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Stability for iterative roots of piecewise monotonic functions

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Abstract
Global C^0 and local C^1 stability of iterative roots for monotonic functions defined on a compact interval, as well as global C^1 instability under some assumptions, are well-known facts. In this paper, we investigate the stability of iterative roots for piecewise monotonic functions with nonmonotonicity height equal to 1. We prove the roots are C^1 locally stable and C^0 global stable with the same extension.

MSC: 39B22; 37E05

Keywords: iterative root; stability; nonmonotonicity height; extension

1 Introduction

Given a Banach space X and $r \geq 0$, $C^r(X)$ is defined as the set of all C^r self-mappings on X. An iterative root of order $k \in \mathbb{N}$ of $F \in C^r(X)$ is a function $f \in C^r(X)$ that satisfies

$$f^k(x) = F(x), \quad \forall x \in X, \tag{1.1}$$

where f^k denotes the kth iterate of f. Being an important problem, iterative roots is connected to the research of embedding flow and topological conjugacy in dynamical systems [1, 2], which is also involved in the study of functional equations [3, 4]. To find solutions of equation (1.1) has a long history since 200 years ago [5–9]. In addition to monotonic mappings [6, 7], plenty results were obtained for the iterative roots of piecewise monotonic functions [8–12].

Let $I := [a, b]$ be an interval. A point $x_0 \in (a, b)$ is referred as a *fort* of continuous mapping $F : I \to I$ when F is strictly monotonic in no neighborhood of x_0. Let $S(F)$ be the set of all forts of F. Then F is called a *piecewise monotonic function* if $N(F) := \#S(f)$ is finite. The set of all such piecewise monotonic self-mappings on I is denoted by $\mathrm{PM}(I, I)$. It is well known that $N(F)$ is nondecreasing under iteration, we define the *nonmonotonicity height* $H(F)$ of F as the smallest integer m such that $N(F^m) = N(F^{m+1})$. When $H(F) = 1$, the problem of iterative roots is reduced to be discussed on the *characteristic interval* (see [8–11]), denoted by $K(F)$. Furthermore, for every continuous iterative root f of F of order $k \geq 2$ there exists a corresponding natural number $\ell(f)$, which maps I into $K(F)$. Such f is called a root of *ℓ-extension* (see [11]). The following result gives the iterative roots of those functions with 1-extension.

Lemma 1.1 (Theorem 3 in [9]) *Suppose $F \in \mathrm{PM}(I,I)$ and $H(F) = 1$. Let $K(F)$ be the characteristic interval of F, $[m,M]$ be the range of F and $[m',M']$ be those of F restricted to $K(F)$. If equation* (1.1) *has a continuous solution $f_0 : K(F) \to K(F)$ maps $[m,M]$ into $[m',M']$, then there exists a continuous function*

$$f(x) := \begin{cases} f_0(x), & x \in K(F), \\ F|_{K(F)}^{-1} \circ f_0 \circ F(x), & x \in I \backslash K(F), \end{cases} \tag{1.2}$$

satisfies $f^k(x) = F(x)$ for all $x \in I$.

Clearly, f defined in (1.2) is a root of 1-extension and $K(f) = K(F)$. Conversely, it is easy to prove that all continuous iterative roots of F of order k with 1-extension are in the form of (1.2).

In addition to the study of the existence of iterative roots, more and more attention was paid to their stability. A local result as regards C^0 stability for a class of strictly monotonic mappings with one fixed point was considered in [13], as well as the global C^0 stability with more than one fixed point in [14]. Recently, the authors of [15] investigated the C^1 stability of iterative roots for increasing functions defined on a compact interval. They proved that those iterative roots are C^1 locally stable but C^1 globally unstable.

In this paper, we consider the stability of iterative roots for piecewise monotonic functions with nonmonotonicity height equal to 1. We prove that those roots with the same extension are locally C^1 stable and globally C^0 stable.

2 C^1 stability

Let $F \in \mathrm{PM}(I,I)$ with $H(F) = 1$ and $K(F) := [a',b'] \subset I$ be its characteristic interval. For each $\lambda \in (0,1)$, let

$$\mathcal{H}_-^2(\lambda) := \{h \in C^2(I) : h(a') = a', h'(a') = \lambda, h'(x) > 0 \text{ and } h(x) < x, \forall x \in (a',b']\},$$
$$\mathcal{H}_+^2(\lambda) := \{h \in C^2(I) : h(b') = b', h'(b') = \lambda, h'(x) > 0 \text{ and } h(x) > x, \forall x \in [a',b')\}.$$

For a given integer $k \geq 2$, as discussed in [7], each function F in class $\bigcup_{\lambda \in (0,1)} \mathcal{H}_-^2(\lambda)$ has a kth order C^1 iterative root f on $K(F)$, *i.e.*, $f^k(x) = F(x)$ for all $x \in K(F)$, which is unique and strictly increasing.

