(y,x)$,
(4) $M(ax,ay)=aM(x,y)$ for all $a>0$.

Example 1.2

(1) $A(x,y)=(x+y)/2$ is the arithmetic mean;
(2) $G(x,y)=\sqrt{xy}$ is the geometric mean;
(3) $H(x,y)=2xy/(x+y)$ is the harmonic mean;
(4) $H_p(x,y)=[(x^p+y^p)/2]^{1/p}$ if $p\neq 0$, and $H_0(x,y)=G(x,y)=\sqrt{xy}$ is the pth Hölder (power) mean;
(5) $L_p(x,y)=(x^{p+1}+y^{p+1})/(x^p+y^p)$ is the pth Lehmer mean.

Definition 1.3 ([14], Definition 2.3) Let I be a subinterval of $(0,\infty)$, $f:I\to(0,\infty)$ be a continuous function, and M and N be any two mean functions. Then f is said to be (M,N)-convex (concave) on I if

$$f\bigl(M(x,y)\bigr)\le(\ge)N\bigl(f(x),f(y)\bigr) \tag{1.3}$$

for all $x,y\in I$. Moreover, if the inequality (1.3) is strict except for $x=y$, then f is said to be strictly (M,N)-convex (concave) on I.

In particular, if both M and N are Hölder means, then (1.3) reduces to

$$f\bigl(H_p(x,y)\bigr)\le(\ge)H_q\bigl(f(x),f(y)\bigr), \tag{1.4}$$

and f is said to be $H_{p,q}$-convex (concave) on I. Also, if inequality (1.4) is strict except for $x=y$, then f is said to be strictly $H_{p,q}$-convex (concave) on I.

Recently, the generalized convexity or concavity has attracted the attention of many mathematicians [30–40]. Baricz [30] proved that $\mathcal{K}(r)$ is strictly $H_{p,p}$-convex on $(0,1)$ for $p\in(0,2]$. Zhang *et al.* [39] improved Baricz's result and proved that $\mathcal{K}(r)$ is strictly $H_{p,q}$-convex on $(0,1)$ for $(p,q)\in\{(p,q)|p\le 2, q\ge 0\}$. In [41], the authors presented the least value p_1 and greatest value p_2 such that $H_p(\mathcal{K}(x),\mathcal{K}(y))\ge\mathcal{K}(H_p(x,y))$ or $H_p(\mathcal{E}(x),\mathcal{E}(y))\le\mathcal{E}(H_p(x,y))$ for all $p\in[p_1,p_2]$ and $x,y\in(0,1)$. Very recently, the $H_{p,q}$-convexity and $H_{p,q}$-concavity of the complete elliptic integrals are discussed in [33, 38].

Theorem 1.4 ([38], Theorem 1.4) *Let*

$$C(q)=\inf_{r\in(0,1)}\bigl\{(q-1)\bigl(\mathcal{E}-r'^2\mathcal{K}\bigr)/\bigl(r'^2\mathcal{K}\bigr)+r^2\bigl(2\mathcal{E}-r'^2\mathcal{K}\bigr)/\bigl[r'^2\bigl(\mathcal{E}-r'^2\mathcal{K}\bigr)\bigr]\bigr\}$$

be a continuous function with $C(q)=2$ for all $q\ge -7/2$ and $C(q)<2$ for all $q<-7/2$. Then the complete elliptic integrals of the first kind $\mathcal{K}(r)$ is strictly $H_{p,q}$-convex if and only if

$$(p,q)\in D=\bigl\{(p,q)|p\le C(q)\bigr\},$$

and there are no values of p and q for which $\mathcal{K}(r)$ is $H_{p,q}$-concave on $(0,1)$.

Theorem 1.5 ([33], Theorem 1.5) *Let*

$$D(q) = \inf_{r\in(0,1)}\left\{r^2\mathcal{E}/\left[r'^2(\mathcal{K}-\mathcal{E})\right] + (1-q)(\mathcal{K}-\mathcal{E})/\mathcal{E}\right\}$$

be a continuous function with $D(q) = 2$ *for all* $q \le 5/2$ *and* $D(q) < 2$ *for all* $q > 5/2$. *Then the complete elliptic integrals of the second kind* $\mathcal{E}(r)$ *is strictly* $H_{p,q}$*-concave if and only if*

$$(p,q) \in D^* = \left\{(p,q)|p \le D(q)\right\},$$

and there are no values of p *and* q *for which* $\mathcal{E}(r)$ *is* $H_{p,q}$*-convex on* $(0,1)$.

The main purpose of this short note is to establish the necessary and sufficient conditions for the convexity or concavity of the complete elliptic integrals of the first and second kind with respect to the Lehmer mean. Our main results are as follows.