Let the norm $\|\cdot\|_r$ be defined by

$$\|F\|_r := \sup_{x\in I}|F(x)| + \cdots + \sup_{x\in I}|F^r(x)|$$

for all $r \in \mathbb{N} \cup \{0\}$ and $F \in C^r(I)$. Based on the determined formula of f, C^1 local stability and C^1 global instability for f were investigated in [15]. The following result shows the stability for those roots.

Lemma 2.1 (Theorem 2.1 in [15]) *Let $F \in \mathcal{H}_-^2(\lambda)$ (or $\mathcal{H}_+^2(\lambda)$) with a given $\lambda \in (0,1)$ and let (F_m) be a sequence of functions in $\mathcal{H}_-^2(\lambda)$ (or $\mathcal{H}_+^2(\lambda)$). If*

$$\lim_{m\to\infty} \|F_m - F\|_2 = 0,$$

then

$$\lim_{m\to\infty} \|\tilde{f}_m - \tilde{f}\|_1 = 0, \tag{2.1}$$

where $\tilde{f}$ *and* $\tilde{f}_m$ *are unique kth order* C^1 *iterative roots of* F *and* F_m*, respectively, defined on* $K(F)$.

Note that a similar result also holds for $F \in \bigcup_{\lambda\in(0,1)} \mathcal{H}_+^2(\lambda)$.

Let

$$\mathcal{PM}_1(I,\lambda)_- := \mathcal{PM}_1(I) \cap \mathcal{H}_-^2(\lambda), \qquad \mathcal{PM}_1(I,\lambda)_+ := \mathcal{PM}_1(I) \cap \mathcal{H}_+^2(\lambda)$$

(see Figures 1 and 2), where

$$\mathcal{PM}_1(I) = \{F \in \mathrm{PM}(I,I) : H(F) = 1 \text{ and } K(F) = [a',b']\}.$$

We first recall some known results. Let $S(F) := \{d_1, d_2, \dots, d_{N(F)}\}$, satisfying $a = d_0 < d_1 < \cdots < d_{N(F)} < d_{N(F)+1} = b$. Furthermore, let $I_i := [d_i, d_{i+1}]$ denotes the closure of ith subinterval and $F_i := F|_{I_i}$. Then $I = \bigcup_{i=0}^{N(F)} I_i$ and F is strictly monotone on I_i.

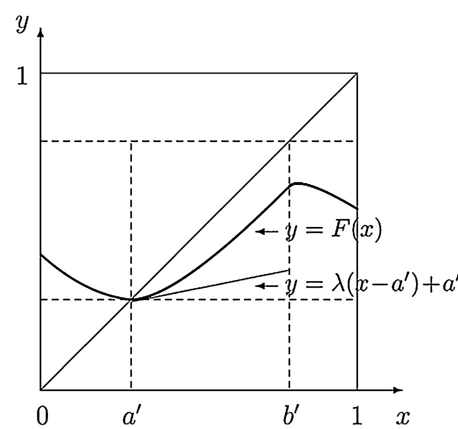

Figure 1 $F \in \mathcal{PM}_1(I,\lambda)_-$.

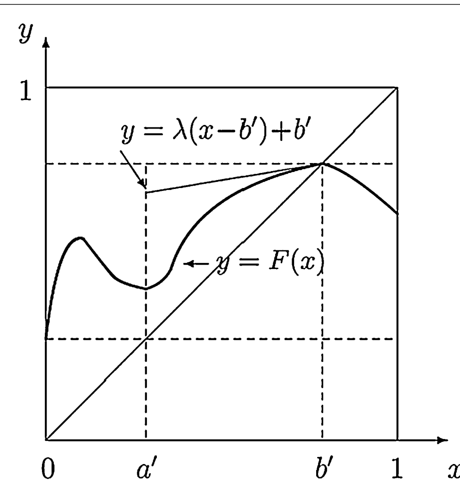

Figure 2 $F \in \mathcal{PM}_1(I,\lambda)_+$.

As shown in [11], for each I_i there exists a sequence $(i_1, i_2, \dots, i_{\tau-1})$ in

$$\{0,1,\dots,N(F)\}^{\tau-1} := \underbrace{\{0,1,\dots,N(F)\} \times \cdots \times \{0,1,\dots,N(F)\}}_{\tau-1},$$

where $\tau \le \min\{k, N(F)\}$ such that

$$I_i \xrightarrow{f} I_{i_1} \xrightarrow{f} \cdots \xrightarrow{f} I_{i_{\tau-1}} \xrightarrow{f} K(F), \tag{2.2}$$

where $I \xrightarrow{f} J$ denotes $f(I) \subset J$. Then (2.2) gives a correspondence $\tau_f : \{0,1,\dots,N(F)\} \to \{1,\dots,\min\{k,N(F)\}\}$ as $i \mapsto \tau$, which is the number that maps I_i into the characteristic interval $K(F)$. In [11], the natural number

$$\ell(f) := \max_{i \in \{0,1,\dots,N(F)\}} \tau_f(i)$$

is referred to as the *pace* of the iterative root f.

The following theorem is our main result in this section.

Theorem 2.1 *Let $F \in \mathcal{PM}_1(I,\lambda)_-$ (or $\mathcal{PM}_1(I,\lambda)_+$) with some $\lambda \in (0,1)$ and let (F_m) be a sequence of functions in $\mathcal{PM}_1(I,\lambda)_-$ (or $\mathcal{PM}_1(I,\lambda)_+$). If*

$$\lim_{m\to\infty} \|F_m - F\|_2 = 0, \tag{2.3}$$

then

$$\lim_{m\to\infty} \|f_m - f\|_1 = 0,$$

where f and f_m are kth order C^1 iterative roots of F and F_m with 1-extension, respectively.

Proof Let $\tilde{F} := F|_{K(F)}$ and $\tilde{F}_m := F_m|_{K(F)}$, for convenience. It suffices to discuss the case that $F \in \mathcal{PM}_1(I,\lambda)_-$ since the proof for the case $F \in \mathcal{PM}_1(I,\lambda)_+$ is similar. We first prove the existence of those iterative roots of $F, F_m \in \mathcal{PM}_1(I,\lambda)_-$ with 1-extension on I. As mentioned before, each function F and F_m has a kth order C^1 iterative root $\tilde{f}$ and $\tilde{f}_m$ on their characteristic interval, respectively. Then $\tilde{f}(a') = \tilde{f}_m(a') = a'$ since a' is a common fixed point of F and F_m. Furthermore, by the definition of $\mathcal{H}^2_-(\lambda)$, we have $\tilde{f}(F([a,b])) \subset [F(a'), F(b')]$, which implies that $\tilde{f}$ satisfies the conditions in Lemma 1.1. Hence, $\tilde{f}$ can be extended as a C^1 iterative root f of F on the whole interval I. Similarly, for each $m \in \mathbb{N}$ there exists a continuous iterative root f_m of F_m, which can be presented by

$$f_m(x) := \begin{cases} \tilde{f}_m(x), & x \in K(F), \\ \tilde{F}_m^{-1} \circ \tilde{f}_m \circ F_m(x), & x \in I \backslash K(F), \end{cases} \tag{2.4}$$

where $\tilde{f}_m^k(x) = \tilde{F}_m(x)$ for all $x \in K(F)$. Then it follows from Lemma 2.1 that

$$\lim_{m\to\infty} \|\tilde{f}_m - \tilde{f}\|_1 = 0. \tag{2.5}$$

Next, we turn to prove the convergence of (f_m) in $I\backslash K(F)$, by (1.2) and (2.4) we have