Theorem 1.6 *The complete elliptic integral of the first kind* $\mathcal{K}(r)$ *is strictly* $L_{\lambda,\lambda}$*-convex on* $(0,1)$ *if and only if* $\lambda \in [-1,0]$.

Theorem 1.7 *The complete elliptic integral of the second kind* $\mathcal{E}(r)$ *is strictly* $L_{\lambda,\lambda}$*-concave on* $(0,1)$ *if and only if* $\lambda \in (-\infty,0]$.

2 A lemma

In order to prove our main results we need a lemma, which we present in this section.

Lemma 2.1 (see [42]) *The inequality*

$$H_{2t+1}(x,y) \le L_t(x,y) \tag{2.1}$$

holds for all $x,y \in \mathbb{R}^+$ *if* $t \in (-1,-1/2) \cup (0,+\infty)$, *and the inequality*

$$H_{2t+1}(x,y) \ge L_t(x,y) \tag{2.2}$$

holds for all $x,y \in \mathbb{R}^+$ *if* $t \in (-\infty,-1) \cup (-1/2,0)$. *Inequality* (2.1) *or* (2.2) *becomes an equality for all* $x,y \in \mathbb{R}^+$ *if* $t = -1$, $-1/2$, *or* 0; *otherwise inequality* (2.1) *or* (2.2) *becomes an equality only when* $x = y$. *Moreover,* $H_{2t+1}(x,y)$ *is the best possible lower* (*or upper*) *Hölder mean bound for* $L_t(x,y)$ *in* (2.1) (*or* (2.2)).

3 Proofs of main results

Proof of Theorem 1.6 Since

$$L_0(x,y) = A(x,y) = H_1(x,y), \qquad L_{-1/2}(x,y) = G(x,y) = H_0(x,y)$$

and

$$L_{-1}(x,y) = H(x,y) = H_{-1}(x,y).$$

Therefore, Theorem 1.4 and (1.4) show that $\{(p,q)|(-1,-1),(0,0),(1,1)\} \subset D$, namely, for $\lambda = 0$, $-1/2$ or -1, the inequality

$$\mathcal{K}\big(L_\lambda(x,y)\big) < L_\lambda\big(\mathcal{K}(x),\mathcal{K}(y)\big)$$

holds for all $x,y \in (0,1)$ with $x \neq y$. Thus $\mathcal{K}(r)$ is strictly $L_{\lambda,\lambda}$-convex on $(0,1)$ for $\lambda = 0$, $-1/2$ and -1.

Next, we divide the proof into four cases.

Case 1. $-1/2 < \lambda < 0$. Then it follows from Theorem 1.4 and (1.4) that $(2\lambda + 1, 0) \in D$. Making use of Lemma 2.1 together with the monotonicity of $\mathcal{K}(r)$ on $(0,1)$, one has

$$\mathcal{K}\big(L_\lambda(x,y)\big) < \mathcal{K}\big(H_{2\lambda+1}(x,y)\big) < H_0\big(\mathcal{K}(x),\mathcal{K}(y)\big) < L_\lambda\big(\mathcal{K}(x),\mathcal{K}(y)\big)$$

for all $x,y \in (0,1)$ with $x \neq y$.

Therefore, $\mathcal{K}(r)$ is strictly $L_{\lambda,\lambda}$-convex on $(0,1)$ for $-1/2 < \lambda < 0$.

Case 2. $-1 < \lambda < -1/2$. Similarly, by Theorem 1.4 and Lemma 2.1 we have $(0, 2\lambda + 1) \in D$ and

$$\mathcal{K}\big(L_\lambda(x,y)\big) < \mathcal{K}\big(H_0(x,y)\big) < H_{2\lambda+1}\big(\mathcal{K}(x),\mathcal{K}(y)\big) < L_\lambda\big(\mathcal{K}(x),\mathcal{K}(y)\big)$$

for all $x,y \in (0,1)$ with $x \neq y$. This implies that $\mathcal{K}(r)$ is strictly $L_{\lambda,\lambda}$-convex on $(0,1)$ for $-1 < \lambda < -1/2$.

Case 3. $\lambda > 0$. For any $0 < y < 1$, letting $x \to 0^+$, then we have

$$\begin{aligned}\mathcal{K}\big(L_\lambda(x,y)\big) - L_\lambda\big(\mathcal{K}(x),\mathcal{K}(y)\big) &= \mathcal{K}\left(\frac{x^{\lambda+1}+y^{\lambda+1}}{x^\lambda+y^\lambda}\right) - \frac{\mathcal{K}^{\lambda+1}(x)+\mathcal{K}^{\lambda+1}(y)}{\mathcal{K}^\lambda(x)+\mathcal{K}^\lambda(y)}\\ &\to \frac{(\pi/2)^\lambda[\mathcal{K}(y)-\pi/2]}{(\pi/2)^\lambda+\mathcal{K}^\lambda(y)} > 0.\end{aligned} \tag{3.1}$$

It follows from (3.1) that there exists $x_0 = x(y) \in (0,1)$ such that $\mathcal{K}(L_\lambda(x,y)) > L_\lambda(\mathcal{K}(x), \mathcal{K}(y))$ for all $x \in (0,x_0)$. Thus $\mathcal{K}(r)$ is not $L_{\lambda,\lambda}$-convex on $(0,1)$ for $\lambda > 0$.