$$\begin{aligned}
\left|f_m'(x)-f'(x)\right| &= \left|\frac{\tilde{f}_m'(F_m(x))F_m'(x)}{\tilde{F}_m'(\tilde{F}_m^{-1}(\tilde{f}_m\circ F_m(x)))}-\frac{\tilde{f}'(F(x))F'(x)}{\tilde{F}'(\tilde{F}^{-1}(\tilde{f}\circ F(x)))}\right| \\
&\leq \left|\frac{\tilde{f}_m'(F_m(x))F_m'(x)}{\tilde{F}_m'(\tilde{F}_m^{-1}(\tilde{f}_m\circ F_m(x)))}-\frac{\tilde{f}_m'(F_m(x))F_m'(x)}{\tilde{F}'(\tilde{F}^{-1}(\tilde{f}\circ F(x)))}\right| \\
&\quad+\left|\frac{\tilde{f}_m'(F_m(x))F_m'(x)}{\tilde{F}'(\tilde{F}^{-1}(\tilde{f}\circ F(x)))}-\frac{\tilde{f}'(F(x))F'(x)}{\tilde{F}'(\tilde{F}^{-1}(\tilde{f}\circ F(x)))}\right| \\
&\leq \left|\frac{\tilde{f}_m'(F_m(x))F_m'(x)}{\tilde{F}_m'(\tilde{F}_m^{-1}(\tilde{f}_m\circ F_m(x)))\tilde{F}'(\tilde{F}^{-1}(\tilde{f}\circ F(x)))}\right|A(x) \\
&\quad+\left|\frac{1}{\tilde{F}'(\tilde{F}^{-1}(\tilde{f}\circ F(x)))}\right|B(x) \qquad (2.6)
\end{aligned}$$

for every $x\in I\backslash K(F)$, where

$$A(x)=\left|\tilde{F}_m'\big(\tilde{F}_m^{-1}\big(\tilde{f}_m\circ F_m(x)\big)\big)-\tilde{F}'\big(\tilde{F}^{-1}\big(\tilde{f}\circ F(x)\big)\big)\right|$$

and

$$B(x)=\left|\tilde{f}_m'\big(F_m(x)\big)F_m'(x)-\tilde{f}'\big(F(x)\big)F'(x)\right|.$$

In the following, we will estimate the limit of $A(x)$ and $B(x)$. Since

$$\begin{aligned}
A(x) &\leq \left|\tilde{F}_m'\big(\tilde{F}_m^{-1}\big(\tilde{f}_m\circ F_m(x)\big)\big)-\tilde{F}'\big(\tilde{F}_m^{-1}\big(\tilde{f}_m\circ F_m(x)\big)\big)\right| \\
&\quad+\left|\tilde{F}'\big(\tilde{F}_m^{-1}\big(\tilde{f}_m\circ F_m(x)\big)\big)-\tilde{F}'\big(\tilde{F}^{-1}\big(\tilde{f}\circ F(x)\big)\big)\right| \\
&\leq \|F_m-F\|_2+\left|\tilde{F}'\big(\tilde{F}_m^{-1}\big(\tilde{f}_m\circ F_m(x)\big)\big)-\tilde{F}'\big(\tilde{F}_m^{-1}\big(\tilde{f}\circ F_m(x)\big)\big)\right| \\
&\quad+\left|\tilde{F}'\big(\tilde{F}_m^{-1}\big(\tilde{f}\circ F_m(x)\big)\big)-\tilde{F}'\big(\tilde{F}^{-1}\big(\tilde{f}\circ F_m(x)\big)\big)\right| \\
&\quad+\left|\tilde{F}'\big(\tilde{F}^{-1}\big(\tilde{f}\circ F_m(x)\big)\big)-\tilde{F}'\big(\tilde{F}^{-1}\big(\tilde{f}\circ F(x)\big)\big)\right| \\
&\leq \|F_m-F\|_2+\left|\tilde{F}'\big(\tilde{F}_m^{-1}\big(\tilde{f}_m\circ F_m(x)\big)\big)-\tilde{F}'\big(\tilde{F}_m^{-1}\big(\tilde{f}\circ F_m(x)\big)\big)\right| \\
&\quad+\left|\tilde{F}'\circ\tilde{F}^{-1}\circ\tilde{F}\circ\tilde{F}_m^{-1}\big(\tilde{f}\circ F_m(x)\big)-\tilde{F}'\circ\tilde{F}^{-1}\circ\tilde{F}_m\circ\tilde{F}_m^{-1}\big(\tilde{f}\circ F_m(x)\big)\right| \\
&\quad+\left|\tilde{F}'\big(\tilde{F}^{-1}\big(\tilde{f}\circ F_m(x)\big)\big)-\tilde{F}'\big(\tilde{F}^{-1}\big(\tilde{f}\circ F(x)\big)\big)\right|
\end{aligned}$$

by (2.3) and the facts that $\tilde{F}'\circ\tilde{F}_m^{-1}$ and $\tilde{F}'\circ\tilde{F}^{-1}$ are uniformly continuous, we have $A(x)\to 0$ uniformly in I as $m\to\infty$.