Case 4. $\lambda < -1$. For $0 < x < 1$, letting $y \to 1^-$, then one has

$$\mathcal{K}\big(L_\lambda(x,y)\big) - L_\lambda\big(\mathcal{K}(x),\mathcal{K}(y)\big) \to \mathcal{K}\left(\frac{1+x^{\lambda+1}}{1+x^\lambda}\right) - \mathcal{K}(x) > 0,$$

where we unitize $(1 + x^{\lambda+1})/(1 + x^\lambda) > x$.

Making use of the analogous arguments in Case 3 we conclude that $\mathcal{K}(r)$ is not $L_{\lambda,\lambda}$-convex on $(0,1)$ for $\lambda < -1$. □

Proof of Theorem 1.7 Clearly, by Theorem 1.5 we know that

$$\big\{(p,q)|(-1,-1),(0,0),(1,1)\big\} \subset D^*.$$

Thus, $\mathcal{E}(r)$ is strictly $L_{\lambda,\lambda}$-concave on $(0,1)$ for $\lambda = 0$, $-1/2$ or -1.

Next, we divide the proof into three cases.

Case I. $-1/2 < \lambda < 0$ or $\lambda < -1$. Then it is easy to check that $(2\lambda + 1, 2\lambda + 1) \in D^*$. Thus Theorem 1.5 and Lemma 2.1 together with the monotonicity of $\mathcal{E}(r)$ lead to the conclusion that

$$\mathcal{E}\big(L_\lambda(x,y)\big) > \mathcal{E}\big(H_{2\lambda+1}(x,y)\big) > H_{2\lambda+1}\big(\mathcal{E}(x),\mathcal{E}(y)\big) > L_\lambda\big(\mathcal{E}(x),\mathcal{E}(y)\big)$$

for all $x, y \in (0,1)$ with $x \neq y$.

Therefore, $\mathcal{E}(r)$ is strictly $L_{\lambda,\lambda}$-concave on $(0,1)$ for $-1/2 < \lambda < 0$ or $\lambda < -1$.

Case II. $-1 < \lambda < -1/2$. Then by Theorem 1.5 and Lemma 2.1 we get

$$\mathcal{E}\big(L_\lambda(x,y)\big) > \mathcal{E}\big(H_0(x,y)\big) > H_0\big(\mathcal{E}(x),\mathcal{E}(y)\big) > L_\lambda\big(\mathcal{E}(x),\mathcal{E}(y)\big)$$

for all $x, y \in (0,1)$ with $x \neq y$.

Therefore, $\mathcal{E}(r)$ is strictly $L_{\lambda,\lambda}$-concave on $(0,1)$ for $-1 < \lambda < -1/2$.

Case III. $\lambda > 0$. For any $0 < y < 1$, letting $x \to 0^+$, then we have

$$\begin{aligned}\mathcal{E}\big(L_\lambda(x,y)\big) - L_\lambda\big(\mathcal{E}(x),\mathcal{E}(y)\big) &= \mathcal{E}\left(\frac{x^{\lambda+1}+y^{\lambda+1}}{x^\lambda+y^\lambda}\right) - \frac{\mathcal{E}^{\lambda+1}(x)+\mathcal{E}^{\lambda+1}(y)}{\mathcal{E}^\lambda(x)+\mathcal{E}^\lambda(y)} \\ &\to \frac{(\pi/2)^\lambda[\mathcal{E}(y)-\pi/2]}{(\pi/2)^\lambda+\mathcal{E}^\lambda(y)} < 0. \end{aligned} \tag{3.2}$$

It follows from (3.2) that there exists $x_1 = x(y) \in (0,1)$ such that $\mathcal{E}(L_\lambda(x,y)) < L_\lambda(\mathcal{E}(x),\mathcal{E}(y))$ for all $x \in (0,x_1)$. Thus $\mathcal{E}(r)$ is not $L_{\lambda,\lambda}$-concave on $(0,1)$ for $\lambda > 0$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Computation Sciences, Hunan City University, Yiyang, 413000, China. [2]Department of Mathematics, Hangzhou Normal University, Hangzhou, 310036, China.

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