Moreover, by the definition of $B(x)$, we obtain

$$\begin{aligned}
B(x) &\leq \left|\tilde{f}_m'\big(F_m(x)\big)F_m'(x)-\tilde{f}_m'\big(F_m(x)\big)F'(x)\right|+\left|\tilde{f}_m'\big(F_m(x)\big)F'(x)-\tilde{f}'\big(F(x)\big)F'(x)\right| \\
&\leq \left|\tilde{f}_m'\big(F_m(x)\big)\right|\left|F_m'(x)-F'(x)\right|+\left|F'(x)\right|\left|\tilde{f}_m'\big(F_m(x)\big)-\tilde{f}'\big(F(x)\big)\right| \\
&\leq \left|\tilde{f}_m'\big(F_m(x)\big)\right|\|F_m-F\|_2 \\
&\quad+\left|F'(x)\right|\big(\left|\tilde{f}_m'\big(F_m(x)\big)-\tilde{f}'\big(F_m(x)\big)\right|+\left|\tilde{f}'\big(F_m(x)\big)-\tilde{f}'\big(F(x)\big)\right|\big) \\
&\leq \left|\tilde{f}_m'\big(F_m(x)\big)\right|\|F_m-F\|_2+\left|F'(x)\right|\big(\|\tilde{f}_m-\tilde{f}\|_1+\left|\tilde{f}'\big(F_m(x)\big)-\tilde{f}'\big(F(x)\big)\right|\big).
\end{aligned}$$

Notice $\tilde{f}$ and $\tilde{f}_m$ are C^1 differentiable. Then it follows from (2.1)-(2.3) that $B(x) \to 0$ uniformly in I when $m \to \infty$.

On the other side, note that $\tilde{F}'(x), \tilde{F}'_m(x) > 0$ for all $x \in K(F)$, which implies

$$0 < \sup_{x\in I}\left|\frac{1}{\tilde{F}'(\tilde{F}^{-1}(\tilde{f}(F(x))))}\right| < \infty$$

and

$$0 \le \sup_{x\in I}\left|\frac{\tilde{f}'_m(F_m(x))F'_m(x)}{\tilde{F}'_m(\tilde{F}_m^{-1}(\tilde{f}_m(F_m(x))))\tilde{F}'(\tilde{F}^{-1}(\tilde{f}(F(x))))}\right| < \infty.$$

Therefore, in view of (2.6), it gives

$$\lim_{m\to\infty}\left\|f'_m - f'\right\|_0 = 0.$$

We conclude $\lim_{m\to\infty}\|f_m - f\|_0 = 0$. The proof of Theorem 2.1 is completed. □

Theorem 2.1 shows the C^1 stability of iterative roots with 1-extension since the form of roots is determined uniquely by (1.2). Conversely, we conclude the C^1 instability for those roots with different extensions. Moreover, according to the construction of iterative roots with large extensions (see [11]), we find that the mode of the roots is not unique, which leads to the C^1 instability for those iterative roots in different modes. Similar to the proof for the 1-extension in Theorem 2.1, we have the following result for larger extensions.

Theorem 2.2 *Let $F \in \mathcal{PM}_1(I,\lambda)_-$ (or $\mathcal{PM}_1(I,\lambda)_+$) with some $\lambda \in (0,1)$ and let (F_m) be a sequence of functions in $\mathcal{PM}_1(I,\lambda)_-$ (or $\mathcal{PM}_1(I,\lambda)_+$). If*

$$\lim_{m\to\infty}\|F_m - F\|_2 = 0,$$

and F, F_m has a kth order C^1 iterative root f and f_m with the same mode of extension, then

$$\lim_{m\to\infty}\|f_m - f\|_1 = 0.$$

3 Hyers-Ulam stability

In this section we prove the Hyers-Ulam stability of equation (1.1).

Suppose $F \in \mathrm{PM}(I,I)$ and I_i is an open interval between two consecutive forts (or endpoints) of F. Recall $I = \bigcup_{i=0}^{N(F)} \mathrm{cl}(I_i)$ and we let $I(F) := \{I_i : i = 0,1,\ldots,N(F)\}$.

Theorem 3.1 *Let $F \in \mathrm{PM}(I,I)$ with $H(F) = 1$ be given. If the function $f_s \in \mathrm{PM}(I,I)$ is Lipschitzian with constants $m > 0$, $M > 0$ such that*

$$m|x-y| \le \left|f_s(x) - f_s(y)\right| \le M|x-y| \tag{3.1}$$

for every $x, y \in K(F)$, and satisfies:

(A1) $H(f_s) = 1$ *and* $K(f_s) = K(F)$;
(A2) $f_s^k(x) = F(x)$ *for all* $x \in K(F)$, *and* f_s *maps* $K(F)$ *onto itself homeomorphically*;
(A3) $\|f_s^k - F\|_0 \le \delta$ *for a constant* $\delta > 0$,

then equation (1.1) *has a solution* $f \in \mathrm{PM}(I,I)$ *such that*

$$\|f_s - f\|_0 \le \frac{1+M}{m^k}\delta. \tag{3.2}$$

Proof This proof is based on the construction of iterative roots of $F \in \mathrm{PM}(I,I)$ with $H(F) = 1$. Let

$$F_s(x) := f_s^k(x), \quad \forall x \in I. \tag{3.3}$$

It follows from (A1) that $H(F_s) = 1$ and $K(F)$ is also the characteristic interval of F_s by the iterating mode of f_s. Thus, from the proof of Theorem 1.1 in [11] and (1.2), each kth order continuous iterative root f_s of F_s is extended from that on $K(F)$ by the following formula:

$$f_s(x) = F_s|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F_s|_{I_i}(x), \quad \forall x \in I_i \in I(F) \backslash \{K(F)\}. \tag{3.4}$$

Hence, the desired function $f \in \mathrm{PM}(I,I)$ can be defined by

$$f(x) := \begin{cases} f_s(x), & x \in K(F), \\ F|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F|_{I_i}(x), & x \in I_i \in I(F)\backslash\{K(F)\}. \end{cases} \tag{3.5}$$

Since F_s maps $K(F)$ onto itself homeomorphically by (A2), it means that $F_s|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F|_{I_i}$ is well defined. Since $f_s^k(x) = F(x)$ for all $x \in K(F)$, one can check that f defined in (3.5) is a solution of equation (1.1).

In order to prove (3.2), it suffices to prove

$$\|f_s - f\|_0 \le \frac{1+M}{m^k}\delta, \quad \forall x \in I_i \in I(F). \tag{3.6}$$

Obviously, (3.6) holds for every $x \in K(F)$. We next claim that, for every $x \in I_i$,

(K1) $\left|F_s|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F|_{I_i}(x) - F_s|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F_s|_{I_i}(x)\right| \le \dfrac{M}{m^k}\delta,$

(K2) $\left|F|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F|_{I_i}(x) - F_s|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F|_{I_i}(x)\right| \le \dfrac{1}{m^k}\delta.$

Actually, it follows from (3.1), (A3), and (3.3) that, for every $x \in I_i$,

$$\left|f_s|_{K(F)} \circ F|_{I_i}(x) - f_s|_{K(F)} \circ F_s|_{I_i}(x)\right| \le M\left|F|_{I_i}(x) - F_s|_{I_i}(x)\right| \le M\delta$$

and

$$\begin{aligned} &\left|f_s|_{K(F)} \circ F|_{I_i}(x) - f_s|_{K(F)} \circ F_s|_{I_i}(x)\right| \\ &\quad = \left|F_s \circ F_s|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F|_{I_i}(x) - F_s \circ F_s|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F_s|_{I_i}(x)\right| \\ &\quad \ge m^k\left|F_s|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F|_{I_i}(x) - F_s|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F_s|_{I_i}(x)\right|, \end{aligned}$$

which gives (K1).

On the other hand, in view of (3.1) and (A3) we have

$$\begin{aligned}\delta &\geq \left|F_s \circ F|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F|_{I_i}(x) - F \circ F|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F|_{I_i}(x)\right| \\ &= \left|F_s \circ F|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F|_{I_i}(x) - F_s \circ F_s|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F|_{I_i}(x)\right| \\ &\geq m^k \left|F|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F|_{I_i}(x) - F_s|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F|_{I_i}(x)\right|\end{aligned}$$

for every $x \in I$ and thus (K2) is proved.

Therefore, consider every $x \in I_i$, it follows from (K1) and (K2) that

$$\begin{aligned}\left|f_s(x) - f(x)\right| &= \left|F|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F|_{I_i}(x) - F_s|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F_s|_{I_i}(x)\right| \\ &\leq \left|F|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F|_{I_i}(x) - F_s|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F|_{I_i}(x)\right| \\ &\quad + \left|F_s|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F|_{I_i}(x) - F_s|_{K(F)}^{-1} \circ f_s|_{K(F)} \circ F_s|_{I_i}(x)\right| \\ &\leq \frac{1+M}{m^k}\delta.\end{aligned}$$

Thus (3.2) is proved. The proof of Theorem 3.1 is completed. □

Theorem 3.2 *Let $F \in \mathrm{PM}(I,I)$ with $H(F) = 1$ be given. If the function $f_s \in \mathrm{PM}(I,I)$ is Lipschitzian with constants $m > 0$, $M > 0$ such that*

$$m|x-y| \leq \left|f_s(x) - f_s(y)\right| \leq M|x-y|$$

for every $x, y \in K(F)$, and satisfies:

(A1) *for each $I_i \in I(F)$ there exists a positive integer $\tau \leq \min\{k, N(F)\}$ such that $f_s^\tau(I_i) \subset K(F)$ and $S(f_s^k) = S(F)$;*

(A2) *$f_s^k(x) = F(x)$ for all $x \in K(F)$, and f_s maps $K(F)$ onto itself homeomorphically;*

(A3) *$\|f_s^k - F\|_0 \leq \delta$ for a constant $\delta > 0$,*

then equation (1.1) *has a solution $f \in \mathrm{PM}(I,I)$ such that*

$$\|f_s - f\|_0 \leq \frac{1+M}{m^k}\delta.$$

The proof is similar to that of Theorem 3.1.

4 Examples

Example 4.1 Consider the mapping $F_1 : [-\frac{1}{2}, 1] \to [-\frac{1}{2}, 1]$, defined by

$$F_1(x) = \begin{cases} -x^3 + \frac{1}{2}x^2 + \frac{1}{4}x, & \forall x \in [-\frac{1}{2}, 0), \\ \frac{1}{2}x^2 + \frac{1}{4}x, & \forall x \in [0,1]. \end{cases}$$

Clearly, $F_1 \in C^2([-\frac{1}{2}, 0)) \cup C^2([0,1])$ and F_1 maps $[-\frac{1}{2}, 1]$ into $[0,1]$. Thus, $K(F_1) = [0,1]$. Moreover, $\lambda = \frac{1}{4}$ and all conditions in $\mathcal{PM}_1(I,\lambda)_-$ are satisfied on $K(F_1)$. Therefore, by Theorem 2.1 the C^1 iterative root of F_1 with 1-extension is C^1 stable.

Example 4.2 Define the mapping $F_2 : [0,1] \to [0,1]$ by

$$F_2(x) = \begin{cases} x, & \forall x \in [0, \frac{1}{2}], \\ 2x^2 - 2x + \frac{1}{2}, & \forall x \in (\frac{1}{2}, 1]. \end{cases}$$

Obviously, $H(F_2) = 1$ and $K(F_2) = [0, \frac{1}{2}]$. In order to demonstrate the validity of conditions in Theorem 3.1, consider the function $f_s : [0,1] \to [0,1]$:

$$f_s(x) = \begin{cases} x, & \forall x \in [0, \frac{1}{2}], \\ -x+1, & \forall x \in (\frac{1}{2}, 1], \end{cases}$$

which is Lipschitzian with the constants $m = \frac{49}{50}$, $M = \frac{51}{50}$ on $K(F_2)$, such that

$$\frac{49}{50}|x-y| \le \left|f_s(x) - f_s(y)\right| \le \frac{51}{50}|x-y|.$$

Moreover, one can check that conditions (A1) and (A2) in Theorem 3.1 are true for f_s. We further calculate that

$$f_s^2(x) = \begin{cases} x, & \forall x \in [0, \frac{1}{2}], \\ -x+1, & \forall x \in (\frac{1}{2}, 1]. \end{cases}$$

Hence, it follows that

$$\left|f_s^2(x) - F_2(x)\right| = \left|2x^2 - x - \frac{1}{2}\right| \le \frac{5}{8}, \quad \forall x \in [0,1].$$

Therefore, by Theorem 3.1, equation (1.1) has a solution $f \in \mathrm{PM}([0,1],[0,1])$ such that $\|f_s - f\|_0 \le \frac{12{,}625}{9{,}604}$ for all $x \in [0,1]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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